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Partial Differential Equations — *Continuous dependence on the data for nonlinear elliptic equations via symmetrization*, by MARIA FRANCESCA BETTA and ANNA MERCALDO.

Dedicated to the memory of Renato Caccioppoli

ABSTRACT. — We prove the continuous dependence on the data of weak solutions to Dirichlet problem for nonlinear elliptic equations with a first order term and datum in dual spaces of classical Sobolev spaces. We deduce uniqueness results.

KEY WORDS: Rearrangements, nonlinear elliptic equations, continuous dependence on the data, uniqueness.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 35J25, 35J60.

1. INTRODUCTION

In this paper we are interested in continuous dependence on the data and uniqueness of weak solutions to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\mathbf{a}(x,\nabla u)) + B(x,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$,

$$\mathbf{a} : (x,\xi) \in \Omega \times \mathbb{R}^N \to \mathbf{a}(x,\xi) = (a_i(x,\xi)) \in \mathbb{R}^N$$

and

$$B : (x,\xi) \in \Omega \times \mathbb{R}^N \to B(x,\xi) \in \mathbb{R}$$

are Carathéodory functions, f belongs to the dual space $W^{-1,p'}(\Omega)$ of $W_0^{1,p}(\Omega)$, for some $p \in]1,+\infty[$.

Standard assumptions which assure the existence of a weak solution to problem (1.1) are the ellipticity of the operator

$$\mathbf{a}(x,\xi)\cdot\xi \geq \lambda|\xi|^p, \quad \lambda > 0, \tag{1.2}$$

the growth conditions on $\mathbf{a}$ and B

$$|\mathbf{a}(x,\xi)| \le c[|\xi|^{p-1} + a_0(x)], \quad c > 0, \quad a_0 \in L^{p'}(\Omega), \tag{1.3}$$

$$|B(x,\xi)| \le B|\xi|^{p-1}, \quad B > 0, \tag{1.4}$$

and the monotonicity of $\mathbf{a}$

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\xi')) \cdot (\xi - \xi') > 0, \quad \xi \ne \xi', \tag{1.5}$$

for a.e. $x \in \Omega$, for all $\xi, \xi' \in \mathbb{R}^N$.

Under these assumptions a weak solution to problem (1.1) exists (cf. [8], [9], [12]), that is a function $u \in W_0^{1,p}(\Omega)$ exists such that

$$\int_\Omega \mathbf{a}(x,\nabla u) \cdot \nabla\varphi\, dx + \int_\Omega B(x,\nabla u)\varphi\, dx = \langle f, \varphi\rangle, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{1.6}$$

As far as uniqueness is concerned, more restricitive assumptions on the structure of the operator are required such as a monotonicity condition on $\mathbf{a}$ stronger then (1.5)

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\xi')) \cdot (\xi - \xi') \ge \alpha(\varepsilon + |\xi| + |\xi'|)^{p-2}|\xi - \xi'|^2, \quad \xi, \xi' \in \mathbb{R}^N, \tag{1.7}$$

where $\alpha > 0$, $\varepsilon > 0$ if $p \ge 2$ or $\varepsilon = 0$ if $p < 2$, and a local Lipschitz continuity condition on B

$$|B(x,\xi) - B(x,\xi')| \le b(\eta + |\xi| + |\xi'|)^{p-2}|\xi - \xi'|, \quad \xi, \xi' \in \mathbb{R}^N, \tag{1.8}$$

where $b > 0$ and $\eta = 0$ if $p \ge 2$ or $\eta > 0$ if $p < 2$.

Uniqueness results for weak solutions to (1.1) are proved under similar assumptions in [4], [6], [7], [11] and also in [1], [16] where they are obtained as a consequence of a comparison principle.

The aim of this paper is to prove the continuous dependence on the data and to deduce the uniqueness of a weak solution to (1.1) under the structural assumptions (1.7) and (1.8). Our approach is based on the classical symmetrization methods (cf. [13], [18]) which make use of isoperimetric inequalities and properties of rearrangements (see also [2], [5], [10]).

