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Mathemtical Analysis — *Optimal regularity results in spaces of Hölder continuous functions for some infinite dimensional Ornstein-Uhlenbeck semigroup*, by Giuseppe Da Prato.

Dedicated to the memory of Renato Caccioppoli

Abstract. — We consider the elliptic equation $\lambda\varphi - L\varphi = f$ where $\lambda > 0$, f is θ-Hölder continuous and L is an Ornstein-Uhlenbeck operator in a Hilbert space H. We show that the mapping $D^2\varphi$ (with values in the space of Hilbert–Schmidt operators on H) is θ-Hölder continuous.

Key words: PDEs with infinitely many variables, Schauder estimates, Ornstein-Uhlenbeck semigroup.

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1. Introduction and setting of the problem

Let H be a separable real Hilbert space (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$). We are given a linear operator $A : D(A) \subset H \to H$ such that

Hypothesis 1.1.

(i) *A is self–adjoint and there exists $\omega > 0$ such that*

$$\langle Ax, x\rangle \le -\omega|x|^2, \quad x \in D(A). \tag{1.1}$$

(ii) *A^{-1} is of trace class.*

As well known, Hypothesis 1.1 implies that there exists a complete orthonormal system (e_k) in H and a sequence of real numbers (a_k) greater than ω such that

$$Ae_k = -a_k e_k, \quad \forall k \in \mathbb{N} \tag{1.2}$$

and

$$\mathrm{Tr}[-A^{-1}] = \sum_{k=1}^{\infty} \frac{1}{a_k} < +\infty.$$

Under Hypothesis 1.1 we can consider the Ornstein–Uhlenbeck semigroup defined by (see [6])

$$R_t\varphi(x) = \int_H \varphi(e^{tA}x + y)N_{Q_t}(dy), \quad \forall t > 0,\ x \in H. \tag{1.3}$$

Here $\varphi : H \to \mathbb{R}$ is any continuous function with e.g. polynomial growth (that is such that $|\varphi(x)| \le M(1 + |x|^n)$ for all $x \in H$ and some $M > 0$, $n \in \mathbb{N}$) and N_{Q_t} is the Gaussian measure in H with mean 0 and covariance operator Q_t given by

$$Q_t = -\frac{1}{2}A^{-1}(1 - e^{2tA}), \quad \forall t \ge 0.$$

Note that the Gaussian measure N_{Q_t} is well defined since A^{-1}, and consequently Q_t, is of trace class.

Let us define the infinitesimal generator L of R_t through its Laplace transform (as in [2]) setting for any $\lambda > 0$ and for any continuous function $f : H \to \mathbb{R}$ with polynomial growth

$$(\lambda - L)^{-1}f(x) = \int_0^\infty e^{-\lambda t}R_t f(x)\,dt, \quad \forall x \in H. \tag{1.4}$$

The operator L acts as a concrete differential operator on the space $\mathscr{E}_A(H)$ of all *exponential functions* defined as the linear span of all real parts of functions φ_h of the form

$$\varphi_h(x) = e^{i\langle x,h\rangle}, \quad \forall x \in H, \tag{1.5}$$

where h varies in $D(A)$. It is not difficult in fact to check that

$$L\varphi = \frac{1}{2}\,\mathrm{Tr}[D^2\varphi] + \langle x, AD\varphi\rangle, \quad \forall \varphi \in \mathscr{E}_A(H). \tag{1.6}$$

This paper is devoted to the study of the elliptic equation

$$\lambda\varphi - L\varphi = f, \tag{1.7}$$

where $\lambda > 0$ is a given number and f is a given function in a suitable functional space. As we shall see there is a dramatic difference between the case when H is finite or infinite dimensional. In order to better illustrate this difference it is convenient to recall what happens when f belongs to $L^2(H,\mu)$ where μ is the unique invariant measure of R_t, $t \ge 0$. The short Section 2 is devoted to recall the main results in this case. Finally, Section 3 is devoted to study (1.7) in spaces of Hölder continuous functions. We first recall previous optimal regularity result proved in [1] and [3] and then we present a new optimal regularity result. This last result will allow us to take into account a new kind of perturbations of the Ornstein–Uhlenbeck diffusion process for which it is will possible to prove existence and uniqueness of an associated martingale problems, arguing as in [13]. These facts will be the object of a future paper.

REMARK 1.2. R_t is the transition semigroup of the diffusion process $X(t)$, $t \geq 0$, the solution to the differential stochastic equation

$$\begin{cases} dX(t) = AX(t)\,dt + dW(t), & t \geq 0, \\ X(0) = x \in H, \end{cases} \tag{1.8}$$

where $W(t)$ is a cylindrical Wiener process in some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in H. We can take $W(t)$ as

$$\langle W(t), z\rangle = \sum_{k=0}^{\infty} \beta_k \langle z, e_k\rangle, \quad \forall z \in H,$$

where (β_k) is a family of mutually independent standard Brownian motions on $(\Omega, \mathscr{F}, \mathbb{P})$. Then we have

$$R_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad t \geq 0,\ x \in H,\ \varphi \in C_b(H),$$

where $\mathbb{E}$ denotes the expectation.

2. OPTIMAL REGULARITY RESULTS FOR $f \in L^2(H, \mu)$

By (1.3) it follows easily that $\mu = N_{Q_\infty}$, where

$$Q_\infty = -\frac{1}{2}A^{-1},$$

is the unique invariant measure for R_t, $t \geq 0$, that is

$$\int_H R_t\varphi(x)\mu(dx) = \int_H \varphi(x)\mu(dx),$$

for all $\varphi : H \to \mathbb{R}$ continuous and bounded. So, R_t can be uniquely extended to $L^2(H, \mu)$ (even to $L^p(H, \mu)$ for any $p \geq 1$) which we shall denote by R_t^2. The infinitesimal generator of R_t^2 will be denoted by L_2.

