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Partial Differential Equations — *The "ergodic limit" for a viscous Hamilton-Jacobi equation with Dirichlet conditions*, by Alessio Porretta.

Dedicated to the memory of Renato Caccioppoli

Abstract. — We study the limit, when λ tends to 0, of the solutions u_λ of the Dirichlet problem

$$\begin{cases} -\Delta u + \lambda u + |\nabla u|^q = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when $1 < q \le 2$ and f is bounded. In case the limit problem does not have any solution, we prove that u_λ has a complete blow-up ($u_\lambda \to -\infty$) and its behaviour is described in terms of the corresponding ergodic problem with state constraint conditions. In particular, λu_λ converges to the ergodic constant c_0 and $u_\lambda + \|u_\lambda\|_\infty$ converges to the boundary blow-up solution v_0 of the ergodic problem associated to the stochastic optimal control with state constraint.

Key words: Ergodic limit, blow-up, viscous Hamilton–Jacobi equations.

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1. Introduction and statement of the results

Let $\Omega \subset \mathbf{R}^N$ be a C^2, bounded domain, and let f belong to $L^\infty(\Omega)$. It is well-known that, when $q > 1$, the problem

$$\begin{cases} -\Delta \varphi + |\nabla \varphi|^q = f(x) & \text{in } \Omega, \\ \varphi = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

may have no solution. One way to realize that is to look at the case $q = 2$. Through a standard change of unknown, that case can be reduced to a linear problem, and the existence of solutions is related to eigenvalues. Therefore there are simple situations when existence fails, as in the following classical example suggested in [17].

Example 1.1. *When $q = 2$, $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution of* (1.1) *if and only if $\psi = e^{-\varphi} - 1$ is a solution of*

$$\begin{cases} -\Delta \psi + f(x)(\psi + 1) = 0 & \textit{in } \Omega, \\ \psi \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{1.2}$$

If we have $f(x) \leq -\lambda_1(-\Delta, \Omega)$ (first eigenvalue of $-\Delta$ in Ω), then φ is negative and then ψ would be a positive supersolution of $-\Delta\psi = \lambda_1\psi + \lambda_1$, which is impossible.

We will come back later (see Remark 1.1) to this special case $q = 2$ to give a more precise statement whether there exist solutions or not (see also [1], [3], [17]).

However, even if the case $q = 2$ is simpler because of the change of unknown, the possible failure of existence of solutions of (1.1) is a general fact due to the superlinear character of the lower order term. As soon as $q > 1$, it is necessary that f satisfies some smallness condition in order that problem (1.1) may admit a solution, see e.g. [2], [16]. In this last paper, as well as in [3], [12], [13], [15], [17], [20], the existence is proved when f is sufficiently small in some suitable norm.

On the other hand, for any $\lambda > 0$ and $q \leq 2$ there exists a solution to the problem

$$\begin{cases} -\Delta u + \lambda u + |\nabla u|^q = f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{1.3}$$

by classical results (see e.g. [4], [11], [17]). It is then a natural question to understand what happens to the solutions of (1.3) when λ goes to zero, especially when the limit problem does not have any solution.

The aim of this paper is to answer this question and in particular to describe the possibly singular behaviour of u_λ in case there is no solution of (1.1). It is a minor problem in which sense a solution of (1.1) should be considered and here we deal with weak solutions belonging to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, we assume in all the paper that $\Omega \subset \mathbb{R}^N$ is a bounded connected open set of class C^2. The main result that we prove is the following

Theorem 1.1. *Assume that $1 < q \leq 2$, and $f \in L^\infty(\Omega)$. For $\lambda > 0$, let u_λ be the solution of* (1.3)*. Then we have*

(i) *If problem* (1.1) *has a solution $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $u_\lambda \to \varphi$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ as $\lambda \to 0$.*
(ii) *If problem* (1.1) *has no solution $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then we have, as $\lambda \to 0$,*

$$u_\lambda(x) \to -\infty \quad \text{for every } x \in \Omega,$$
$$\lambda u_\lambda \to c_0 \quad \text{locally uniformly in } \Omega,$$

where c_0 is the unique constant such that the problem

$$\begin{cases} -\Delta v + |\nabla v|^q + c_0 = f(x) & \text{in } \Omega, \\ \lim_{x \to \partial\Omega} v(x) = +\infty \end{cases} \tag{1.4}$$

admits a solution $v \in W^{2,p}_{loc}(\Omega)$ for every $p < \infty$.

Moreover, if we set $v_\lambda = u_\lambda + \|u_\lambda\|_\infty$, then

$$v_\lambda \to v_0 \quad \text{locally uniformly in } \Omega,$$

where v_0 *is the unique solution of* (1.4) (*in* $W^{2,p}_{loc}(\Omega)$ $\forall p < \infty$) *such that* $\min_{\Omega} v_0(x) = 0$.

The above result shows that when there is no solution of (1.1) then u_λ blows-up completely and its behaviour is described in terms of the couple (c_0, v_0) solution of (1.4). This latter problem is usually called an ergodic problem: the unknowns are both the constant c_0 and the solution v. As far as this problem is concerned, we rely on a fundamental result proved by J. M. Lasry and P. L. Lions:

Thm ([18], *Theorem VI.I*): *Let* $1 < q \leq 2$ *and* $f \in L^\infty(\Omega)$. *There exists a unique constant* c_0 *such that* (1.4) *has a solution* $v \in W^{2,p}_{loc}(\Omega)$ ($\forall p < \infty$)*; moreover,* v *is unique up to an additive constant.*

The full comprehension of Theorem 1.1, as well as of the role of problem (1.4), goes back to the stochastic interpretation of (1.3). Let us recall that if X_t is a stochastic process solution of the SDE

$$dX_t = a(X_t)\,dt + \sqrt{2}\,dB_t, \quad X_0 = x \in \Omega,$$

where B_t is a standard Brownian motion, then, thanks to the dynamic programming principle, the solution u_λ of (1.3) can be represented as the value function of an optimal control problem:

$$u_\lambda(x) = \inf_{a \in \mathscr{A}} E_x\left\{\int_0^{\tau_x}\left[f(X_t) + \frac{q-1}{q^{q/(q-1)}}|a(X_t)|^{q/(q-1)}\right]\mathrm{e}^{-\lambda t}\,dt\right\}, \tag{1.5}$$

where E_x is the conditional expectation with respect to $X_0 = x$, τ_x is the first exit time from Ω and $a(\cdot)$ belongs to a set $\mathscr{A}$ of admissible control laws (or, otherwise said, $a_t = a(X_t)$ is an admissible control).