We point out that condition (1.7) is guaranteed if $\mathbf{a}(x,0) = 0$ and the following ellipticity condition holds

$$\sum_{ij=1}^{n} \frac{\partial a_i}{\partial z_j}(x,z)\xi_i\xi_j \ge (\varepsilon + |z|)^{p-2}|\xi|^2, \quad \xi \in \mathbb{R}^N.$$

Roughly speaking this means that the operator $\mathbf{a}$ can be reduced to a linear degenerate elliptic operator whose degenerancy is linked to the first order terms of problem (1.1). The model we have in mind is $\mathbf{a}(x,\xi) = (\varepsilon + |\nabla u|^2)^{(p-2)/2}\nabla u$, which yields the so-called p-Laplace operator when $p \le 2$ by our assumptions on ε. This linearization process suggests to require that the datum f belongs to a weighted dual space $H^{-1}(\Omega, m)$ for a suitable weight m linked to the degenerancy of the operator (cf. [15]). Actually, at least when $p > 2$, we assume that data of (1.1) belong to the smaller dual space $H^{-1}(\Omega)$. Such an hypothesy seems to be necessary in order to prove the continuous dependence on the data. No further restrictions are required when $p \leq 2$: under this assumption we prove the continuous dependence of weak solutions on data belonging to $W^{-1,p'}(\Omega)$.

Our main results are the following

THEOREM 1.1. *Let u, v be weak solutions to problem* (1.1) *with data $f, g \in H^{-1}(\Omega)$ respectively. Assume* (1.2), (1.3), (1.4), (1.7), (1.8) *and*

$$2 \leq p < \frac{2N}{N-2},$$

if $N \geq 3$ and $2 \leq p < +\infty$, if $N = 2$. Then the following inequality holds true

$$\| |\nabla u - \nabla v| \|_{L^p}^p \leq C \|f - g\|_{H^{-1}}^2, \tag{1.9}$$

where C is a positive constant which depends on N, $|\Omega|$, p, α, b, ε, $\|f\|_{H^{-1}}$ and $\|g\|_{H^{-1}}$; however it is bounded when f and g belong to bounded subset of $H^{-1}(\Omega)$.

THEOREM 1.2. *Let u, v be weak solutions to problem* (1.1) *with data $f, g \in W^{-1,p'}(\Omega)$ respectively. Assume* (1.2), (1.3), (1.4), (1.7), (1.8) *and*

$$\frac{2N}{N+2} < p < 2.$$

Then the following inequality holds true

$$\| |\nabla u - \nabla v| \|_{L^p} \leq C \|f - g\|_{W^{-1,p'}}, \tag{1.10}$$

where C is a positive constant which depends on N, $|\Omega|$, p, α, b, η, $\|f\|_{W^{-1,p'}}$ and $\|g\|_{W^{-1,p'}}$; however it is bounded when f and g belong to bounded subset of $W^{-1,p'}(\Omega)$.

Obviously Theorems 1.1 and 1.2 imply in turn uniqueness of weak solutions to (1.1). They improve, at least when $p < 2$, well-known results contained in [6], [11] and [16], since we find a larger range of the values of p for which uniqueness holds.

2. POINTWISE ESTIMATES

The proofs of Theorems 1.1 and 1.2 are based on a pointwise estimate for the decreasing rearrangement of $u - v$, difference of two weak solutions u, v to (1.1) corresponding to the data f, g respectively.

We recall that the decreasing rearrangement of a measurable function w defined in Ω is the function

$$w^*(s) = \sup\{t \geq 0 : \mu(t) > s\}, \quad s \in [0, |\Omega|],$$

where μ denotes its distribution function

$$\mu(t) = |\{x \in \Omega : |w(x)| > t\}|, \quad t \geq 0.$$

The estimate of the decreasing rearrangement of $u - v$ is proved by adapting classical symmetrization methods introduced in [13], [18] and extended to degenerate elliptic operators in [3].

