Calculus of Variations — *Multiplicity of global minima for parametrized functions*, by BIAGIO RICCERI.

Dedicated to the memory of Renato Caccioppoli

ABSTRACT. — Let X be a topological space, I a real interval and Ψ a real-valued function on $X \times I$. In this paper, we prove that if Ψ is lower semicontinuous and inf-compact in X, quasi-concave and continuous in I and satisfies $\sup_I \inf_X \Psi < \inf_X \sup_I \Psi$, then there exists $\lambda^* \in I$ such that $\Psi(\cdot, \lambda^*)$ has at least two global minima. An application involving the integral functional of the calculus of variations is also presented.

KEY WORDS: Multiplicity, global minimum, parametric optimization, minimax inequality.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 49J27, 49J35, 49J45, 90C31.

1. INTRODUCTION

If we wished to summarize with a few words the object of this paper, we could say that it deals with eigenvalues for global minima.

To explain this, let us introduce the notion of an eigenvalue in the most general setting.

So, let X, Y, Λ be three non-empty sets, y a point of Y, and $G : X \times \Lambda \to Y$ a function such that, for each $\lambda \in \Lambda$, the equation

$$G(x, \lambda) = y$$

has at least one solution in X.

We then call eigenvalue for this equation any $\lambda^* \in \Lambda$ for which the equation

$$G(x, \lambda^*) = y$$

has at least two solutions in X.

Clearly, the most classical particular case of this general scheme is when X, Y are two vector spaces, $\Lambda = \mathbf{C}$ (or $\Lambda = \mathbf{R}$), $y = 0$ and

$$G(x, \lambda) = T_1(x) + \lambda T_2(x),$$

where T_1, T_2 are two linear operators. The general theory coming out in that case is one of the milestones in both linear algebra and analysis.

In recent years, much work has also been done in the nonlinear case. In this connection, an account can be found, for instance, in [1]. However, it seems that, among the various research directions, the one followed in the present paper has not been explored yet.

Actually, our aim is to give a contribution to the study of the above eigenvalue problem in the case where $Y = \mathbf{R}$, $y = 0$, Λ is a real interval and

$$G(x, \lambda) = \Psi(x, \lambda) - \inf_{u \in X} \Psi(u, \lambda),$$

for a given $\Psi : X \times \Lambda \to \mathbf{R}$.

Our main result is Theorem 1. It ensures the existence of an eigenvalue provided that Ψ is lower semicontinuous and inf-compact in X, quasi-concave and continuous in Λ and satisfies $\sup_\Lambda \inf_X \Psi < \inf_X \sup_\Lambda \Psi$. We then reformulate Theorem 1 in the case where Ψ is affine in λ and $\Lambda =]0, +\infty[$, so obtaining Theorem 2. Finally, we present two applications of Theorem 2 the last of which deals with the integral functional of the calculus of variations.

2. RESULTS

For a generic function $f : X \to \mathbf{R}$, the sets of the type $f^{-1}(]-\infty, \rho])$ $(\rho \in \mathbf{R})$ are called sub-level sets. If X is a convex set in a vector space, f is said to be quasi-concave if $f^{-1}([\rho, +\infty[)$ is convex for all $\rho \in \mathbf{R}$.

For reader's convenience, we now recall a result from [4] that will be used later.

For a generic set $S \subseteq X \times I$, for each $(x, \lambda) \in X \times I$, we set

$$S_x = \{\mu \in I : (x, \mu) \in S\}$$

and

$$S^\lambda = \{u \in X : (u, \lambda) \in S\}.$$

THEOREM A ([4], Theorem 2.3). *Let X be a topological space, $I \subseteq \mathbf{R}$ a compact interval and $S, T \subseteq X \times I$. Assume that S is connected and $S^\lambda \neq \emptyset$ for all $\lambda \in I$, while T_x is non-empty and connected for all $x \in X$, and T^λ is open for all $\lambda \in I$.*
  Then, one has $S \cap T \neq \emptyset$.

Our main result reads as follows:

THEOREM 1. *Let X be a topological space, $I \subseteq \mathbf{R}$ an open interval and $\Psi : X \times I \to \mathbf{R}$ a function satisfying the following conditions:*

(a) for each $x \in X$, the function $\Psi(x, \cdot)$ is quasi-concave and continuous;
(b) for each $\lambda \in I$, the function $\Psi(\cdot, \lambda)$ has compact and closed sub-level sets;
(c) one has

$$\sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda).$$

Under such hypotheses, there exists $\lambda^* \in I$ such that the function $\Psi(\cdot, \lambda^*)$ has at least two global minima.