The following result can be found in [7], see also [4, (10.55)].

PROPOSITION 2.1. *Let $\lambda > 0$ and $f \in L^2(H, \mu)$. Then equation* (1.8) *has a unique solution $\varphi \in D(L_2)$ with the following properties*

$$\varphi \in W^{2,2}(H, \mu), \tag{2.1}$$

$$(-A)^{1/2} D\varphi \in L^2(H, \mu; H). \tag{2.2}$$

Moreover the following identity holds.

$$\int_H (L_2\varphi)^2\,d\mu = \frac{1}{2}\int_H \mathrm{Tr}[(D^2\varphi)^2]\,d\mu + \int_H |(-A)^{1/2}D\varphi|^2\,d\mu. \tag{2.3}$$

Notice that if the dimension of H is finite, equation (1.8) reduces to

$$\lambda\varphi - \frac{1}{2}\Delta\varphi - \langle x, AD\varphi\rangle = f,$$

so that, by (2.1) it follows that both terms

$$\Delta\varphi, \quad \langle x, AD\varphi\rangle$$

belong to $L^2(H,\mu)$. Nothing similar happens if the dimension of H is infinite. In this case we have no information on the terms

$$\frac{1}{2}\mathrm{Tr}[D^2\varphi], \quad \langle x, AD\varphi\rangle,$$

we know only that the sum of these two terms is meaningful. However, the weaker informations (2.1) and (2.2) are available. When H is infinite dimensional A is unbounded and so, identity (2.3) shows that they are in a sense optimal.

3. Optimal regularity results in space of Hölder continuous functions

3.1. Introduction

Here we consider equation (1.8) when f belongs to the space of all θ-Hölder continuous and bounded real functions on H, which we denote by $C_b^\theta(H)$.

We start by recalling some known results.

Theorem 3.1. *Assume that Hypothesis* 1.1 *holds. Let* $\theta \in (0,1)$, $f \in C_b^\theta(H)$, $\lambda > 0$ *and let* $\varphi = (\lambda - L)^{-1}f$ *be the solution to* (1.8). *Then the following statements hold.*

(i) *φ belongs to $C_b^{2+\theta}(H)$ and there exists $M > 0$ (independent on λ and on f) such that*

$$\|\varphi\|_{C_b^{2+\theta}(H)} \le M\|f\|_{C_b^\theta(H)}. \tag{3.1}$$

(ii) *For all $x \in H$ we have $D\varphi(x) \in D((-A)^{1/2})$ and $(-A)^{1/2}D\varphi \in C_b^\theta(H)$. Moreover, there exists $M_1 > 0$ (independent on λ and on f) such that*

$$\|(-A)^{1/2}D\varphi\|_{C_b^\theta(H)} \le M_1\|f\|_{C_b^\theta(H)}. \tag{3.2}$$

For a precise definition of $C_b^\theta(H)$ and $C_b^{2+\theta}(H)$ see the end of this subsection. The Schauder estimate (i) was proved in [1] whereas (ii) was proved in [3]. Clearly (ii) is a counterpart of (2.2) in the Hölder setting. The main result of this paper is the proof of a counterpart of (2.1), namely that if $f \in C_b^\theta(H)$ then

(iii) $D^2\varphi \in C_b^\theta(H, L_2(H))$ *and there exists* $M_3 > 0$ (*independent on* λ *and on* f) *such that*

$$\|D^2\varphi\|_{C_b^\theta(H, L_2(H))} \le M_3 \|f\|_{C_b^\theta(H)}. \tag{3.3}$$

Remark 3.2. When H is finite-dimensional, the Schauder estimates (3.1) were proved in [5]. Even in this case they are not consequence of the general results in [8] because the Ornstein–Uhlenbeck operator has unbounded coefficients.

Remark 3.3. A result similar to (iii) was proved for the Gross Laplacian by [11].

Let us finish this section by giving some notation and by recalling the definition of interpolation spaces needed in what follows.

3.1.1 Notations. In all the paper H is a separable Hilbert space, $A : D(A) \subset H \to H$ is a linear operator fulfilling Hypothesis 1.1 and (e_h) is an orthonormal basis defined by (1.2). For each $x \in H$ and any $h \in \mathbb{N}$ we set $x_h = \langle x, e_h \rangle$.

By $L_2(H)$ we denote the Hilbert space of all Hilbert–Schmidt operators from H into H endowed with the inner product

$$\langle T, S \rangle = \mathrm{Tr}[TS^*], \quad \forall T \in L_2(H)$$

and the norm

$$\|T\|_{L_2(H)}^2 = \mathrm{Tr}[TT^*] = \sum_{h,k=1}^{\infty} |\langle Te_h, e_k \rangle|^2, \quad \forall T \in L_2(H).$$

Let E be a Banach space. We shall denote by $C_b(H; E)$ the Banach space of all uniformly continuous and bounded functions from H into E endowed with the norm $\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|_E$. For any $k \in \mathbb{N}$ we denote by $C^k(H; E)$ the space of all mappings $\varphi : H \to E$ which are uniformly continuous and bounded together with their derivatives up to the k-th order. $C^k(H; E)$ is a Banach space with the norm

$$\|\varphi\|_k = \sum_{h=1}^{k} \sup_{x \in H} \|D^h\varphi(x)\|.$$

Here $D^h\varphi(x)$ is the derivative of φ at x of order h and $\|D^h\varphi(x)\|$ is the usual norm of the h-linear form $D^h\varphi(x)$.