The limit of λu_λ when $\lambda \to 0$ is usually called the ergodic limit, as it is related to the properties of ergodicity of the process X_t and to the large time behaviour of the corresponding evolution problem (see e.g. [5], [9], [10]). This is well known and extensively studied in case of periodic boundary conditions or in the whole space $\mathbb{R}^N$, which of course are natural settings to study ergodicity.

In case of the exit time problem (corresponding to Dirichlet boundary conditions), formula (1.5) suggests that when $\lambda \to 0$ the function u_λ should remain bounded *unless the exit time* $\tau_x \to +\infty$. This case corresponds to the so-called state constraint problem. In the case of Brownian motion, the state constraint problem was studied by J. M. Lasry and P. L. Lions in [18], where in particular they prove the above mentioned ergodic result, i.e. the existence and uniqueness of the couple (c_0, v) solution of (1.4).

Therefore, in view of the stochastic interpretation of (1.3), there is no surprise that the singular behaviour of u_λ is described by the couple (c_0, v_0) of the state constraint problem. Indeed, formula (1.5) suggests the following: when the function f is strongly negative inside Ω, then the minimizing control will tend to keep the process in the interior preventing it from reaching the boundary and this leads the exit time problem to a state constraint condition.

From a purely PDE point of view, the behaviour described in (ii) of Theorem 1.1 is a consequence of interior gradient bounds; ∇u_λ remains uniformly bounded in the interior independently of the boundary condition and of the L^∞ bound of u_λ. This explains why λu_λ converges to a constant. It is remarkable to note that the existence of solutions of (1.1) depends itself on this constant, which is the unique ergodic constant c_0 of problem (1.4) (note that c_0 depends on f). Indeed, we have

PROPOSITION 1.1. *Assume that* $1 < q \le 2$, *and* $f \in L^\infty(\Omega)$. *Then problem* (1.1) *has a solution* $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *if and only if* $c_0 > 0$.

Moreover, in the case $c_0 > 0$, φ *is the unique solution in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

In Section 2 we give a proof of Proposition 1.1 which follows from the stability result of Theorem 1.1. The same conclusion of Proposition 1.1 is proved in [8] for viscosity solutions, using a slightly different argument. Let us stress that this kind of result is strongly related to the basic principle (already stated in [19]) that a solution of (1.1) exists if and only if there exists a subsolution and to the fact that $c_0 = \sup\{c : \exists\varphi : -\Delta\varphi + |\nabla\varphi|^q + c \le f(x)\}$ (see Corollary 2.2). It is not difficult from this characterization to recognize that c_0 plays the role of an eigenvalue, which is exactly the case when $q = 2$.

REMARK 1.1. In the case $q = 2$ we have $c_0 = \lambda_1(-\Delta + f, \Omega)$, i.e. it turns out that the ergodic constant is nothing but the first eigenvalue of the operator $-\Delta + f$. In particular, when $q = 2$ it is easy to prove that there exists a solution of (1.1) if and only if $\lambda_1(-\Delta + f, \Omega) > 0$, just by using the linear theory. Indeed, if there exists φ solution of (1.1), then $\tilde\psi = e^{-\varphi}$ is a positive solution of

$$-\Delta\tilde\psi + f\tilde\psi = 0,$$

and therefore $0 \le \lambda_1$. On the other hand we cannot have $\lambda_1 = 0$; otherwise this means that the first positive eigenfunction ψ_1 satisfies

$$\begin{cases} -\Delta\psi_1 + f(x)\psi_1 = 0 & \text{in } \Omega, \\ \psi_1 \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases}$$

hence $\int_\Omega f\psi_1\,dx = \int_\Omega \partial_\nu\psi_1\,d\sigma < 0$. In particular f is not orthogonal to ψ_1 and no solution can exist of (1.2). Therefore, if problem (1.1) has a solution one has necessarily $\lambda_1(-\Delta + f, \Omega) > 0$. The converse is also obviously true; if $\lambda_1(-\Delta + f, \Omega) > 0$ then $\psi \mapsto -\Delta\psi + f\psi$ defines a coercive bilinear form and a solution of (1.2) exists by Lax-Milgram theorem. □

As a consequence of Proposition 1.1, we can rephrase the result of Theorem 1.1 in the following way.

COROLLARY 1.1. *Assume that* $1 < q \le 2$, *and* $f \in L^\infty(\Omega)$. *Let* u_λ *be the solution of* (1.3), *and* c_0 *the ergodic constant of problem* (1.4). *Then we have*

(i) *If* $c_0 > 0$ *then* $u_\lambda \to \varphi$ *in* $H_0^1(\Omega) \cap L^\infty(\Omega)$ *as* $\lambda \to 0$, *where* φ *is the unique solution of* (1.1) *in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

(ii) *If* $c_0 \le 0$, *then we have, as* $\lambda \to 0$,

$$u_\lambda(x) \to -\infty \quad \text{for every } x \in \Omega,$$
$$\lambda u_\lambda \to c_0 \quad \text{locally uniformly in } \Omega,$$

and if we set $v_\lambda = u_\lambda + \|u_\lambda\|_\infty$ *then*

$$v_\lambda \to v_0 \quad \text{locally uniformly in } \Omega,$$

where v_0 *is the unique solution of* (1.4) *such that* $\min_\Omega v_0(x) = 0$.

When $f \in W^{1,\infty}(\Omega)$, we also give estimates on the rate of convergence of λu_λ to c_0, or equivalently, on the growth of $u_\lambda - \frac{c_0}{\lambda}$. The expert reader will recognize that this step is strictly related to the so-called corrector problem in homogenization or to the large time profile of the solutions of the evolution problem. Indeed, the following result follows the ideas introduced in [8] to estimate the blow-up rate, when $t \to +\infty$, of the solutions of the evolution problem. It is interesting to note how the blow-up rate changes in the borderline case $c_0 = 0$.

THEOREM 1.2. *Assume that* $1 < q \le 2$, *and* $f \in W^{1,\infty}(\Omega)$. *Let* u_λ *be the solution of* (1.3), *and* c_0 *the ergodic constant of problem* (1.4). *Then, for any compact set* $K \subset \Omega$ *there exists a constant* C_K *such that, as* $\lambda \to 0$:

(i) *if* $c_0 < 0$ *then*

$$\begin{cases} \|\lambda u_\lambda - c_0\|_{L^\infty(K)} \le C_K \lambda & \text{when } \frac{3}{2} \le q \le 2 \\ \|\lambda u_\lambda - c_0\|_{L^\infty(K)} \le C_K \lambda^{(q-1)/(2-q)} & \text{when } 1 < q < \frac{3}{2} \end{cases} \tag{1.6}$$

(ii) *if* $c_0 = 0$ *then*

$$\begin{cases} \|\lambda u_\lambda\|_{L^\infty(K)} \le C_K \lambda |\log \lambda| & \text{when } q = 2 \\ \|\lambda u_\lambda\|_{L^\infty(K)} \le C_K \lambda^{q-1} & \text{when } 1 < q < 2 \end{cases} \tag{1.7}$$

REMARK 1.2. If $c_0 > 0$, then by Corollary 1.1 u_λ is bounded and converges to the unique solution of (1.1), then of course $\lambda u_\lambda \to 0$ in this case and, trivially, we have $\|\lambda u_\lambda\|_{L^\infty(\Omega)} \le C\lambda$.