Lemma 2.1. *Let u, v be weak solutions to problem* (1.1) *with data $f, g \in H^{-1}(\Omega)$ respectively. Assume* (1.2), (1.3), (1.4), (1.7), (1.8) *and*

$$2 \leq p < \frac{2N}{N-2},$$

with $N \geq 3$. Then we have

$$(u - v)^*(s) \leq C\|f - g\|_{H^{-1}}\, s^{-(N-2)/2N}, \quad s \in (0, |\Omega|), \tag{2.1}$$

where C is a positive constant which depends on N, $|\Omega|$, p, α, b, ε, $\|f\|_{H^{-1}}$ and $\|g\|_{H^{-1}}$; however it is bounded when f and g belong to bounded subset of $H^{-1}(\Omega)$.

Lemma 2.2. *Let u, v be weak solutions to problem* (1.1) *with data $f, g \in W^{-1,p'}(\Omega)$ respectively. Assume* (1.2), (1.3), (1.4), (1.7), (1.8) *and*

$$\frac{2N}{N+2} < p < 2.$$

Then we have

$$(u - v)^*(s) \leq C\|f - g\|_{W^{-1,p'}} s^{-(N-p)/Np}, \quad s \in (0, |\Omega|), \tag{2.2}$$

where C is a positive constant which depends on N, $|\Omega|$, p, α, b, η, $\|f\|_{W^{-1,p'}}$ and $\|g\|_{W^{-1,p'}}$; however it is bounded when f and g belong to bounded subset of $W^{-1,p'}(\Omega)$.

Proof of Lemma 2.1. Denote $w = u - v$, $h = f - g$ and $H \in (L^2(\Omega))^N$ the vector field such that

$$h = -\mathrm{div}(H). \tag{2.3}$$

For any fixed $t \in]0, \operatorname{ess\,sup} w[$ and $k > 0$ we consider the function

$$\varphi = \begin{cases} k \operatorname{sign} w & \text{if } |w| > t + k, \\ w - t \operatorname{sign} w & \text{if } t < |w| \le t + k, \\ 0 & \text{otherwise,} \end{cases}$$

as test function in (1.6) with datum f, g respectively. Then we subtract the equations and we divide by k,

$$\begin{aligned} &\frac{1}{k}\int_{t<|w|\le t+k} [\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla v)] \cdot \nabla w \, dx \\ &\quad = \int_{|w|>t+k} [B(x,\nabla u) - B(x,\nabla v)] \operatorname{sign} w \, dx \\ &\qquad + \frac{1}{k}\int_{t<|w|\le t+k} [B(x,\nabla u) - B(x,\nabla v)](w - t \operatorname{sign} w) \, dx \\ &\qquad + \frac{1}{k}\int_{t<|w|\le t+k} H \cdot \nabla w \, dx. \end{aligned}$$

By assumptions (1.7) and (1.8), using Hölder inequality and letting k goes to zero, we obtain

$$\begin{aligned} &-\frac{d}{dt}\int_{|w|>t} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \\ &\quad \le \frac{b}{\alpha}\int_{|w|>t} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w| \, dx + \frac{1}{\alpha \varepsilon^{(p-2)/2}} \left(-\frac{d}{dt}\int_{|w|>t} |H|^2 \, dx\right)^{1/2} \\ &\quad \times \left(-\frac{d}{dt}\int_{|w|>t} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx\right)^{1/2}. \end{aligned} \tag{2.4}$$

On the other hand by Schwarz and isoperimetric inequalities, it follows

$$\begin{aligned} &N\omega_N^{1/N} \mu(t)^{1-1/N} \\ &\quad \le -\frac{d}{dt}\int_{|w|>t} |\nabla w| \, dx \\ &\quad \le \frac{(-\mu'(t))^{1/2}}{\varepsilon^{(p-2)/2}} \left(-\frac{d}{dt}\int_{|w|>t} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx\right)^{1/2}, \end{aligned} \tag{2.5}$$

where ω_N denotes the measure of the unit ball of $\mathbb{R}^N$.