Finally, if $\theta \in (0,1)$, we shall denote by $C_b^\theta(H;E)$ (resp. $C_b^{k+\theta}(H;E)$, $k \in \mathbb{N}$) the subspace of $C_b(H;E)$ (resp. $C^k(H;E)$) consisting of all functions $\varphi : H \to E$ such that

$$[\varphi]_\theta := \sup_{\substack{x,y\in H\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\theta} < +\infty,$$

(respectively,

$$[\varphi]_{k+\theta} := \sup_{\substack{x,y\in H\\ x\neq y}} \frac{\|D^k\varphi(x)-D^k\varphi(y)\|}{|x-y|^\theta} < +\infty.)$$

$C_b^\theta(H;E)$ is a Banach space with the norm

$$\|\varphi\|_\theta := \|\varphi\|_0 + [\varphi]_\theta, \quad \varphi \in C_b^\theta(H;E).$$

When $E=\mathbb{R}$ we shall write $C_b^k(H;\mathbb{R}) = C_b^k(H)$ and $C_b^{k+\theta}(H;\mathbb{R}) = C_b^{k+\theta}(H)$.

3.1.2 Interpolation spaces. We shall use the K method for real interpolation spaces, see e.g. [12]. Let X and Y be Banach spaces such that $Y \subset X$ with continuous embedding. For any $t>0$ and any $x \in H$ define

$$K(t,x) = \inf\{\|a\|_X + t\|b\|_Y : x = a+b, a \in X, b \in Y\}.$$

Then, for arbitrary $\theta \in (0,1)$, set

$$\|x\|_{(X,Y)_{\theta,\infty}} = \sup_{t>0} t^{-\theta} K(t,x),$$

$$(X,Y)_{\theta,\infty} = \{x \in X : \|x\|_{(X,Y)_{\theta,\infty}} < +\infty\}.$$

As is easily seen $(X,Y)_{\theta,\infty}$, endowed with the norm

$$\|x\|_{(X,Y)_{\theta,\infty}},$$

is a Banach space.

Remark 3.4. It is not difficult to check that the following statement (i):

(i) For all $t>0$ there exist $a_t \in X$ and $b_t \in Y$ such that $x = a_t + b_t$ and

$$\|a_t\|_X + t\|b_t\|_Y \le Lt^\theta,$$

implies that

(ii) $x \in (X,Y)_{\theta,\infty}$ and $\|x\|_{(X,Y)_{\theta,\infty}} \le L$.

Conversely, statement (ii) implies that $\forall \varepsilon > 0$, $\forall t>0$ there exist $a_t \in X$ and $b_t \in Y$ such that $x = a_t + b_t$ and

$$\|a_t\|_X + t\|b_t\|_Y \le (L+\varepsilon)t^\theta.$$

Let us recall the basic interpolation theorem, see e.g. [12].

THEOREM 3.5. *Let X, X_1, Y, Y_1 be Banach spaces such that $Y \subset X$, $Y_1 \subset X_1$ with continuous embeddings. Let moreover T be a linear mapping $T : X \to X_1$, $T : Y \to Y_1$, such that for some $M, N > 0$*

$$\|Tx\|_{X_1} \le M\|x\|_X, \quad \|Ty\|_{Y_1} \le N\|y\|_Y.$$

Then T maps $(X, Y)_{\theta,\infty}$ into $(X_1, Y_1)_{\theta,\infty}$, and

$$\|Tx\|_{(X_1,Y_1)_{\theta,\infty}} \le M^{1-\theta}N^{\theta}\|x\|_{(X,Y)_{\theta,\infty}}, \quad x \in (X, Y)_{\theta,\infty}.$$

We shall need also the following result, see [1].

THEOREM 3.6. *Let K be a separable Hilbert space. Then we have*

$$(C_b(K), C_b^1(K))_{\theta,\infty} = C_b^{\theta}(K), \quad \forall\theta \in (0,1). \tag{3.4}$$

Moreover there exists a positive constant κ_θ such that

$$\frac{1}{\kappa_\theta}\|\varphi\|_{C_b^{\theta}(K)} \le \|\varphi\|_{(C_b(K),C_b^1(K))_{\theta,\infty}} \le \kappa_\theta\|\varphi\|_{C_b^{\theta}(K)}. \tag{3.5}$$

REMARK 3.7. Let $\varphi \in C_b(K)$ and let $\theta \in (0,1)$. By Remark 3.4 to prove that $\varphi \in C_b^{\theta}(K)$ it is enough to prove that for any $t \in (0,1]$ there exist $a_t \in C_b(K)$ and $b_t \in C_b^1(K)$ such that $\varphi = a_t + b_t$ and

$$\|a_t\|_0 \le \kappa t^{\theta}, \quad \|b_t\|_1 \le \kappa t^{\theta-1} \tag{3.6}$$

for a suitable positive constant κ.