Let us conclude this introduction with a few more comments. First of all, an obvious remark is that the study of the limit of

$$\begin{cases} -\Delta u + \lambda u = |\nabla u|^q + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases}$$

is contained in the previous statements up to replacing u with $-u$. In this case, the singular behaviour means that $u_\lambda(x) \to +\infty$ everywhere.

We also stress that the above results still hold for any Dirichlet condition $u = g$ on $\partial\Omega$, at least if g is a continuous function. In this case, a suitable setting seems to be that of viscosity solutions (see Theorem 2.1). On the other hand, the extension of such results to more general f and/or to more general operators is more delicate and will be dealt with in a next work. Of course, some of the above results can be extended without problems to more general situations (e.g. inhomogeneous diffusions), but a complete description as it is given in Theorem 1.1 is not obvious (unless for smooth diffusions) and needs in any case a more general version of the ergodic theorem of [18]. Actually, the aim of the present paper is to make it completely clear what happens in the model case (i.e. for the Laplace operator) in order to serve as a guideline for the study of more general situations. Finally, let us point out that the present study is motivated and closely related to the study of large time behaviour of solutions of the time-dependent version of (1.1), which is treated in [8].

2. Proof of the results

2.1. *Proof of Theorem 1.1 and Proposition 1.1*

Let us recall that a comparison principle holds for weak subsolutions and supersolutions belonging to $H^1(\Omega) \cap L^\infty(\Omega)$ of the problem

$$\begin{cases} \lambda v - \Delta v + |\nabla v|^q = f(x) & \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

when $\lambda > 0$ and $f \in L^\infty(\Omega)$. We warn the reader that this is no longer true for simply $H_0^1(\Omega)$ solutions, and we refer to [6], [7] for comparison principle and uniqueness results for weak solutions. Such uniqueness results are more delicate when $\lambda = 0$, but we will see later that the comparison principle holds in that case too. Moreover, we stress that it is possible to use different formulations of such problems (solutions in $W^{2,p}(\Omega)$, or viscosity solutions), but clearly all formulations ensuring the validity of weak maximum principle eventually coincide.

In particular, problem (1.3) admits a unique solution, which is actually more regular and satisfies the following gradient bound, which is an essential tool in the study of the ergodic limit.

Lemma 2.1. *Let $q > 1$, and let $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the solution of* (1.3). *Then $u_\lambda \in W^{2,p}(\Omega)$ for every $p < \infty$ and we have, for every $x \in \Omega$,*

$$|\nabla u_\lambda(x)| \le \frac{K}{d(x)^{1/(q-1)}}$$

where K depends only on $\|f\|_{L^\infty(\Omega)}$, q, Ω, and $d(x)$ denotes the distance of x from the boundary.

PROOF. The uniqueness of u_λ implies that it coincides with the unique solution in $W^{2,p}(\Omega)$ ($\forall p < \infty$), whose existence is proved e.g. in [4], [17]. Moreover, we have $\lambda\|u_\lambda\|_\infty \le \|f\|_\infty$ by the weak maximum principle. Then we apply Theorem IV.I in [18] to deduce the gradient bound. □

As a consequence of the above estimate we have the following

PROPOSITION 2.1. *Let u_λ be the solution of* (1.3), *and set*

$$v_\lambda = u_\lambda + \|u_\lambda\|_\infty.$$

Then v_λ is bounded in $W^{1,\infty}_{loc}(\Omega)$.

PROOF.

Step 1. We claim that for any $\theta \in (0,1)$ there exists δ_0, depending only on θ, q, Ω, $\|f\|_\infty$ such that, for every $\lambda > 0$,

$$u_\lambda(x) \ge -d(x)^\theta - \sup_{\{d(x)=\delta_0\}} u_\lambda^- \quad \forall x : d(x) \le \delta_0. \tag{2.1}$$

Indeed, take $\psi(x) = -d(x)^\theta$, with $\theta \in (0,1)$ and δ_0 sufficiently small so that $d(x)$ is smooth when $d(x) < \delta_0$. Then

$$\begin{aligned}
&-\Delta\psi + \lambda\psi + |\nabla\psi|^q - f(x) \\
&\quad = -\theta(1-\theta)d^{\theta-2} + \theta d^{\theta-1}\Delta d - \lambda d^\theta + \theta^q d^{(\theta-1)q} - f(x) \\
&\quad \le -\theta d^{\theta-2}[(1-\theta) - d\Delta d - \theta^{q-1}d^{2-q+\theta(q-1)}] + \|f\|_\infty
\end{aligned}$$

Since $1 < q \le 2$, we have that ψ is a subsolution in the subset $\{x \in \Omega : d(x) < \delta_0\}$, for some $\delta_0 > 0$ depending only on θ, q, Ω, f. Since $\psi - \sup_{\{d(x)=\delta_0\}} u_\lambda^-$ is still a subsolution, we conclude by comparison that our claim holds true.

Step 2. First observe that u_λ is bounded from above; indeed, we have $-\Delta u_\lambda^+ \le |f(x)|$, hence $\|u_\lambda^+\|_\infty \le c\|f\|_\infty$. Therefore, we deduce from Step 1 that, for some constant C_0,

$$\|u_\lambda\|_\infty \le C_0 + \sup_{\{d(x)\ge\delta_0\}} u_\lambda^-$$

hence there exists x_λ such that $d(x_\lambda) \ge \delta_0$ such that

$$0 \le v_\lambda(x) \le C_0 + u_\lambda(x) - u_\lambda(x_\lambda)$$

Using Lemma 2.1 we deduce that v_λ is locally uniformly bounded. Since $\nabla v_\lambda = \nabla u_\lambda$, again from Lemma 2.1 we deduce that $|\nabla v_\lambda|$ is locally uniformly bounded too, hence we conclude. □

We are ready to prove our main result.

Proof of Theorem 1.1:

Proof of (i). We prove actually the following claim: *if problem* (1.1) *admits a subsolution in* $H_0^1(\Omega) \cap L^\infty(\Omega)$, *then a subsequence of* u_λ *converges in* $H_0^1(\Omega) \cap L^\infty(\Omega)$ *to a solution* φ *of* (1.1).