Therefore by (2.4) and (2.5), we get

$$
\Big(-\frac{d}{dt}\int_{|w|>t}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\Big)^{1/2}
\le \frac{b\,(-\mu'(t))^{1/2}}{\alpha N\omega_N^{1/N}\varepsilon^{(p-2)/2}\mu(t)^{1-1/N}}\int_{|w|>t}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|\,dx
+\frac{1}{\alpha\varepsilon^{(p-2)/2}}\Big(-\frac{d}{dt}\int_{|w|>t}|H|^2\,dx\Big)^{1/2}. \tag{2.6}
$$

Now we evaluate the first integral in the right-hand side of (2.6). By Schwarz inequality and coarea formula, we get

$$
\begin{aligned}
\int_{|w|>t}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|\,dx
&= \int_t^{+\infty}\Big(-\frac{d}{d\tau}\int_{|w|>\tau}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|\,dx\Big)\,d\tau\\
&\le \int_t^{+\infty}\Big(-\frac{d}{d\tau}\int_{|w|>\tau}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\Big)^{1/2}\\
&\quad\times\Big(-\frac{d}{d\tau}\int_{|w|>\tau}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}\,dx\Big)^{1/2}\,d\tau.
\end{aligned} \tag{2.7}
$$

Denote by $\underline{K}, \underline{H} : [0, |\Omega|) \to \mathbb{R}$ the functions which satisfy the following equalities

$$
\underline{K}(\mu(t))(-\mu'(t)) = -\frac{d}{dt}\int_{|w|>t}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}\,dx, \tag{2.8}
$$

$$
\underline{H}(\mu(t))(-\mu'(t)) = -\frac{d}{dt}\int_{|w|>t}|H|^2\,dx. \tag{2.9}
$$

Properties of such functions have been studied in [3], [17] (see also [14]).

Collecting (2.6), (2.7), (2.8) and (2.9), we get

$$
\begin{aligned}
&\Big(-\frac{d}{dt}\int_{|w|>t}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\Big)^{1/2}\\
&\quad\le \frac{b\,(-\mu'(t))^{1/2}}{\alpha N\omega_N^{1/N}\varepsilon^{(p-2)/2}\mu(t)^{1-1/N}}\int_t^{+\infty}\Big(-\frac{d}{d\tau}\int_{|w|>\tau}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\Big)^{1/2}\\
&\quad\quad\times(\underline{K}(\mu(\tau))^{1/2}(-\mu'(\tau))^{1/2}\,d\tau+\frac{1}{\alpha\varepsilon^{(p-2)/2}}(\underline{H}(\mu(t))^{1/2}(-\mu'(t))^{1/2}.
\end{aligned}
$$

By Gronwall Lemma, we deduce

$$
\begin{aligned}
(2.10)\qquad &\Big(-\frac{d}{dt}\int_{|w|>t}(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\Big)^{1/2}\\
&\le \frac{(\underline{H}(\mu(t))^{1/2}(-\mu'(t))^{1/2}}{\alpha\varepsilon^{(p-2)/2}}+\frac{b\,(-\mu'(t))^{1/2}}{\alpha^2 N\omega_N^{1/N}\varepsilon^{p-2}\mu(t)^{1-1/N}}\\
&\times\int_t^{+\infty}(\underline{H}(\mu(\tau))\underline{K}(\mu(\tau)))^{1/2}(-\mu'(\tau))\\
&\times\exp\Big(\frac{b}{\alpha N\omega_N^{1/N}\varepsilon^{(p-2)/2}}\int_t^{\tau}\frac{(\underline{K}(\mu(\sigma)))^{1/2}}{\mu(\sigma)^{1-1/N}}(-\mu'(\sigma))\,d\sigma\Big)\,d\tau.
\end{aligned}
$$