3.2. Estimates

We assume here that Hypothesis 1.1 holds. Under this assumption for any $t > 0$ and any $\varphi \in C_b(H)$ we have that $R_t\varphi \in C_b^{\infty}(H)$, see [7]. Moreover, the following expressions hold for the three first derivatives of $R_t\varphi$.

$$\langle DR_t\varphi(x), \alpha\rangle = \int_H \langle\Lambda_t\alpha, Q_t^{-1/2}y\rangle\varphi(e^{tA}x + y)N_{Q_t}(dy), \quad \forall x, \alpha \in H, \tag{3.7}$$

$$\begin{aligned}\langle D^2R_t\varphi(x)\cdot\alpha, \beta\rangle = &\int_H \langle\Lambda_t\alpha, Q_t^{-1/2}y\rangle\langle\Lambda_t\beta, Q_t^{-1/2}y\rangle\varphi(e^{tA}x + y)N_{Q_t}(dy)\\ &- \langle\Lambda_t\alpha, \Lambda_t\beta\rangle R_t\varphi(x), \quad \forall x, \alpha, \beta \in H.\end{aligned} \tag{3.8}$$

and, for any $x, \alpha, \beta, \gamma \in H$,

$$(3.9)\qquad \begin{aligned} &D^3R_t\varphi(x)(\alpha,\beta,\gamma)\\ &= \int_H \langle\Lambda_t\alpha, Q_t^{-1/2}y\rangle\langle\Lambda_t\beta, Q_t^{-1/2}y\rangle\langle\Lambda_t\gamma, Q_t^{-1/2}y\rangle\varphi(e^{tA}x+y)N_{Q_t}(dy)\\ &\quad - (\langle\Lambda_t\alpha,\Lambda_t\beta\rangle D_\gamma R_t\varphi(x) + \langle\Lambda_t\alpha,\Lambda_t\gamma\rangle D_\beta R_t\varphi(x)\\ &\quad + \langle\Lambda_t\beta,\Lambda_t\gamma\rangle D_\alpha R_t\varphi(x)). \end{aligned}$$

Here we have set

$$(3.10)\qquad \Lambda_t = Q_t^{-1/2}e^{tA} = \sqrt{2}(-A)^{1/2}e^{tA}(1-e^{2tA})^{-1/2}.$$

Lemma 3.8. *There exist* $c_1 > 0$ *such that*

$$(3.11)\qquad \|\Lambda_t\| \le c_1 t^{-1/2},\quad \forall t>0,$$

Proof. It is enough to notice that

$$\begin{aligned} \|\Lambda_t\| &= \sup_{k\in\mathbb{N}} \sqrt{2a_k}e^{-ta_k}(1-e^{-2ta_k})^{-1/2}\\ &\le t^{-1/2}\sup_{\xi>0}\sqrt{2\xi}e^{-\xi}(1-e^{-2\xi})^{-1/2},\quad t>0. \end{aligned}$$ □

Lemma 3.9. *Let* $\varphi \in C_b(H)$ *and* $t>0$*. Then* $D^2R_t\varphi \in C_b(H;L_2(H))$ *and there exists* $d_1 > 0$ *such that*

$$(3.12)\qquad \|D^2R_t\varphi(x)\|_{L_2(H)} \le d_1 t^{-1}\|\varphi\|_0,\quad \forall t>0,\ x\in H.$$

Proof. By (3.8) we have for all $h,k\in\mathbb{N}$

$$\begin{aligned} \langle D^2R_t\varphi(x)\cdot e_h, e_k\rangle &= \int_H \langle\Lambda_t e_h, Q_t^{-1/2}y\rangle\langle\Lambda_t e_k, Q_t^{-1/2}y\rangle\varphi(e^{tA}x+y)N_{Q_t}(dy)\\ &\quad - \langle\Lambda_t e_h,\Lambda_t e_k\rangle R_t\varphi(x), \end{aligned}$$

which can be written as

$$\begin{aligned} \langle D^2R_t\varphi(x)\cdot e_h, e_k\rangle &= \Lambda_{t,h}\Lambda_{t,k}\lambda_h(t)^{-1/2}\lambda_k(t)^{-1/2}\int_H y_hy_k\varphi(e^{tA}x+y)N_{Q_t}(dy)\\ &\quad - \Lambda_{t,h}^2\delta_{h,k}\int_H\varphi(e^{tA}x+y)N_{Q_t}(dy), \end{aligned}$$

where $y_k = \langle y, e_k\rangle$ for all $k\in\mathbb{N}$ and for $t>0$, $\Lambda_{t,k}$, $k\in\mathbb{N}$, is the sequence of eigenvalues of Λ_t defined by,

$$(3.13)\qquad \Lambda_t e_k = \Lambda_{t,k}e_k,\quad \forall t>0,\ k\in\mathbb{N},$$

whereas $\lambda_k(t)$, $h \in \mathbb{N}$, are the sequence of eigenvalues of Q_t,

$$Q_t e_k = \lambda_k(t) e_k, \quad h \in \mathbb{N}. \tag{3.14}$$

In order to estimate $\|D^2 R_t \varphi(x)\|_{L_2(H)}$ we proceed as in [7, Lemma 6.2.7], introducing a suitable orthonormal system in $L^2(H, N_{Q_t})$. More precisely, for any $t > 0$ we define

$$\Phi_{h,k}(t) = \begin{cases} 2^{-1/2}(\lambda_h^{-1}(t) y_h^2 - 1), & \text{if } h = k, \\ \lambda_h^{-1/2}(t)\lambda_k^{-1/2}(t) y_h y_k & \text{if } h \neq k. \end{cases} \tag{3.15}$$

(It is not difficult to check that $(\Phi_{h,k}(t))$ is indeed orthonormal in $L^2(H, N_{Q_t})$ for any $t > 0$.)