Indeed, assume that there exists a subsolution $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.1), then $\psi - \|\psi\|_\infty$ is a subsolution of (1.3). Since the comparison principle holds in $H_0^1(\Omega) \cap L^\infty(\Omega)$, this implies that $u_\lambda \geq \psi - \|\psi\|_\infty \geq -2\|\psi\|_\infty$. On the other hand since $-\Delta u_\lambda^+ \leq |f|$, we have $\|u_\lambda^+\|_\infty \leq c\|f\|_\infty$, so that we conclude that u_λ remains bounded in $L^\infty(\Omega)$. By standard results (see e.g. [11]), it follows that u_λ is relatively compact in $H_0^1(\Omega)$ hence it converges, up to a subsequence, to a function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution of (1.1). Moreover, the convergence of u_λ also holds in $L^\infty(\Omega)$, since the uniform bound of u_λ also implies that u_λ is bounded in $W^{2,p}(\Omega)$ ($\forall p < \infty$) by classical results (see e.g. [4], Proposition 2), hence u_λ is relatively compact in $L^\infty(\Omega)$. In particular, we also deduce that $\varphi \in W^{2,p}(\Omega)$.

This proves our claim. To conclude the proof of part (i) we only need to know that φ is the unique solution of (1.1) in $H_0^1(\Omega) \cap L^\infty(\Omega)$, a fact that will be proved in Proposition 1.1 below. The uniqueness of φ implies that the whole sequence u_λ converges.

Proof of (ii). First observe that, as $\lambda \to 0$, we have that $\|u_\lambda\|_\infty \to \infty$; indeed, any subsequence of $\{u_\lambda\}$ cannot be bounded otherwise (up to a new subsequence) it would converge to a solution of (1.1), a fact which contradicts our assumption. Moreover, since $\|u_\lambda^+\|_\infty \leq c\|f\|_\infty$, this implies that we have, for λ small enough and converging to zero:

$$\|u_\lambda\|_\infty = \|u_\lambda^-\|_\infty \to \infty.$$

Let us recall that in consequence of maximum principle we also have

$$\lambda\|u_\lambda\|_\infty \leq \|f\|_\infty. \tag{2.2}$$

Define now $v_\lambda = u_\lambda + \|u_\lambda\|_\infty$, hence v_λ solves

$$-\Delta v_\lambda + \lambda v_\lambda + |\nabla v_\lambda|^q = f(x) + \lambda\|u_\lambda\|_\infty \tag{2.3}$$

which implies because of (2.2)

$$-\Delta v_\lambda + \lambda v_\lambda + |\nabla v_\lambda|^q \geq -2\|f\|_\infty.$$

Now let $q < 2$; in the domain $\{x \in \Omega : d(x) < \delta_0\}$, consider the function $\psi = \sigma\big(d(x) + \frac{1}{n}\big)^{-\alpha} - M$, where

$$\alpha = \frac{2-q}{q-1}, \quad M = \sigma\Big(\delta_0 + \frac{1}{n}\Big)^{-\alpha} + \sup_{d(x)=\delta_0} v_\lambda$$

and δ_0 is to be chosen (sufficiently small so that $d(x)$ is smooth in this domain). Computing we have

$$-\Delta\psi + \lambda\psi + |\nabla\psi|^q = \alpha\sigma\Big(d(x)+\frac{1}{n}\Big)^{-\alpha-2}\Big[-(\alpha+1)+\Big(d(x)+\frac{1}{n}\Big)\Delta d + (\alpha\sigma)^{q-1}\Big(d(x)+\frac{1}{n}\Big)^{2+\alpha-(\alpha+1)q} + \frac{\lambda}{\alpha}\Big(d(x)+\frac{1}{n}\Big)^2\Big] - \lambda M$$

where we used that $|\nabla d(x)| = 1$. The value of $\alpha = \frac{2-q}{q-1}$ implies that $2+\alpha = (\alpha+1)q$ hence we get

$$-\Delta\psi + \lambda\psi + |\nabla\psi|^q = \alpha\sigma\Big(d(x)+\frac{1}{n}\Big)^{-\alpha-2}\Big[-(\alpha+1)+\Big(d(x)+\frac{1}{n}\Big)\Delta d + (\alpha\sigma)^{q-1} + \frac{\lambda}{\alpha}\Big(d(x)+\frac{1}{n}\Big)^2\Big] - \lambda M$$

Choosing σ such that $(\alpha\sigma)^{q-1} < \alpha+1$, then δ_0 and n sufficiently small, we obtain that

$$-\Delta\psi + \lambda\psi + |\nabla\psi|^q \le -2\|f\|_\infty$$

in $\{d(x) < \delta_0\}$. The value of M implies that $\psi \le v_\lambda$ on $\{d(x) = \delta_0\}$, and, if λ is small, we have $v_\lambda \ge \psi$ on $\partial\Omega$ as well. We conclude that

$$v_\lambda \ge \psi \quad \text{in } \{d(x) < \delta_0\}.$$

Observe that M depends on λ (and n) but is uniformly bounded, since v_λ is locally uniformly bounded, hence there exists some constant K such that

$$v_\lambda \ge \sigma\Big(d(x)+\frac{1}{n}\Big)^{-\alpha} - K \quad \text{in } \{d(x) < \delta_0\}.$$

Now, by Proposition 2.1, there exists a subsequence of λ (not relabeled) and a (nonnegative) function $v_0 \in W^{1,\infty}_{loc}(\Omega)$ such that $v_\lambda \to v_0$ locally uniformly in Ω as $\lambda \to 0$. We deduce that

$$v_0 \ge \sigma\Big(d(x)+\frac{1}{n}\Big)^{-\alpha} - K \quad \text{in } \{d(x) < \delta_0\},$$

which implies, after letting $n \to \infty$, that $v_0(x) \to +\infty$ as $x \to \partial\Omega$. When $q = 2$, the same conclusion can be obtained using $\psi = -\sigma\log\big(d(x)+\frac{1}{n}\big) - M$ as a comparison function.

Moreover, by elliptic regularity, v_λ is bounded in $W^{2,p}_{loc}(\Omega)$, and standard compactness results allow us to pass to the limit in the equation (2.3) satisfied by v_λ. Finally, in view of (2.2) we have that, still up to subsequences,

$$\lambda\|u_\lambda\|_\infty \to -c_0$$

for some constant $c_0 \leq 0$, and we conclude that v_0 satisfies (1.4). Note also that λu_λ itself converges to c_0 locally uniformly, since $\lambda u_\lambda = \lambda v_\lambda - \lambda\|u_\lambda\|_\infty$ and $\lambda v_\lambda \to 0$ because v_λ is locally bounded. Moreover, since $u_\lambda(x) = v_\lambda(x) - \|u_\lambda\|_\infty$, we have $u_\lambda(x) \to -\infty$ for every $x \in \Omega$.