Taking into account (2.5), we obtain

$$
\begin{aligned}
1 &\le \frac{(\underline{H}(\mu(t)))^{1/2}(-\mu'(t))}{\alpha N\omega_N^{1/N}\varepsilon^{p-2}\mu(t)^{1-1/N}}+\frac{b\,(-\mu'(t))}{\alpha^2 N^2\omega_N^{2/N}\varepsilon^{(3/2)(p-2)}\mu(t)^{2-2/N}}\\
&\times\int_t^{+\infty}(\underline{H}(\mu(\tau))\underline{K}(\mu(\tau)))^{1/2}(-\mu'(\tau))\\
&\times\exp\Big(\frac{b}{\alpha N\omega_N^{1/N}\varepsilon^{(p-2)/2}}\int_t^{\tau}\frac{(\underline{K}(\mu(\sigma)))^{1/2}}{\mu(\sigma)^{1-1/N}}(-\mu'(\sigma))\,d\sigma\Big)\,d\tau,
\end{aligned}
$$

from which in a standard way we get

$$
\begin{aligned}
(2.11)\qquad -\frac{dw^*}{dr}(r) &\le \frac{\underline{H}(r)^{1/2}}{\alpha N\omega_N^{1/N}\varepsilon^{p-2}}r^{-1+1/N}\\
&+\frac{b}{\alpha^2N^2\omega_N^{2/N}\varepsilon^{(3/2)(p-2)}}r^{-2+2/N}\int_0^r(\underline{H}(\sigma)\underline{K}(\sigma))^{1/2}\\
&\times\exp\Big(\frac{b}{\alpha N\omega_N^{1/N}\varepsilon^{(p-2)/2}}\int_\sigma^r\frac{(\underline{K}(z))^{1/2}}{z^{1-1/N}}\,dz\Big)\,d\sigma.
\end{aligned}
$$

for $r\in(0,|\Omega|)$.

Now we evaluate the integral in the right-hand side of (2.11). To this aim we recall that the functions $\underline{K}$, $\underline{H}$ are weak limit of functions having the same rearrangement as $(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}$ and $|H|^2$ respectively. Therefore the Lebesgue norms of $\underline{K}$ and $\underline{H}$ can be estimated from above by the same norm of $(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}$ and $|H|^2$ respectively. This implies that $\underline{K}$ belongs to $L^{p/(p-2)}(0,|\Omega|)$ and $\underline{H}$ to $L^1(0,|\Omega|)$ respectively. Therefore, using Hölder inequality, since $p<\frac{2N}{N-2}$, we have

$$(2.12)\qquad \int_0^{|\Omega|} \frac{(\underline{K}(z))^{1/2}}{z^{1-1/N}}\,dz \le \Big(\int_\Omega (\varepsilon + |\nabla u| + |\nabla v|)^p\,dx\Big)^{(p-2)/2p} \times \Big(\int_0^{|\Omega|} \frac{1}{z^{((N-1)/N)(2p/(p+2))}}\,dz\Big)^{(p+2)/2p} < +\infty,$$

and

$$(2.13)\qquad \int_0^r (\underline{H}(\sigma)\underline{K}(\sigma))^{1/2}\,d\sigma \le \Big(\int_\Omega |H|^2\,dx\Big)^{1/2}\Big(\int_\Omega (\varepsilon + |\nabla u| + |\nabla v|)^p\,dx\Big)^{(p-2)/2p} r^{1/p}.$$

Denote by C a positive constant which depends only on the data and which can vary from line to line.

A priori estimates for the gradients of weak solutions to (1.1) are well-known (cf. Lemma 3.1 in [8] or [9]), that is

$$(2.14)\qquad \|\,|\nabla u|\,\|_{L^p} \le C\|f\|_{W^{-1,p'}}^{p'/p}.$$

Moreover, since $p \ge 2$,

$$(2.15)\qquad \|f\|_{W^{-1,p'}} \le C\|f\|_{H^{-1}}.$$

Combining (2.11), (2.12), (2.13), (2.14) and (2.15), we have the following differential inequality

$$(2.16)\qquad -\frac{dw^*}{dr}(r) \le C[\underline{H}(r)]^{1/2} r^{1/N-1} + C\|\,|H|\,\|_{L^2} r^{-2+2/N+1/p}, \quad r \in (0, |\Omega|).$$