Now let $h = k \in \mathbb{N}$ and write

$$\begin{aligned} \langle D^2 R_t \varphi(x) \cdot e_k, e_k \rangle &= \sqrt{2}\Lambda_{t,k}^2 \int_H \Phi_{k,k}(t) \varphi(e^{tA}x + y) N_{Q_t}(dy) \\ &= \sqrt{2}\Lambda_{t,k}^2 \langle \Phi_{k,k}(t), \varphi(e^{tA}x + \cdot) \rangle_{L^2(H, N_{Q_t})}. \end{aligned}$$

Recalling that $|\Lambda_{t,k}^2| \leq \|\Lambda_t\|^2$ for all $k \in \mathbb{N}$ and all $t > 0$ we have

$$|\langle D^2 R_t \varphi(x) \cdot e_k, e_k \rangle|^2 \leq 2\|\Lambda_t\|^4 |\langle \Phi_{k,k}(t), \varphi(e^{tA}x + \cdot) \rangle_{L^2(H, N_{Q_t})}|^2.$$

Summing up on k we deduce by the Parseval inequality that

$$\sum_{k=1}^{\infty} |\langle D^2 R_t \varphi(x) \cdot e_k, e_k \rangle|^2 \leq 2\|\Lambda_t\|^4 \int_H |\varphi(e^{tA}x + y)|^2 N_{Q_t}(dy) = 2\|\Lambda_t\|^4 \|\varphi\|_0^2.$$

Now from (3.11) we have

$$\sum_{k=1}^{\infty} |\langle D^2 R_t \varphi(x) \cdot e_k, e_k \rangle|^2 \leq 2c_1^4 t^{-2} \|\varphi\|_0^2. \tag{3.16}$$

Let now $h \neq k \in \mathbb{N}$ and write

$$\begin{aligned} \langle D^2 R_t \varphi(x) \cdot e_h, e_k \rangle &= \Lambda_{t,h}\Lambda_{t,k} \int_H \Phi_{h,k}(t) \varphi(e^{tA}x + y) N_{Q_t}(dy) \\ &= \Lambda_{t,h}\Lambda_{t,k} \langle \Phi_{h,k}(t), \varphi(e^{tA}x + \cdot) \rangle_{L^2(H, N_{Q_t})}. \end{aligned}$$

Proceeding as before we see that

$$|\langle D^2 R_t \varphi(x) \cdot e_h, e_k \rangle|^2 \leq \|\Lambda_t\|^4 |\langle \Phi_{h,k}(t), \varphi(e^{tA}x + \cdot) \rangle_{L^2(H, N_{Q_t})}|^2.$$

By the Parseval inequality we deduce that

$$\sum_{h,k=1,h\neq k}^{\infty} |\langle D^2 R_t\varphi(x)\cdot e_h, e_k\rangle|^2 \le \|\Lambda_t\|^4\|\varphi\|_0^2.$$

Now from (3.11) we have

$$\sum_{h,k=1}^{\infty} |\langle D^2 R_t\varphi(x)\cdot e_h, e_k\rangle|^2 \le c_1^4 t^{-2}\|\varphi\|_0^2. \tag{3.17}$$

By (3.16) and (3.17) it follows that

$$\mathrm{Tr}[(D^2 R_t\varphi(x))^2] \le 2c_1^4 t^{-2}\|\varphi\|_0^2, \tag{3.18}$$

which proves the result with $d_1 = \sqrt{2}c_1^2$. However, it remains to show that $D^2R_t\varphi \in C_b(H, L_2(H))$. To this purpose let us introduce a one-to-one mapping $\psi : \mathbb{N}\times\mathbb{N} \mapsto \mathbb{N}$. For any $x, y \in H$ and any $N \in \mathbb{N}$ we have

$$\begin{aligned}\|D^2R_t\varphi(x) - D^2R_t\varphi(y)\|_{L_2(H)}^2 &= \sum_{(h,k):\psi(h,k)=1}^{N} (D_hD_kR_t\varphi(x) - D_hD_kR_t\varphi(y))^2\\ &\quad + \sum_{(h,k):\psi(h,k)=N+1}^{\infty} (D_hD_kR_t\varphi(x) - D_hD_kR_t\varphi(y))^2\\ &:= I_1 + I_2.\end{aligned}$$

Now I_2 can be made arbitrarily small by (3.18) choosing N sufficiently large, then I_1 goes to zero when y is close to x because all partial derivatives of $R_t\varphi$ are Lipschitz continuous. The proof is complete. □

Now we prove

LEMMA 3.10. *Let $\varphi \in C_b^1(H)$ and $t > 0$. Then $D^2R_t\varphi \in C_b(H; L_2(H))$ and there exists $d_2 > 0$ such that*

$$\|D^2R_t\varphi(x)\|_{L_2(H)} \le d_2 t^{-1/2}\|\varphi\|_1, \quad \forall t > 0,\ x \in H. \tag{3.19}$$

PROOF. Let $\varphi \in C_b^1(H)$, $t > 0$. Then, differentiating (1.3) with respect to x yields

$$\langle DR_t\varphi(x), \alpha\rangle = \int_H \langle D\varphi(e^{tA}x + y), e^{tA}\alpha\rangle N_{Q_t}(dy), \quad \forall t > 0,\ x, \alpha \in H.$$

Now, using (3.7) with $\langle D\varphi(e^{tA}x + \cdot), e^{tA}\alpha\rangle$ replacing φ, yields

$$\langle D^2R_t\varphi(x)\alpha, \beta\rangle = \int_H \langle \Lambda_t\beta, Q_t^{-1/2}y\rangle\langle D\varphi(e^{tA}x + y), e^{tA}\alpha\rangle N_{Q_t}(dy),$$
$$\forall t > 0,\ x, \alpha, \beta \in H.$$

Consequently for any $h, k \in \mathbb{N}$

$$\langle D^2 R_t\varphi(x)e_h, e_k\rangle = \Lambda_{t,k} e^{-ta_h} \int_H \lambda_k(t)^{-1/2} y_k D_h\varphi(e^{tA}x + y) N_{Q_t}(dy),$$
$$\forall t > 0,\ x \in H,$$

where $\Lambda_{t,k}$ were defined in (3.13). Setting

$$\Psi_k(t) = \lambda_k(t)^{-1/2} y_k, \quad t > 0,\ k \in \mathbb{N},$$

we can write the above identity as

$$\langle D^2 R_t\varphi(x)e_h, e_k\rangle = \Lambda_{t,k} e^{-ta_h} \langle \Psi_k(t), D_h\varphi(e^{tA}x + \cdot)\rangle_{L^2(H, N_{Q_t})}.$$