Finally, we claim that $\min_\Omega v_0 = 0$. Indeed, since, for λ small,

$$v_\lambda = u_\lambda - \min_\Omega u_\lambda$$

we clearly have $\min_\Omega v_\lambda(x) = 0 = v_\lambda(x_\lambda)$ for some point $x_\lambda \in \Omega$ such that $\min_\Omega u_\lambda = u_\lambda(x_\lambda)$. If $\{x_\lambda\}$ remains in a compact subset of Ω, we deduce that $\min_\Omega v_0 = 0$ as a consequence of the local uniform convergence of v_λ. Otherwise, (always up to subsequences) we have $d(x_\lambda) \to 0$; however, from (2.1) this means that there exist y_λ such that $d(y_\lambda) = \delta_0$ and

$$u_\lambda(x_\lambda) \geq -d(x_\lambda)^\theta - u_\lambda^-(y_\lambda).$$

Since $u_\lambda(x) \to -\infty$ everywhere (and locally uniformly), we deduce that

$$v_\lambda(y_\lambda) = u_\lambda(y_\lambda) - u_\lambda(x_\lambda) \leq u_\lambda(y_\lambda) + d(x_\lambda)^\theta + u_\lambda^-(y_\lambda) = d(x_\lambda)^\theta \to 0$$

which means that there exists a point y_0 such that $d(y_0) = \delta_0$ and $v_0(y_0) = 0$ (since v_0 is nonnegative). This proves that $\min_\Omega v_0 = 0$.

To conclude, we use Theorem VI.I in [18] which says that c_0 is unique (i.e. the unique constant such that (1.4) may have solution) and that problem (1.4) has a unique solution up to addition of a constant. In particular, we deduce that v_0 is the unique solution such that $\min_\Omega v_0 = 0$. The uniqueness of c_0 and v_0 implies that the whole sequences v_λ and λu_λ converge to v_0 and to c_0 respectively. □

Remark 2.1. In the proof of part (i) we actually proved that the existence of a subsolution of (1.1) in $H_0^1(\Omega) \cap L^\infty(\Omega)$ implies the existence of a solution. In Proposition 1.1 we complete this argument showing that this also implies the uniqueness in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Remark 2.2. With the same arguments as above, in the case (ii) of Theorem 1.1 we can prove that if we fix any point $x_0 \in \Omega$, then $u_\lambda(x) - u_\lambda(x_0)$ converges to the unique solution of (1.4) such that $v(x_0) = 0$. This is also a typical statement for the ergodic limit. However, the convergence of $u_\lambda + \|u_\lambda\|_\infty$ seems more interesting here since it better shows that the blow-up propagates from the interior.

We end this subsection by giving a simple proof of Proposition 1.1 in consequence of the study of the ergodic limit. A different proof is given in [8], in the framework of viscosity solutions.

Proof of Proposition 1.1:

Assume that there exists a solution $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.1), and let v be a solution of (1.4). Assume by contradiction that $c_0 \leq 0$, then v is a supersolution of

(1.1) and $\varphi - v$ solves in the weak sense

$$-\Delta(\varphi - v) + q|\nabla v|^{q-2}\nabla v\nabla(\varphi - v) \le 0.$$

Since $v \to +\infty$ on the boundary, $\varphi - v$ has a maximum inside Ω, which we can assume to be positive replacing v with $v - k$ for a constant k. Using that $v \in W^{1,\infty}_{loc}(\Omega)$, we can apply the strong maximum principle (see [14], Theorem 8.19) and we get that $\varphi - v$ is a constant, which is impossible. Therefore we must have $c_0 > 0$.

Conversely, assume that $c_0 > 0$. We are going to prove not only that problem (1.1) has a solution but actually that there exists a solution of

$$\begin{cases} -\Delta u + |\nabla u|^q = f(x)(1+\delta) & \text{in } \Omega, \\ u \in H^1_0(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{2.4}$$

for any δ such that $0 \le \delta < \frac{c_0}{\|f\|_\infty}$. Indeed, for $\lambda \ge 0$ consider the problem

$$\begin{cases} -\Delta u_\lambda + \lambda u_\lambda + |\nabla u_\lambda|^q = f(x) - \delta\|f\|_\infty & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Assume that there is no solution when $\lambda = 0$: then applying Theorem 1.1 we deduce that $u_\lambda(x) \to -\infty$ for every $x \in \Omega$ and that $\lambda u_\lambda \to c_\delta$ locally uniformly, where c_δ is the unique constant such that the problem

$$\begin{cases} -\Delta v + |\nabla v|^q + c_\delta = f(x) - \delta\|f\|_\infty & \text{in } \Omega, \\ v \to +\infty \quad \text{as } x \to \partial\Omega \end{cases}$$

admits a solution. The uniqueness of the ergodic constant implies $c_\delta = c_0 - \delta\|f\|_\infty$. On the other hand, since $u_\lambda \to -\infty$, we deduce that $c_\delta \le 0$, hence $c_0 \le \delta\|f\|_\infty$, which is not possible as soon as $\delta < \frac{c_0}{\|f\|_\infty}$. Therefore, we proved that there exists a solution to problem (2.5) when $\lambda = 0$, for any $\delta \in [0, \frac{c_0}{\|f\|_\infty})$. Taking $\delta = 0$, this already proves that (1.1) admits a solution. For $\delta > 0$, the solution of (2.5) with $\lambda = 0$ is clearly a subsolution of problem (2.4), and then we can deduce (see Remark 2.1) that problem (2.4) also admits a solution.

Observe now that, defining a new function $z = \frac{u}{1+\delta}$, the existence of a solution of (2.4) implies the existence of a solution of problem

$$\begin{cases} -\Delta z + (1+\mu)|\nabla z|^q = f(x) & \text{in } \Omega, \\ z \in H^1_0(\Omega) \cap L^\infty(\Omega), \end{cases}$$

for μ positive and sufficiently small. By Theorem 2.5 in [7] we conclude that the comparison principle holds for problem (1.1) in $H^1_0(\Omega) \cap L^\infty(\Omega)$ and in particular that there exists a unique solution φ of (1.1) in $H^1_0(\Omega) \cap L^\infty(\Omega)$. □

Note that, in the above analysis, we proved that $c_0 > 0$ is necessary and sufficient for the existence of weak subsolutions to problem (1.1), and by solving problem (2.5) with $\lambda = 0$ we showed in that case the existence of a strict subsolution. Applying Theorem 2.5 in [7], this implies that the comparison principle holds for problem (1.1) in $H_0^1(\Omega) \cap L^\infty(\Omega)$. It could be useful to state explicitly this result.