Finally we integrate such an inequality between s and $|\Omega|$ and, by Hölder inequality and the property of $\underline{H}$ stated above, we get

$$w^*(s) \le C\|\,|H|\,\|_{L^2}\Big[\int_s^{|\Omega|} r^{-2+2/N+1/p}\,dr + \Big(\int_s^{|\Omega|} r^{2/N-2}\,dr\Big)^{1/2}\Big].$$

This yields (2.1). □

Remark 2.1. The previous proof can be easily adapted to the case when $N = 2$. Indeed the integration of differential inequality (2.16) yields the following pointwise estimate of $(u - v)^*$, which replaces (2.1),

$$(u - v)^*(s) \le C\|f - g\|_{H^{-1}}\Big[|\Omega|^{1/p} + \Big(\log\frac{|\Omega|}{s}\Big)^{1/2}\Big], \quad s \in (0, |\Omega|).$$

The proof of Lemma 2.2 is analogous to the proof of Lemma 2.1.

PROOF OF LEMMA 2.2. Denote $w = u - v$, $h = f - g$ and $H \in (L^{p'}(\Omega))^N$ the vector field such that (2.3) holds. As for (2.4), we get

$$-\frac{d}{dt}\int_{|w|>t}\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx \le \frac{b}{\alpha}\int_{|w|>t}\frac{|\nabla w|}{(\eta+|\nabla u|+|\nabla v|)^{2-p}}dx + \frac{1}{\alpha}\Big(-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{2-p}|H|^2\,dx\Big)^{1/2} \times\Big(-\frac{d}{dt}\int_{|w|>t}\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\Big)^{1/2}.$$

On the other hand by Schwarz and isoperimetric inequalities, it follows

$$N\omega_N^{1/N}\mu(t)^{1-1/N} \le \Big(-\frac{d}{dt}\int_{|w|>t}\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\Big)^{1/2} \times\Big(-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{2-p}\,dx\Big)^{1/2}. \tag{2.17}$$

Hence we have

$$\Big(-\frac{d}{dt}\int_{|w|>t}\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\Big)^{1/2} \le \frac{b}{\alpha N\omega_N^{1/N}\mu(t)^{1-1/N}}\Big(-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{2-p}\,dx\Big)^{1/2} \times\int_{|w|>t}\frac{|\nabla w|}{(\eta+|\nabla u|+|\nabla v|)^{2-p}}dx + \frac{1}{\alpha}\Big(-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{2-p}|H|^2\,dx\Big)^{1/2}. \tag{2.18}$$

By Schwarz inequality and coarea formula, since $\eta > 0$, we have

$$\int_{|w|>t}\frac{|\nabla w|}{(\eta+|\nabla u|+|\nabla v|)^{2-p}}dx \le \frac{1}{\eta^{(2-p)/2}}\int_t^{+\infty}\Big(-\frac{d}{d\tau}\int_{|w|>\tau}\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\Big)^{1/2}(-\mu'(\tau))^{1/2}\,d\tau. \tag{2.19}$$

We denote by $\overline{K}, \overline{H} : [0, |\Omega|) \to \mathbb{R}$ the functions which satisfy the following equalities

$$\overline{K}(\mu(t))(-\mu'(t)) = -\frac{d}{dt}\int_{|w|>t} (|\nabla u| + |\nabla v|)^{2-p}\, dx, \tag{2.20}$$

$$\overline{H}(\mu(t))(-\mu'(t)) = -\frac{d}{dt}\int_{|w|>t} (|\nabla u| + |\nabla v|)^{2-p}|H|^2\, dx. \tag{2.21}$$

Collecting (2.18), (2.19), (2.20) and (2.21), we get

$$\begin{aligned}
&\Big(-\frac{d}{dt}\int_{|w|>t} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\, dx\Big)^{1/2} \\
&\quad\le \frac{b(\overline{K}(\mu(t)))^{1/2}(-\mu'(t))^{1/2}}{\alpha\eta^{(2-p)/2} N\omega_N^{1/N}\mu(t)^{1-1/N}} \int_t^{+\infty} \Big(-\frac{d}{d\tau}\int_{|w|>\tau} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\, dx\Big)^{1/2} \\
&\quad\quad \times (-\mu'(\tau))^{1/2}\, d\tau + \frac{1}{\alpha}(\overline{H}(\mu(t)))^{1/2}(-\mu'(t))^{1/2}.
\end{aligned}$$