It follows that

$$|\langle D^2 R_t\varphi(x)e_h, e_k\rangle|^2 \le \|\Lambda_t\|^2 \, |\langle \Psi_h(t), D_h\varphi(e^{tA}x + \cdot)\rangle|^2_{L^2(H, N_{Q_t})}.$$

Now, summing up on k and taking into account that the system $(\Psi_h(t))$ is orthonormal on $L^2(H, N_{Q_t})$, we see by the Parseval inequality and (3.11) that

$$\sum_{k=1}^{\infty} |\langle D^2 R_t\varphi(x)e_h, e_k\rangle|^2 \le c_1^2 t^{-1} \int_H |D_h\varphi(e^{tA}x + y)|^2 N_{Q_t}(dy) \le c_1^2 t^{-1} \|\varphi\|_1.$$

Equation (3.19) follows summing up on h and taking $d_2 = c_1$. □

Corollary 3.11. *Let $\varphi \in C_b^\theta(H)$, $\theta \in (0,1)$ and $t > 0$. Then $D^2 R_t\varphi \in C_b(H; L_2(H))$ and we have*

$$\|D^2 R_t\varphi\|_{C_b(H;L_2(H))} \le c_\theta t^{\theta/2-1} \|\varphi\|_\theta, \quad t > 0, \tag{3.20}$$

where $c_\theta = d_1^{1-\theta} d_2^\theta \kappa_\theta$ and κ_θ is defined in (3.5).

Proof. Let $t > 0$ be fixed and denote by γ the mapping

$$\gamma : C_b(H) \to C_b(H; L_2(H)), \quad \varphi \mapsto D^2 R_t\varphi.$$

From Lemmas 3.9 and 3.10 it follows that

(i) γ maps $C_b(H)$ into $C_b(H; L_2(H))$ with norm less than $d_1 t^{-1}$,
(ii) γ maps $C_b^1(H)$ into $C_b(H; L_2(H))$ with norm less than $d_2 t^{-1/2}$.

Consequently, by Theorem 3.6, we have that γ maps $(C_b(H), C_b^1(H))_{\theta,\infty}$ into $C_b(H; L_2(H))$ with norm less than $(d_1 t^{-1})^{1-\theta} (d_2 t^{-1/2})^\theta$. Therefore

$$\|\gamma(\varphi)\|_{C_b(H;L_2(H))} \le (d_1 t^{-1})^{1-\theta} (d_3 t^{-1/2})^\theta \|\varphi\|_{(C_b(H), C_b^1(H))_{\theta,\infty}}.$$

On the other hand by Theorem 3.5 we have

$$(C_b(H), C_b^1(H))_{\theta,\infty} = C_b^\theta(H),$$

and so the conclusion follows from (3.5). □

LEMMA 3.12. *Let* $\varphi \in C_b(H)$ *and* $t > 0$. *Then* $D^2 R_t\varphi \in C_b^1(H; L_2(H))$ *and there exists* $d_3 > 0$ *such that*

$$\|DD^2 R_t\varphi(x)\|_{L_2(H)} \le d_3 t^{-3/2}\|\varphi\|_0, \quad t > 0. \tag{3.21}$$

PROOF. Let $\varphi \in C_b(H)$. Then we have

$$\|DD^2 R_t\varphi(x)\|_{L_2(H)} = \sum_{h,k,l=1}^{\infty} |D^3 R_t\varphi(x)(e_h, e_k, e_l)|^2. \tag{3.22}$$

On the other hand, by (3.9) we have

$$\begin{aligned} &D^3 R_t\varphi(x)(e_h, e_k, e_l) \\ &= \Lambda_{t,h}\Lambda_{t,k}\Lambda_{t,l} \int_H \lambda_h(t)^{-1/2} y_h \lambda_k(t)^{-1/2} y_k \lambda_l(t)^{-1/2} y_l \varphi(e^{tA}x + y) N_{Q_t}(dy) \\ &- \Lambda_{t,h}^2 \Lambda_{t,l}\delta_{h,k} \int_H \lambda_l(t)^{-1/2} y_l \varphi(e^{tA}x + y) N_{Q_t}(dy) \\ &- \Lambda_{t,l}^2 \Lambda_{t,k}\delta_{h,l} \int_H \lambda_k(t)^{-1/2} y_k \varphi(e^{tA}x + y) N_{Q_t}(dy) \\ &- \Lambda_{t,k}^2 \Lambda_{t,l}\delta_{k,l} \int_H \lambda_h(t)^{-1/2} y_h \varphi(e^{tA}x + y) N_{Q_t}(dy). \end{aligned} \tag{3.23}$$

Now we define an orthonormal system on $L^2(H, N_{Q_t})$ setting

$$\zeta_{h,k,l} = \begin{cases} (\lambda_h(t)\lambda_k(t)\lambda_l(t))^{-1/2} y_h y_k y_l, & \text{if } h \ne k \ne l, \\ 3^{-1/2}(\lambda_h^2(t)\lambda_l(t))^{-1/2} y_h^2 y_l - \lambda_l(t)^{-1/2} y_l, & \text{if } h = k \ne l, \\ 3^{-1/2}(\lambda_h^2(t)\lambda_k(t))^{-1/2} y_h^2 y_k - \lambda_k(t)^{-1/2} y_l, & \text{if } h = l \ne k \\ 3^{-1/2}(\lambda_k^2(t)\lambda_h(t))^{-1/2} y_h y_k^2 - \lambda_h(t)^{-1/2} y_l, & \text{if } k = l \ne h \end{cases}$$