Corollary 2.1. *Let* $1 < q \le 2$ *and* $f \in L^\infty(\Omega)$*. If* $u_1, u_2 \in H^1(\Omega) \cap L^\infty(\Omega)$ *are respectively a weak subsolution and supersolution of problem* (1.1)*, then we have* $u_1 \le u_2$ *a.e. in* Ω*.*

We state now another straightforward consequence of Proposition 1.1, which gives a characterization of the ergodic constant c_0.

Corollary 2.2. *Let* $1 < q \le 2$, $f \in L^\infty(\Omega)$*, and let* c_0 *be the ergodic constant* (*i.e. the unique constant such that* (1.4) *has a solution*)*. Then we have*

$$\begin{aligned} c_0 &= \sup\{c \in \mathbb{R} : \exists \varphi \in H^1(\Omega) \cap L^\infty(\Omega) \text{ such that } -\Delta\varphi + |\nabla\varphi|^q + c \le f\} \\ &= \sup\{c \in \mathbb{R} : \exists \varphi \in W^{2,p}(\Omega)\ (\forall p < \infty) \text{ such that } -\Delta\varphi + |\nabla\varphi|^q + c \le f\} \\ &= \sup\{c \in \mathbb{R} : \exists \varphi \in C(\overline{\Omega}) \text{ such that } -\Delta\varphi + |\nabla\varphi|^q + c \le f \text{ in viscosity sense}\} \end{aligned}$$

and moreover c_0 *is not attained.*

Proof. Since the ergodic constant corresponding to $f - c$ is $c_0 - c$, we proved in Proposition 1.1 that there exists a subsolution (or a solution) of

$$\begin{cases} -\Delta\varphi + |\nabla\varphi|^q + c = f(x) & \text{in } \Omega, \\ \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases}$$

if and only if $c_0 - c > 0$, hence we conclude. The equivalence between all different formulations is just a consequence of the comparison principle. The characterization of c_0 in terms of viscosity solutions is also proved in [8]. □

2.2. Rate of Convergence and Proof of Theorem 1.2

We give here an error estimate for the convergence of λu_λ to c_0. This will follow from the next lemma, where we use the same ideas introduced in [8] for the asymptotic behaviour of the evolution problem. Let us recall that we denote by v_0 the unique solution of (1.4) such that $\min_\Omega v_0 = 0$.

Lemma 2.2. *Let* $f \in W^{1,\infty}(\Omega)$*, and let* u_λ *be the solution of* (1.3)*. Then we have:*

(i) *If* $c_0 < 0$*, there exists a constant* $M > 0$ *such that*

$$\begin{cases} u_\lambda - \frac{c_0}{\lambda} \ge \gamma(\lambda) v_0(x + \mu(\lambda) n(x)) - M & \text{if } \frac{3}{2} \le q \le 2, \\ u_\lambda - \frac{c_0}{\lambda} \ge \gamma(\lambda) v_0(x + \mu(\lambda) n(x)) - M\lambda^{-(3-2q)/(2-q)} & \text{if } 1 < q < \frac{3}{2} \end{cases} \tag{2.6}$$

where $n(x)$ *is a vector field such that* $n(x) \cdot \nabla d(x) > 0$*, and where* $\gamma(\lambda) \to 1$, $\mu(\lambda) \to 0$ *as* $\lambda \to 0$*.*

(ii) *If $c_0 = 0$, there exists a constant $M > 0$ such that*

$$\begin{cases} u_\lambda \geq \gamma(\lambda) v_0(x + \mu(\lambda) n(x)) - M|\log \lambda| & \text{if } q = 2, \\ u_\lambda \geq \gamma(\lambda) v_0(x + \mu(\lambda) n(x)) - M\lambda^{-(2-q)} & \text{if } 1 < q < 2 \end{cases} \tag{2.7}$$

where $\gamma(\lambda) \to 1$, $\mu(\lambda) \to 0$ as $\lambda \to 0$.

PROOF. Consider here $d(x)$ to be the signed distance function, which is negative when $x \notin \overline{\Omega}$. Let us fix a $\bar{\delta} > 0$ such that $d(x)$ is C^2 in $\{x \in \mathbb{R}^N : |d(x)| < \bar{\delta}\}$ and take a smooth function $\tilde{d}(x)$ such that $\tilde{d}(x) = d(x)$ for $|d(x)| < \frac{\bar{\delta}}{2}$ and $\tilde{d}(x)$ is constant for $|d(x)| > \bar{\delta}$. Consider now the vector field

$$n_k(x) = \int_{\mathbb{R}^N} \nabla \tilde{d}(y) \rho_k(x - y)\, dy$$

where ρ_k is a standard mollifying kernel (supported in the ball $B_{1/k}(0)$). Of course we have $n_k \in C^\infty$ and, using the properties of $d(x)$ and in particular that $d \in C^2$, we have

$$|n_k| \leq 1, \quad |Dn_k| \leq \|D^2 d\|_\infty, \quad |D^2 n_k| \leq k\|D^2 d\|_\infty. \tag{2.8}$$

Moreover n_k is clearly an approximation of $\nabla \tilde{d}(x)$, in particular

$$|n_k(x) - \nabla \tilde{d}(x)| \leq \frac{\|D^2 d\|_\infty}{k}. \tag{2.9}$$

Then we consider the function

$$v(x) = \gamma(v_0(x + \mu n_k(x)) - L)$$

where $\gamma, \mu \in (0, 1)$ will be fixed later suitably depending on λ, and where L is an additive constant to be chosen. Observe that since

$$d(x + \mu n_k(x)) = d(x) + \mu \nabla d(x) \cdot n_k(x) + O(\mu^2)$$

if we choose $k \geq 2\|D^2 d\|_\infty$ we get using (2.9)

$$d(x) + \frac{1}{2}\mu + O(\mu^2) \leq d(x + \mu n_k(x)) \leq d(x) + \frac{3}{2}\mu + O(\mu^2)$$

for any x: $d(x) \leq \frac{\bar{\delta}}{2}$. In particular, we can take μ small enough in a way that $x + \mu n_k(x) \in \Omega$ for every $x \in \Omega$. In the following, the value of k is fixed e.g. as $2\|D^2 d\|_\infty$.