Now we apply Gronwall lemma and use (2.17) again. Therefore, as in the proof of Lemma 2.1, we obtain

$$\begin{aligned}
-\frac{dw^*}{dr}(r) \le{}& \frac{b\overline{K}(r) r^{-2+2/N}}{\alpha^2\eta^{(2-p)/2} N^2\omega_N^{2/N}} \int_0^r (\overline{H}(\sigma))^{1/2} \\
&\times \exp\Big(\frac{b}{\alpha N\omega_N^{1/N}\eta^{(p-2)/2}} \int_\sigma^r \frac{(\overline{K}(z))^{1/2}}{z^{1-1/N}}\, dz\Big)\, d\sigma \\
&+ \frac{1}{\alpha N\omega_N^{1/N}} (\overline{K}(r))^{1/2}(\overline{H}(r))^{1/2} r^{-1+1/N},
\end{aligned} \tag{2.22}$$

for $r \in (0, |\Omega|)$. Let us evaluate the integral in the right-hand side of (2.22). By the property of $\overline{H}$ and $\overline{K}$ stated above, $\overline{K}$ belongs to $L^{p/(2-p)}(0, |\Omega|)$ and $\overline{H}$ to $L^1(0, |\Omega|)$. Therefore, using Hölder inequality, since $p > \frac{2N}{N+2}$, we have

$$\begin{aligned}
\int_0^{|\Omega|} \frac{(\overline{K}(z))^{1/2}}{z^{1-1/N}}\, dz \le{}& \Big(\int_\Omega (|\nabla u| + |\nabla v|)^p\, dx\Big)^{(2-p)/2p} \\
&\times \Big(\int_0^{|\Omega|} \frac{1}{z^{((N-1)/N)(2p/(3p-2))}}\, dz\Big)^{(3p-2)/2p} < +\infty
\end{aligned} \tag{2.23}$$

and

$$\int_0^r (\overline{H}(\sigma))^{1/2}\, d\sigma \le \Big(\int_\Omega (|\nabla u| + |\nabla v|)^{2-p}|H|^2\, dx\Big)^{1/2} r^{1/2}. \tag{2.24}$$

Taking into account (2.22), (2.23), (2.24) and the a priori estimates (2.14), we get

$$-\frac{dw^*}{dr}(r) \le C\Big[\Big(\int_\Omega (|\nabla u| + |\nabla v|)^{2-p}|H|^2\,dx\Big)^{1/2}\overline{K}(r)r^{-2+2/N+1/2} + (\overline{K}(r))^{1/2}(\overline{H}(r))^{1/2}r^{1/N-1}\Big], \tag{2.25}$$

for $r \in (0, |\Omega|)$. Now we integrate such an inequality between s and $|\Omega|$ and then we use Hölder inequality and the a priori estimates (2.14). Therefore we get

$$\begin{aligned} w^*(s) \le C\Big(\int_\Omega (|\nabla u| + |\nabla v|)^{2-p}|H|^2\,dx\Big)^{1/2}\|\overline{K}\|^{1/2}_{L^{p/(2-p)}} \\ \times\Big[\|\overline{K}\|^{1/2}_{L^{p/(2-p)}}\Big(\int_s^{|\Omega|} r^{(-3/2+2/N)(p/2(p-1))}\,dr\Big)^{2(p-1)/p} \\ + \Big(\int_s^{|\Omega|} r^{(1/N-1)(p/(p-1))}\,dr\Big)^{(p-1)/p}\Big]. \end{aligned} \tag{2.26}$$

Since $H \in L^{p'}(\Omega)$, using Hölder inequality, we get

$$\int_\Omega (|\nabla u| + |\nabla v|)^{2-p}|H|^2\,dx \le \|\,|\nabla u| + |\nabla v|\,\|^{2-p}_{L^p}\|H\|^2_{L^{p'}}.$$