PROOF. Fix r satisfying

$$(1) \qquad \sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda) < r < \inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda).$$

Arguing by contradiction, assume that, for each $\lambda \in I$, the function $\Psi(\cdot, \lambda)$ has a unique global minimum, say $\hat{x}_\lambda$. Let us show that the map $\lambda \to \hat{x}_\lambda$ is continuous. To this end, it is clearly enough to show that if $\{\lambda_n\}$ is a sequence in I converging to $\hat{\lambda} \in I$, then $\hat{x}_{\hat{\lambda}}$ is a cluster point of $\{\hat{x}_{\lambda_n}\}$. Fix a compact interval $[a, b] \subset I$ such that $\lambda_n \in [a, b]$ for all $n \in \mathbf{N}$. From (a) it easily follows that

$$(2) \qquad \bigcup_{\lambda \in [a,b]} \{x \in X : \Psi(x, \lambda) \le r\}$$
$$= \{x \in X : \Psi(x, a) \le r\} \cup \{x \in X : \Psi(x, b) \le r\}.$$

Note that, by (1), the sequence $\{\hat{x}_{\lambda_n}\}$ lies in the set on the left-hand side of (2) which, by (b), is compact. As a consequence, this sequence admits a cluster point $\hat{y} \in X$. Thus, $(\hat{y}, \hat{\lambda})$ is a cluster point in $X \times I$ for the sequence $\{(\hat{x}_{\lambda_n}, \lambda_n)\}$. Now, observe that, by (a), (b) and Lemma 4 of [6], the function Ψ turns out to be lower semicontinuous in $X \times I$. We claim that

$$(3) \qquad \Psi(\hat{y}, \hat{\lambda}) \le \limsup_{n \to \infty} \Psi(\hat{x}_{\lambda_n}, \lambda_n).$$

Assume the contrary. Fix η satisfying

$$\limsup_{n \to \infty} \Psi(\hat{x}_{\lambda_n}, \lambda_n) < \eta < \Psi(\hat{y}, \hat{\lambda}).$$

Then, there are a neighbourhood U of $(\hat{y}, \hat{\lambda})$ and $v \in \mathbf{N}$ such that

$$(4) \qquad \Psi(\hat{x}_{\lambda_n}, \lambda_n) < \eta < \Psi(x, \lambda)$$

for all $n > v$ and all $(x, \lambda) \in U$. But, since $(\hat{y}, \hat{\lambda})$ is a cluster point of $\{(\hat{x}_{\lambda_n}, \lambda_n)\}$, there is $n_1 > v$ such that $(\hat{x}_{\lambda_{n_1}}, \lambda_{n_1}) \in U$, against (4). Now, fix $x \in X$. Taking (3) into account, we have

$$\Psi(\hat{y}, \hat{\lambda}) \le \limsup_{n \to \infty} \Psi(\hat{x}_{\lambda_n}, \lambda_n) \le \lim_{n \to \infty} \Psi(x, \lambda_n) = \Psi(x, \hat{\lambda}).$$

That is, $\hat{y}$ is a global minimum of $\Psi(\cdot, \hat{\lambda})$, and so we have $\hat{y} = \hat{x}_{\hat{\lambda}}$. Thus, $\hat{x}_{\hat{\lambda}}$ is a cluster point of $\{\hat{x}_{\lambda_n}\}$, as desired. Now, let $\{I_n\}$ be an increasing sequence of compact intervals such that $I = \bigcup_{n \in \mathbf{N}} I_n$. We claim that there is $n \in \mathbf{N}$ such that

(5)
$$\sup_{\lambda \in I_n} \inf_{x \in X} \Psi(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I_n} \Psi(x, \lambda).$$

Assume the contrary, that is

$$\sup_{\lambda \in I_n} \inf_{x \in X} \Psi(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in I_n} \Psi(x, \lambda)$$

for all $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, put

$$C_n = \left\{ x \in X : \sup_{\lambda \in I_n} \Psi(x, \lambda) \le r \right\}.$$