We compute now the equation for v. Since $\nabla v = \gamma(I + \mu Dn_k(x)) \nabla v_0(x + \mu n_k(x))$, using (2.8), and setting $C_0 = \|D^2 d\|_\infty$, we have

$$
\begin{aligned}
-\Delta v + |\nabla v|^q &\le -\gamma\Delta v_0 + \gamma^q(1+\mu C_0)^q|\nabla v_0|^q \\
&\quad + \gamma\mu|Dn_k(x)|(2+\mu C_0)|D^2 v_0| + \gamma\mu\,\mathrm{tr}(D^2 n_k \nabla v_0) \\
&\le -\gamma\Delta v_0 + \gamma^q(1+\mu C_0)^q|\nabla v_0|^q \\
&\quad + \gamma\mu|Dn_k(x)|(2+\mu C_0)|D^2 v_0| + \gamma\mu k C_0|\nabla v_0|
\end{aligned}
$$

where the argument of v_0 is $x + \mu n_k(x)$. Since v_0 is a solution of (1.4) we obtain

$$
\begin{aligned}
-\Delta v + |\nabla v|^q &\le \gamma(f - c_0) - \gamma|\nabla v_0|^q + \gamma^q(1+\mu C_0)^q|\nabla v_0|^q \\
&\quad + \gamma\mu|Dn_k(x)|(2+\mu C_0)|D^2 v_0| + \gamma\mu k C_0|\nabla v_0|
\end{aligned}
$$

hence

$$
\begin{aligned}
&\lambda v - \Delta v + |\nabla v|^q \\
&\quad \le \gamma(f - c_0) - \gamma|\nabla v_0|^q + \gamma^q(1+\mu C_0)^q|\nabla v_0|^q \\
&\qquad + \gamma\mu|Dn_k(x)|(2+\mu C_0)|D^2 v_0| + \gamma\mu k C_0|\nabla v_0| + \lambda\gamma(v_0 - L)
\end{aligned}
\tag{2.10}
$$

Let us now use the asymptotic behaviour near the boundary of the function v_0; indeed, from [18] we know that

$$
\begin{cases} v_0(x) \sim -\log d(x), & \text{if } q = 2, \\ v_0(x) \sim C^* d(x)^{-\alpha}, & \text{if } 1 < q < 2, \end{cases}
\tag{2.11}
$$

where $\alpha = \frac{2-q}{q-1}$ and C^* is a given positive constant. Moreover, from the asymptotic behaviour of ∇v_0 given in [21], we can deduce (see also [8], Lemma 2.1) that

$$
|D^2 v_0| \le K|\nabla v_0|^q
\tag{2.12}
$$

in some neighborhood of $\partial\Omega$ and for some constant $K > 0$, while

$$
|v_0| \le K(|\nabla v_0| + 1) \quad \text{in } \Omega.
\tag{2.13}
$$

Since $n_k(x)$ is supported in $\left\{d(x) < \bar{\delta} + \frac{1}{k}\right\}$, without loss of generality we can suppose that (2.12) holds true in the support of $n_k(x)$. Therefore we can use (2.12) and also (2.13) in the inequality (2.10), and we end up with

$$
\begin{aligned}
&\lambda v - \Delta v + |\nabla v|^q \\
&\quad \le \gamma(f - c_0) - \gamma[1 - \gamma^{q-1}(1+\mu C_0)^q - K\mu|Dn_k(x)|(2+\mu C_0)]|\nabla v_0|^q \\
&\qquad + \gamma\mu k C_0|\nabla v_0| + \lambda\gamma(K|\nabla v_0| + K - L)
\end{aligned}
$$

hence (using also (2.8)) there exists a constant C_1, independent on γ and μ, such that

$$
\begin{aligned}
\lambda v - \Delta v + |\nabla v|^q &\le \gamma(f - c_0) - \gamma(1 - \gamma^{q-1} - C_1\mu)|\nabla v_0|^q \\
&\quad + \gamma\mu k C_0|\nabla v_0| + \lambda\gamma(K|\nabla v_0| + K - L).
\end{aligned}
$$

Choosing $L > K$ and using Young's inequality we obtain,

$$\begin{aligned}\lambda v - \Delta v + |\nabla v|^q \le \gamma(f - c_0) - \gamma\Big(\frac{1-\gamma^{q-1}}{2} - C_1\mu\Big)|\nabla v_0|^q \\ + C\gamma(1-\gamma^{q-1})\Big(\frac{\mu}{1-\gamma^{q-1}}\Big)^{q/(q-1)} \\ + C\gamma(1-\gamma^{q-1})\Big(\frac{\lambda}{1-\gamma^{q-1}}\Big)^{q/(q-1)}\end{aligned}$$

for some constant $C > 0$. Moreover, since f is Lipschitz and $|n_k| \le 1$ we have

$$\gamma(f(x + \mu n_k(x)) - c_0) \le f(x) - c_0 + (1-\gamma)\|f - c_0\|_\infty + \mu\|\nabla f\|_\infty$$

hence

$$\begin{aligned}\lambda v - \Delta v + |\nabla v|^q \le (f - c_0) + (1-\gamma)\|f - c_0\|_\infty + \mu\|\nabla f\|_\infty \\ - \gamma\Big(\frac{1-\gamma^{q-1}}{2} - C_1\mu\Big)|\nabla v_0|^q \\ + C\gamma(1-\gamma^{q-1})\Big[\Big(\frac{\mu}{1-\gamma^{q-1}}\Big)^{q/(q-1)} + \Big(\frac{\lambda}{1-\gamma^{q-1}}\Big)^{q/(q-1)}\Big]\end{aligned}$$

Choose now γ such that

$$1 - \gamma^{q-1} = \max((2C_1 + 1)\mu, \lambda). \tag{2.14}$$

Note that $0 < \gamma < 1$ as soon as $(2C_1 + 1)\mu < 1$ and $\lambda < 1$. Then we deduce

$$\lambda v - \Delta v + |\nabla v|^q \le (f - c_0) + (1-\gamma)\|f - c_0\|_\infty + \mu\|\nabla f\|_\infty + 2C(1-\gamma^{q-1})$$

and since $\mu \le (1-\gamma^{q-1}) \le (1-\gamma)$ we get

$$\lambda v - \Delta v + |\nabla v|^q \le (f - c_0) + \tilde{K}(1-\gamma) \tag{2.15}$$

where $\tilde{K} = \|f - c_0\|_\infty + \|\nabla f\|_\infty + 2C$. In particular, we have obtained that $v + \frac{c_0}{\lambda} - \tilde{K}\frac{(1-\gamma)}{\lambda}$ is a subsolution of the same equation of u_λ.