Combining this inequality, (2.26) and the a priori estimates (2.14), we get (2.2). □

3. Continuous dependence on the data

The pointwise estimates proved in the previous section imply estimates in Lebesgue spaces of $u - v$ in terms of the norms in dual space of the data. Indeed under the assumptions of Lemma 2.1 (see also Remark 2.1), we have the following estimate of L^p-norm of $u - v$

$$\|u - v\|_{L^p} \le C\|f - g\|_{H^{-1}}, \tag{3.1}$$

while under the assumptions of Lemma 2.2,

$$\|u - v\|_{L^2} \le C\|f - g\|_{W^{-1,p'}}. \tag{3.2}$$

These estimates play an important role in the proof of the continuous dependence of the weak solutions to (1.1) on the data.

Proof of Theorem 1.1. Denote $h = f - g$ and $H \in (L^2(\Omega))^N$ the vector field defined by (2.3). We consider $w = u - v$ as test function in (1.1) with data f and g respectively. Then we subtract the two equations and, using (1.7) and (1.8), we get

$$\alpha\int_\Omega(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx$$
$$\leq b\int_\Omega(|\nabla u|+|\nabla v|)^{p-2}|\nabla w|\,|w|\,dx+\int_\Omega|H|\,|\nabla w|\,dx.$$

By Hölder inequality we have

$$\begin{aligned}(3.3)\qquad \alpha\int_\Omega&(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\\ &\leq b\Big(\int_\Omega(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\Big)^{1/2}\\ &\quad\times\Big(\int_\Omega(|\nabla u|+|\nabla v|)^p\,dx\Big)^{(p-2)/2p}\Big(\int_\Omega|w|^p\,dx\Big)^{1/p}\\ &\quad+\frac{1}{\varepsilon^{(p-2)/2}}\Big(\int_\Omega|H|^2\,dx\Big)^{1/2}\Big(\int_\Omega(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\Big)^{1/2}.\end{aligned}$$

On the other hand, since $p\geq 2$,

$$\int_\Omega|\nabla u-\nabla v|^p\,dx\leq\int_\Omega(\varepsilon+|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx.$$

Therefore combining (3.1), (3.3) and (2.14), we get (1.9). □

The proof of Theorem 1.2 is similar to the previous proof.

Proof of Theorem 1.2. As in the proof of Theorem 1.1, we consider $w=u-v$ as test function, then we subtract the two equations and we use (1.7) and (1.8)

$$\alpha\int_\Omega\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\leq b\int_\Omega\frac{|\nabla w|\,|w|}{(\eta+|\nabla u|+|\nabla v|)^{2-p}}dx+\int_\Omega|H|\,|\nabla w|\,dx,$$

where $H\in(L^{p'}(\Omega))^N$ is the vector field defined by (2.3) holds.

By Schwarz inequality we have

$$\begin{aligned}(3.4)\qquad \alpha\int_\Omega&\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\\ &\leq\frac{b}{\eta^{(2-p)/2}}\Big(\int_\Omega\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\Big)^{1/2}\Big(\int_\Omega|w|^2\,dx\Big)^{1/2}\\ &\quad+\Big(\int_\Omega(|\nabla u|+|\nabla v|)^{2-p}|H|^2\,dx\Big)^{1/2}\Big(\int_\Omega\frac{|\nabla w|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\Big)^{1/2}.\end{aligned}$$

On the other hand by Hölder inequality

$$\int_\Omega (|\nabla u| + |\nabla v|)^{2-p} |H|^2 \, dx \le \| \, |\nabla u| + |\nabla v| \, \|_{L^p}^{2-p} \|H\|_{L^{p'}}^2. \tag{3.5}$$

Finally, since $p < 2$,

$$\int_\Omega |\nabla w|^p \, dx \le \Big(\int_\Omega \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \Big)^{p/2} \Big(\int_\Omega (|\nabla u| + |\nabla v|)^p \, dx \Big)^{(2-p)/2}. \tag{3.6}$$

Combining (3.2), (3.4), (3.5), (3.6) and the a priori estimates (2.14), we get (1.10). □

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