Note that $C_n \ne \emptyset$. Indeed, otherwise, we would have

$$r \le \inf_{x \in X} \sup_{\lambda \in I_n} \Psi(x, \lambda) = \sup_{\lambda \in I_n} \inf_{x \in X} \Psi(x, \lambda) \le \sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda),$$

against (1). Consequently, $\{C_n\}$ is a non-increasing sequence of non-empty compact and closed subsets of X. Therefore, one has $\bigcap_{n \in \mathbf{N}} C_n \ne \emptyset$. Let $x^* \in \bigcap_{n \in \mathbf{N}} C_n$. Then, one has

$$\sup_{\lambda \in I} \Psi(x^*, \lambda) = \sup_{n \in \mathbf{N}} \sup_{\lambda \in I_n} \Psi(x^*, \lambda) \le r$$

and so

$$\inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda) \le r,$$

against (1). So, fix $n \in \mathbf{N}$ for which (5) holds and fix also ρ so that

(6)
$$\sup_{\lambda \in I_n} \inf_{x \in X} \Psi(x, \lambda) < \rho < \inf_{x \in X} \sup_{\lambda \in I_n} \Psi(x, \lambda).$$

Set

$$S = \{ (\hat{x}_\lambda, \lambda) : \lambda \in I_n \}$$

and

$$T = \{ (x, \lambda) \in X \times I_n : \Psi(x, \lambda) > \rho \}.$$

Since the map $\lambda \to \hat{x}_\lambda$ is continuous, the set S is connected. By (6), we also have $\Psi(\hat{x}_\lambda, \lambda) < \rho$ for all $\lambda \in I_n$, and so $S \cap T = \emptyset$. By (a), (b) and (6) again, it follows that T_x is non-empty and connected for all $x \in X$, while T^λ is open for all $\lambda \in I_n$. Thus, S and T satisfy the assumptions of Theorem A, and hence it should be $S \cap T \ne \emptyset$. This contradiction ends the proof. $\qquad \square$

REMARK 1. Essentially the same proof as the one above shows that Theorem 1 is still true if, in (c), we replace "compact and closed" with "sequentially compact and sequentially closed".

REMARK 2. We also notice that the number λ^* in the conclusion of Theorem 1 can be unique. In this connection, the simplest example is as follows.

Let $X = \{x_0, x_1\}$. Consider the function $\Psi : X \times \mathbf{R} \to \mathbf{R}$ defined by

$$\Psi(x, \lambda) = \begin{cases} -\lambda & \text{if } (x, \lambda) \in \{x_0\} \times \mathbf{R} \\ \lambda & \text{otherwise.} \end{cases}$$

Clearly, (a), (b) hold. Concerning (c), note that

$$\sup_{\lambda \in \mathbf{R}} \inf_{x \in X} \Psi(x, \lambda) = 0,$$

while

$$\inf_{x \in X} \sup_{\lambda \in \mathbf{R}} \Psi(x, \lambda) = +\infty.$$

So, the assumptions of Theorem 1 are satisfied. Finally, observe that, for any $\lambda \neq 0$, the function $\Psi(\cdot, \lambda)$ has a unique global minimum (precisely, x_0 if $\lambda > 0$ and x_1 if $\lambda < 0$).

When the function Ψ is affine in λ and $I =]0, +\infty[$, Theorem 1 assumes the following form.

THEOREM 2. *Let X be a topological space and $J, \Phi : X \to \mathbf{R}$ two functions satisfying the following conditions:*

(a_1) *for each $\lambda > 0$, the function $J + \lambda\Phi$ has compact and closed sub-level sets;*
(b_1) *there exist $\rho \in]\inf_X \Phi, \sup_X \Phi[$ and $u_1, u_2 \in X$ such that*

$$\Phi(u_1) < \rho < \Phi(u_2)$$

and

$$\frac{J(u_1) - \inf_{\Phi^{-1}(]-\infty, \rho])} J}{\rho - \Phi(u_1)} < \frac{J(u_2) - \inf_{\Phi^{-1}(]-\infty, \rho])} J}{\rho - \Phi(u_2)}.$$

Under such hypotheses, there exists $\lambda^* > 0$ such that the function $J + \lambda^*\Phi$ has at least two global minima.