Assume first that $h \ne k \ne l$ and write (3.23) as

$$D^3 R_t\varphi(x)(e_h, e_k, e_l) = \Lambda_{t,h}\Lambda_{t,k}\Lambda_{t,l}\langle \zeta_{h,k,l}, \varphi(e^{tA}x + \cdot)\rangle_{L^2(H,N_{Q_t})},$$

which implies

$$|D^3 R_t\varphi(x)(e_h, e_k, e_l)|^2 \le c_1^6 t^{-3} |\langle \zeta_{h,k,l}, \varphi(e^{tA}x + \cdot)\rangle_{L^2(H,N_{Q_t})}|^2.$$

So, by the Parseval inequality

$$\sum_{h,k,l,h\neq k\neq l} |D^3 R_t\varphi(x)(e_h, e_k, e_l)|^2 \leq c_1^6 t^{-3}\|\varphi\|_0^2. \tag{3.24}$$

Let now $h = k \neq l$ and write (3.23) as

$$D^3 R_t\varphi(x)(e_h, e_k, e_l) = 3^{1/2}\Lambda_{t,h}^2\Lambda_{t,l}\langle\zeta_{h,k,l}, \varphi(e^{tA}x + \cdot)\rangle_{L^2(H,N_{Q_t})},$$

which implies

$$|D^3 R_t\varphi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^6 t^{-3}|\langle\zeta_{h,k,l}, \varphi(e^{tA}x + \cdot)\rangle_{L^2(H,N_{Q_t})}|^2.$$

So, by the Parseval inequality

$$\sum_{h,k,l,h=k\neq l} |D^3 R_t\varphi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^6 t^{-3}\|\varphi\|_0^2. \tag{3.25}$$

In a similar way we see that if $h = l \neq k$ we have

$$\sum_{h,k,l,h=l\neq k} |D^3 R_t\varphi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^6 t^{-3}\|\varphi\|_0^2. \tag{3.26}$$

and if $k = l \neq h$ we have

$$\sum_{h,k,l,k=l\neq h} |D^3 R_t\varphi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^6 t^{-3}\|\varphi\|_0^2. \tag{3.27}$$

Taking into account (3.25), (3.26) and (3.27) we end up with

$$\sum_{h,k,l=1}^{\infty} |D^3 R_t\varphi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^3 t^{-3}\|\varphi\|_0^2$$

and so, the conclusion follows since the fact that $D^2 R_t\varphi \in C_b(H, L_2(H))$ can be proved as before. □

LEMMA 3.13. *Let $\varphi \in C_b^1(H)$ and $t > 0$. Then $D^2 R_t\varphi \in C_b^1(H; L_2(H))$ and there exists $d_4 > 0$ such that*

$$\|DD^2 R_t\varphi(x)\|_{L_2(H)} \leq d_4 t^{-1}\|\varphi\|_0, \quad t > 0. \tag{3.28}$$

PROOF. Let $\varphi \in C_b^1(H)$ and $h, k \in \mathbb{N}$. Then, differentiating (1.3) with respect to x in the direction e_h yields

$$\langle DR_t\varphi(x), e_h\rangle = e^{-t\alpha_h}\int_H D_h\varphi(e^{tA}x + y)N_{Q_t}(dy), \quad \forall t > 0,\ x.$$

Now, using (3.8) with $D_h\varphi(e^{tA}x+\cdot)$ replacing φ, yields

$$D^3R_t\varphi(x)(e_h,e_k,e_l)=\int_H\langle\Lambda_te_k,Q_t^{-1/2}y\rangle\langle\Lambda_te_l,Q_t^{-1/2}y\rangle D_h\varphi(e^{tA}x+\cdot)N_{Q_t}(dy)$$
$$-\langle\Lambda_te_k,\Lambda_te_l\rangle R_tD_h\varphi(x),$$

which can be written as

$$\begin{aligned}D^3R_t\varphi(x)&(e_h,e_k,e_l)\\&=e^{-ta_k}\Lambda_{t,k}\Lambda_{t,l}\int_H\lambda_k(t)^{-1/2}\lambda_l(t)^{-1/2}y_ky_lD_h\varphi(e^{tA}x+\cdot)N_{Q_t}(dy)\\&\quad-e^{-ta_k}\Lambda_{t,k}^2\delta_{k,l}\int_HD_h\varphi(e^{tA}x+\cdot)N_{Q_t}(dy).\end{aligned}$$

Let now $k=l$. Then recalling (3.15) we have

$$D^3R_t\varphi(x)(e_h,e_k,e_l)=2^{1/2}e^{-ta_h}\Lambda_{t,k}^2\langle\Phi_{k,k},D_h\varphi(e^{tA}x+\cdot)\rangle_{L^2(H,N_{Q_t})},$$

from which

$$|D^3R_t\varphi(x)(h,k,k)|^2\le 2c_1^4t^2|\langle\Phi_{k,k},D_h\varphi(e^{tA}x+\cdot)\rangle_{L^2(H,N_{Q_t})}|^2$$

and, summing up on k and h

$$\sum_{h,k=1}^{\infty}|D^3R_t\varphi(x)(e_h,e_k,e_l)|^2\le 2c_1^4t^2\|\varphi\|_0^2. \tag{3.29}$$

Finally, if $k\neq l$, then using again by (3.15) we have

$$D^3R_t\varphi(x)(e_h,e_k,e_l)=e^{-ta_h}\Lambda_{t,k}^2\langle\Phi_{k,l},D_h\varphi(e^{tA}x+\cdot)\rangle_{L^2(H,N_{Q_t})},$$

from which

$$|D^3R_t\varphi(x)(e_h,e_k,e_l)|^2\le c_1^2t^2|\langle\Phi_{k,k},D_h\varphi(e^{tA}x+\cdot)\rangle_{L^2(H,N_{Q_t})}|^2$$

and, summing up on k, l and h

$$\sum_{h,k,l=1,k\neq l}^{\infty}|D^3R_t\varphi(x)(e_h,e_k,e_l)|^2\le c_1^2t^2\|\varphi\|_0^2. \tag{3.30}$$

Now the conclusion follows from (3.29) and (3.30). □

Finally we prove.