PROOF. Observe that, in view of Theorem 1 of [2], condition (b_1) is equivalent to the inequality

$$\sup_{\lambda \geq 0} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho)) < \inf_{x \in X} \sup_{\lambda \geq 0} (J(x) + \lambda(\Phi(x) - \rho)).$$

On the other hand, since the function $\lambda \to \inf_{x \in X}(J(x) + \lambda(\Phi(x) - \rho))$ is concave (and real-valued) in $]0, +\infty[$, it is lower semicontinuous in $[0, +\infty[$ and so

$$\sup_{\lambda \geq 0} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho)) = \sup_{\lambda > 0} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho)).$$

Consequently, condition (b_1) is equivalent to the inequality

$$\sup_{\lambda > 0} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho)) < \inf_{x \in X} \sup_{\lambda > 0} (J(x) + \lambda(\Phi(x) - \rho)).$$

Now, we can apply Theorem 1 taking $I =]0, +\infty[$ and

$$\Psi(x, \lambda) = J(x) + \lambda(\Phi(x) - \rho),$$

and the conclusion follows.   □

A suitable application of Theorem 2 gives the following result:

THEOREM 3. *Let S be a topological space and $F, \Phi : S \to \mathbf{R}$ two lower semi-continuous functions satisfying the following conditions:*

(a_2) *the function Φ has compact sub-level sets;*
(b_2) *for some $a > 0$, one has*

$$\inf_{x \in \Phi^{-1}(]a, +\infty[)} \frac{F(x)}{\Phi(x)} = -\infty.$$

Under such hypotheses, for each ρ large enough, there exists $\lambda_\rho^* > 0$ such that the restriction of the function $F + \lambda_\rho^* \Phi$ to $\Phi^{-1}(]-\infty, \rho])$ has at least two global minima.

PROOF. Fix $\rho_0 > \inf_X \Phi$, $x_0 \in \Phi^{-1}(]-\infty, \rho_0[)$ and λ satisfying

$$\lambda > \frac{F(x_0) - \inf_{\Phi^{-1}(]-\infty, \rho_0])} F}{\rho_0 - \Phi(x_0)}.$$

Hence, one has

(7) $$F(x_0) + \lambda \Phi(x_0) < \lambda \rho_0 + \inf_{\Phi^{-1}(]-\infty, \rho_0])} F.$$

Since $\Phi^{-1}(]-\infty, \rho_0])$ is compact, by lower semicontinuity, there is $\hat{x} \in \Phi^{-1}(]-\infty, \rho_0])$ such that

(8) $$F(\hat{x}) + \lambda \Phi(\hat{x}) = \inf_{x \in \Phi^{-1}(]-\infty, \rho_0])} (F(x) + \lambda \Phi(x)).$$

We claim that $\Phi(\hat{x}) < \rho_0$. Arguing by contradiction, assume that $\Phi(\hat{x}) \geq \rho_0$. Then, in view of (7), we would have

$$F(x_0) + \lambda\Phi(x_0) < F(\hat{x}) + \lambda\Phi(\hat{x})$$

against (8). By (b_2), there is a sequence $\{u_n\}$ in $\Phi^{-1}(]a, +\infty[)$ such that

$$\lim_{n\to\infty} \frac{F(u_n)}{\Phi(u_n)} = -\infty.$$

Now, set

$$\gamma = \min\left\{0, \inf_{x \in \Phi^{-1}(]-\infty,\rho_0])} (F(x) + \lambda\Phi(x))\right\}$$

and fix $\hat{n} \in \mathbf{N}$ so that

$$\frac{F(u_{\hat{n}})}{\Phi(u_{\hat{n}})} < -\lambda + \frac{\gamma}{a}.$$

We then have

$$F(u_{\hat{n}}) + \lambda\Phi(u_{\hat{n}}) < \frac{\gamma}{a}\Phi(u_{\hat{n}}) \leq \gamma.$$

Hence, if we put

$$\rho^* = \Phi(u_{\hat{n}}),$$

we have

$$\inf_{x \in \Phi^{-1}(]-\infty,\rho^*])} (F(x) + \lambda\Phi(x)) < \inf_{x \in \Phi^{-1}(]-\infty,\rho_0])} (F(x) + \lambda\Phi(x)).$$

At this point, for each $\rho \geq \rho^*$, we realize that it is possible to apply Theorem 2 taking $X = \Phi^{-1}(]-\infty, \rho])$ and $J = F + \lambda\Phi$. Indeed, with these choices and taking $u_1 = \hat{x}$, $u_2 = u_{\hat{n}}$, the left-hand side of the last inequality in (b_1) is zero, while the right-hand side is positive. Consequently, there exists $\hat{\lambda}_\rho > 0$ such that the restriction of the function $F + \lambda\Phi + \hat{\lambda}_\rho\Phi$ to $\Phi^{-1}(]-\infty, \rho])$ has at least two global minima. So, the conclusion follows taking $\lambda_\rho^* = \lambda + \hat{\lambda}_\rho$. $\qquad\square$

It is worth noticing the following consequence of Theorem 3.

THEOREM 4. *Let S be a cone in a real vector space equipped with a (not necessarily vector) topology and let $F, \Phi : S \to \mathbf{R}$ be two lower semicontinuous functions satisfying the following conditions:*

(a_3) the function Φ is positively homogeneous of degree α and has compact sub-level sets;

(b_3) the function F is positively homogeneous of degree $\beta > \alpha$ and there is $\tilde{x} \in S$ such that $F(\tilde{x}) < 0 < \Phi(\tilde{x})$.

Under such hypotheses, there exists $\rho^* > \inf_S \Phi$ such that the restriction of the function $F + \Phi$ to $\Phi^{-1}(]-\infty, \rho^*])$ has at least two global minima.

PROOF. Clearly, we have

$$\lim_{\lambda \to +\infty} \frac{F(\lambda\tilde{x})}{\Phi(\lambda\tilde{x})} = \lim_{\lambda \to +\infty} \frac{F(\tilde{x})}{\Phi(\tilde{x})} \lambda^{\beta-\alpha} = -\infty.$$

So, the hypotheses of Theorem 3 are satisfied and hence there exist $\rho > \inf_S \Phi$ and $\lambda > 0$ such that the restriction of the function $F + \lambda\Phi$ to $\Phi^{-1}(]-\infty, \rho])$ has at least two global minima, say v_1, v_2. Now, observe that

$$\lambda^{\beta/(\alpha-\beta)}(F(x) + \lambda\Phi(x)) = F(\lambda^{1/(\alpha-\beta)}x) + \Phi(\lambda^{1/(\alpha-\beta)}x)$$

for all $x \in S$. From this, it easily follows that the points $\lambda^{1/(\alpha-\beta)}v_1$ and $\lambda^{1/(\alpha-\beta)}v_2$ are two global minima of the restriction of the function $F + \Phi$ to $\Phi^{-1}(]-\infty, \lambda^{\alpha/(\alpha-\beta)}\rho])$, that is the conclusion. $\square$

REMARK 3. Of course, due to Remark 1, Theorems 2, 3 and 4 are still valid if instead of "closed", "compact", "lower semicontinuous" one assumes "sequentially closed", "sequentially compact", "sequentially lower semicontinuous" respectively.

REMARK 4. We also remark that the number ρ^* in the conclusion of Theorem 4 can be unique. In this connection, a very simple example is provided by taking $S = \mathbf{R}$, $\Phi(x) = x^2$ and $F(x) = -x^3$. Actually, it is seen at once that, if $r > 0$, the restriction of the function $x \to x^2 - x^3$ to $[-r, r]$ has a unique global minimum when $r \neq 1$ and exactly two global minima when $r = 1$.

We conclude by presenting an application of Theorem 2 involving the integral functional of the calculus of variations.

In the sequel, $\Omega \subset \mathbf{R}^n$ is a bounded open set, with smooth boundary, and $p > n$. Therefore, the Sobolev space $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p \, dx + \int_\Omega |u(x)|^p \, dx \right)^{1/p},$$

is compactly embedded in $C^0(\overline{\Omega})$ and hence the constant

$$(9) \qquad c = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\sup_{x \in \Omega} |u(x)|}{\|u\|}$$

is finite.

Recall that a function $f : \Omega \times \mathbf{R}^m \to \ ]-\infty, +\infty]$ is said to be a normal integrand ([5]) if it is $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R}^m)$-measurable and $f(x, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$. Here $\mathscr{L}(\Omega)$ and $\mathscr{B}(\mathbf{R}^m)$ denote the Lebesgue and the Borel σ-algebras of subsets of Ω and $\mathbf{R}^m$, respectively.

Recall that if f is a normal integrand, then, for each measurable function $u : \Omega \to \mathbf{R}^m$, the composite function $x \to f(x, u(x))$ is measurable ([5]).

If $\xi \in \mathbf{R}$, we continue to denote by ξ the constant function on Ω assuming the value ξ.