COROLLARY 3.14. *Let $\varphi \in C_b^\theta(H)$, $\theta \in (0,1)$ and $t > 0$. Then $D^2 R_t\varphi \in C_b^\theta(H; L_2(H))$ and we have*

$$\|D^2 R_t\varphi(x)\|_{C_b^1(H;L_2(H))} \le c_{1,\theta} t^{(\theta-3)/2}\|\varphi\|_\theta, \quad t > 0, \tag{3.31}$$

where $c_{\theta,1} = d_3^{1-\theta} d_4^\theta \kappa_\theta$.

PROOF. Let $t > 0$ be fixed and denote by δ the mapping

$$\delta : C_b(H) \to C_b(H, L_2(H)), \quad \varphi \mapsto D^2 R_t\varphi.$$

From Lemmas 3.12 and 3.13 it follows that

(i) δ maps $C_b(H)$ into $C_b^1(H; L_2(H))$ with norm less than $d_3 t^{-3/2}$,
(ii) δ maps $C_b^1(H)$ into $C_b^1(H; L_2(H))$ with norm less than $d_4 t^{-1}$.

Consequently, by Theorem 3.5, we have that δ maps $(C_b(H), C_b^1(H))_{\theta,\infty}$ into $C_b^1(H; L_2(H))$ with norm $\le (d_3 t^{-3/2})^{1-\theta}(d_4^{-1})^\theta$. Therefore

$$\|\delta(\varphi)\|_{C_b^1(H;L_2(H))} \le (c_4 t^{-3/2})^{1-\theta}(c_5 t^{-1})^\theta \|\varphi\|_{(C_b(H), C_b^1(H))_{\theta,\infty}}.$$

Now the conclusion follows from Theorem 3.5. □

3.3. Proof of the Main Result

We are now ready to prove the main result of the paper. The proof is similar to the finite-dimensional case, see [9].

THEOREM 3.15. *Assume that Hypothesis 1.1 holds. Let $\theta \in (0,1)$, $f \in C_b^\theta(H)$, $\lambda > 0$ and let $\varphi = (\lambda - L)^{-1} f$ be the solution to (1.8). Then we have $D^2\varphi \in C_b^\theta(H; L_2(H))$ and there exists $M_1 > 0$ (independent on λ and on f) such that*

$$\|D^2\varphi\|_{C_b^\theta(H;L_2(H))} \le M_1 \|f\|_{C_b^\theta(H)}. \tag{3.32}$$

PROOF. Let $f \in C_b^\theta(H)$, $\lambda > 0$ and $\varphi = (\lambda - L)^{-1} f$. Then for any $s \ge 0$,

$$D^2 R_s\varphi(x) = \int_0^{+\infty} e^{-\lambda s_1} D^2 R_{s_1} f(x)\, ds_1, \quad x \in H.$$

Proceeding as in [2] it follows that the integral is well defined for each $x \in H$. Following Remark 3.7 we shall look, given $t > 0$, for $a_t \in C_b(H; L_2(H))$ and $b_t \in C_b^1(H; L_2(H))$ such that (3.6) holds. We shall set

$$a_t(x) = \int_0^{t^2} e^{-\lambda s} D^2 R_s f(x)\, ds, \quad x \in H,$$

and

$$b_t(x) = \int_{t^2}^{+\infty} e^{-\lambda s} D^2 R_s f(x)\, ds, \quad x \in H.$$

By arguing as in Lemma 3.9 we see that a_t and b_t are uniformly continuous. Moreover, it is easy to check that

$$\|a_t(x)\|_{L_2(H)} \le \int_0^{t^2} e^{-\lambda s} \|D^2 R_s f(x)\|_{L_2(H)}\, ds, \quad x \in H,$$

so, by (3.20) we deduce

$$\|a_t(x)\|_{L_2(H)} \le \int_0^{t^2} e^{-\lambda s} \|D^2 R_s f\|_{C_b(H;L_2(H))}\, ds, \quad x \in H.$$

Finally, taking the supremum in x yields

$$\|a_t\|_{C_b(H;L_2(H))} \le c_\theta \|f\|_\theta \int_0^{t^2} s^{\theta/2-1}\, ds = \frac{2}{\theta} c_\theta \|f\|_\theta t^\theta. \tag{3.33}$$

In the same way since

$$Db_t(x) = \int_{t^2}^{+\infty} e^{-\lambda s} DD^2 R_s f(x)\, ds, \quad x \in H,$$

we deduce by (3.31) that

$$\|Db_t\|_{C_b(H;L_2(H))} \le c_{1,\theta} \|f\|_\theta \int_{t^2}^{+\infty} s^{(\theta-3)/2}\, ds = \frac{2c_{1,\theta}}{1-\theta} \|f\|_\theta t^{\theta-1}. \tag{3.34}$$

Therefore $D^2 R_t \varphi$ belongs to $(C_b(L_2(H)), C_b^1(L_2(H)))_{\theta,\infty}$ and so to $C_b^\theta(L_2(H))$ by Theorem 3.6. □

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