THEOREM 5. *Let $f : \Omega \times \mathbf{R} \to \ ]-\infty, +\infty]$ and $\varphi : \Omega \times \mathbf{R} \times \mathbf{R}^n \to [0, +\infty]$ be two normal integrands satisfying the following conditions:*

(i) *there exist $v > 0$ such that*

$$v(|\xi|^p + |\eta|^p) \leq \varphi(x, \xi, \eta)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$, and, for each $(x, \xi) \in \Omega \times \mathbf{R}$, the function $\varphi(x, \xi, \cdot)$ is convex in $\mathbf{R}^n$;

(ii) *for each $\varepsilon > 0$, there exists $\gamma_\varepsilon \in L^1(\Omega)$ such that*

$$-\varepsilon|\xi|^p + \gamma_\varepsilon(x) \leq f(x, \xi)$$

for all $(x, \xi) \in \Omega \times \mathbf{R}$;

(iii) *there exist $\xi_1, \rho \in \mathbf{R}$ such that*

$$\int_\Omega \varphi(x, \xi_1, 0)\, dx < \rho, \quad \int_\Omega f(x, \xi_1)\, dx < +\infty$$

and

$$f(x, \xi_1) = \inf_{|\xi| \leq \delta} f(x, \xi)$$

for all $x \in \Omega$, where

$$\delta = c \left(\frac{\rho}{v} \right)^{1/p}$$

and c is given in (9).

Under such hypotheses, for every sequentially weakly closed set $V \subseteq W^{1,p}(\Omega)$ containing the constant ξ_1 and a w for which

$$\int_\Omega \varphi(x, w(x), \nabla w(x))\, dx < +\infty$$

and

$$\int_\Omega f(x, w(x))\, dx < \int_\Omega f(x, \xi_1)\, dx,$$

there exists $\lambda^* > 0$ such that the restriction to V of the functional

$$u \to \int_\Omega f(x, u(x))\, dx + \lambda^* \int_\Omega \varphi(x, u(x), \nabla u(x))\, dx$$

has at least two global minima.

PROOF. For each $u \in W^{1,p}(\Omega)$, set

$$\tilde{J}(u) = \int_\Omega f(x, u(x))\, dx$$

and

$$\tilde{\Phi}(u) = \int_\Omega \varphi(x, u(x), \nabla u(x))\, dx.$$

By a classical result ([3], Theorem 4.6.8), for each $\lambda > 0$ the functional $\tilde{J} + \lambda\tilde{\Phi}$ is sequentially weakly lower semicontinuous. On the other hand, for $\varepsilon \in]0, \lambda v[$, by (*ii*), we have

$$\tilde{J}(u) + \lambda\tilde{\Phi}(u) \geq (\lambda v - \varepsilon)\|u\|^p + \int_\Omega \gamma_\varepsilon(x)\, dx.$$

Consequently, by reflexivity and Eberlein-Smulyan theorem, the sub-level sets of $\tilde{J} + \lambda\tilde{\Phi}$ are weakly compact. Now, let $V \subseteq W^{1,p}(\Omega)$ be as in the conclusion. Set

$$X = \{u \in V : \sup\{\tilde{J}(u), \tilde{\Phi}(u)\} < +\infty\}.$$

Observe that $\xi_1, w \in X$ and that

(10) $\qquad \{u \in X : \tilde{J}(u) + \lambda\tilde{\Phi}(u) \leq r\} = \{u \in V : \tilde{J}(u) + \lambda\tilde{\Phi}(u) \leq r\}$

for all $\lambda > 0$, $r \in \mathbf{R}$. Denote by J and Φ the restrictions to X of $\tilde{J}$ and $\tilde{\Phi}$ respectively. We want to apply Theorem 2 considering X with the relative weak topology. Clearly, in view of (10), (a_1) holds. Concerning (b_1), observe that for each $u \in \Phi^{-1}(]-\infty, \rho])$, by (*i*), one has

$$v\|u\|^p \leq \rho$$

and so

$$\sup_\Omega |u| \leq c\left(\frac{\rho}{v}\right)^{1/p},$$

the above inequalities being strict if $\Phi(u) < \rho$. Then, from this and from (*iii*), it follows that

$$J(\xi_1) = \inf_{\Phi^{-1}(]-\infty,\rho])} J$$

and

$$\Phi(\xi_1) < \rho$$

as well. Consequently, (b_1) is satisfied taking $u_1 = \xi_1$ and $u_2 = w$. So, the conclusion follows directly from Theorem 2. $\square$

REMARK 5. We are not aware of known results close enough to Theorem 5 in order to do a proper comparison.

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