Moreover, for every $x \in \partial\Omega$ we have

$$d(x + \mu n_k(x)) \ge \frac{1}{2}\mu + O(\mu^2),$$

hence, using (2.11), we deduce that there exists a constant K^* such that

$$v_0(x + \mu n_k(x)) \le K^*(\mu^{-\alpha} + 1) \quad \forall x \in \partial\Omega$$

when $\alpha > 0$ (i.e. $q < 2$). In particular we deduce, choosing $L > K^*$, that

$$v(x) \le K^* \mu^{-\alpha} \quad \forall x \in \partial\Omega. \tag{2.16}$$

When $q = 2$ we have

$$v_0(x + \mu n_k(x)) \le -\log \mu + K^*$$

and this time we have $v \le -\log \mu$ for $L > K^*$.

We distinguish now two situations:

(i) if $c_0 < 0$ and $q < 2$ we fix μ so that

$$K^* \mu^{-\alpha} = -\frac{c_0}{\lambda}.$$

Then (2.16) implies that $v + \frac{c_0}{\lambda} - \tilde{K}\frac{(1-\gamma)}{\lambda} \le 0$ on the boundary, and then it is a subsolution of problem (1.3) and by comparison we deduce that

$$u_\lambda(x) \ge \gamma(v_0(x + \mu n_k(x)) - L) + \frac{c_0}{\lambda} - \tilde{K}\frac{(1-\gamma)}{\lambda} \tag{2.17}$$

where $\mu = O(\lambda^{1/\alpha})$ and consequently, from (2.14), we have

$$1 - \gamma = \begin{cases} O(\lambda) & \text{if } \alpha \le 1 \\ O(\lambda^{1/\alpha}) & \text{if } \alpha > 1 \end{cases}$$

Recalling the value of $\alpha = \frac{2-q}{q-1}$ we obtain (2.6) from (2.17). When $q = 2$ we fix μ so that $-\log \mu = -\frac{c_0}{\lambda}$, in a way that $v + \frac{c_0}{\lambda} \le 0$ on the boundary; since this implies $\mu = o(\lambda)$, by (2.14) we have $1 - \gamma = O(\lambda)$ and the conclusion follows from (2.17).

(ii) If $c_0 = 0$, and if $q < 2$, then we fix μ so that

$$K^* \mu^{-\alpha} = \frac{\tilde{K}(2C_1 + 1)\mu}{\lambda},$$

which means $\mu = \left(\frac{K^*}{\tilde{K}(2C_1+1)}\lambda\right)^{1/(1+\alpha)}$. Since $\alpha > 0$ this implies that $\lambda = o(\mu)$, and consequently, from (2.14), we get

$$1 - \gamma^{q-1} = (2C_1 + 1)\mu,$$

hence $K^* \mu^{-\alpha} = \tilde{K}\frac{(1-\gamma^{q-1})}{\lambda} \le \tilde{K}\frac{(1-\gamma)}{\lambda}$. We deduce from (2.16) that $v - \tilde{K}\frac{(1-\gamma)}{\lambda} \le 0$ on the boundary, and since $v - \tilde{K}\frac{(1-\gamma)}{\lambda}$ is a subsolution, we conclude again by comparison that (2.17) holds true, where now $c_0 = 0$ and

$$1 - \gamma = O(\lambda^{1/(1+\alpha)})$$

Recalling the value of α we get (2.7). If $q = 2$, now we fix μ by the implicit relation

$$-\log \mu = \frac{\tilde{K}(2C_1 + 1)\mu}{\lambda},$$

which makes sense when λ, μ are small. This implies $\lambda = o(\mu)$, hence $1 - \gamma^{q-1} = (2C_1 + 1)\mu$ and the above relation gives as before that $v - \tilde{K}\frac{(1-\gamma)}{\lambda} \le 0$ on the boundary. We conclude again that $u \ge v(x) - \tilde{K}\frac{(1-\gamma)}{\lambda}$, and since $\frac{(1-\gamma)}{\lambda} = O(\frac{\mu}{\lambda}) = O(|\log \lambda|)$, we get (2.7). □

We deduce from the above lemma the estimate for the corrector term $u_\lambda - \frac{c_0}{\lambda}$ and therefore the conclusion of Theorem 1.2.

Corollary 2.3. *Assume that* $1 < q \le 2$, *and* $f \in W^{1,\infty}(\Omega)$. *Let* u_λ *be the solution of* (1.3), *and* c_0 *the ergodic constant of problem* (1.4). *Then we have*

(i) *If* $c_0 < 0$ *then*

$$\begin{cases} u_\lambda - \frac{c_0}{\lambda} = O(1) & \text{when } \frac{3}{2} \le q \le 2 \\ u_\lambda - \frac{c_0}{\lambda} = O(\lambda^{-(3-2q)/(2-q)}) & \text{when } 1 < q < \frac{3}{2} \end{cases} \tag{2.18}$$

(ii) *If* $c_0 = 0$ *then*

$$\begin{cases} u_\lambda = O(|\log \lambda|) & \text{when } q = 2 \\ u_\lambda = O(\lambda^{-(2-q)}) & \text{when } 1 < q < 2 \end{cases} \tag{2.19}$$

where the above bounds are to be meant as locally uniform.

In particular, the conclusions of Theorem 1.2 *hold true.*

Proof. The function $v_\lambda = u_\lambda - \frac{c_0}{\lambda}$ solves

$$\begin{cases} \lambda v_\lambda - \Delta v_\lambda + |\nabla v_\lambda|^q + c_0 = f(x) & \text{in } \Omega, \\ v_\lambda = -\frac{c_0}{\lambda} & \text{on } \partial\Omega. \end{cases}$$

Since $v_0 \ge 0$, and since $v_0 \to +\infty$ on the boundary, we have that v_0 is a supersolution of the same problem, hence we deduce that $u_\lambda - \frac{c_0}{\lambda} \le v_0$. Using Lemma 2.2 for the estimate from below we conclude the local uniform bounds stated in (2.18) and (2.19). □

Remark 2.3. *The error estimates of Corollary* 2.3 (*or Theorem* 1.2) *are optimal. If either* $c_0 = 0$ *or* $c_0 < 0$ *and* $q < \frac{3}{2}$, *it is possible that* $u_\lambda - \frac{c_0}{\lambda} \to -\infty$ *locally uniformly with the rate given in* (2.18) *or* (2.19). *Such type of behaviours can be proved e.g. in star-shaped domains in case that* $f(x) - c_0 < -\delta < 0$ *for some constant* δ, *a situation which can actually occurr. We refer the reader to* [8] *where similar examples are constructed for the solutions of evolution problems: the same construction would apply here to show the optimality of the previous bounds.*

2.3. General Dirichlet Conditions

The same results hold true for a nonhomogeneous Dirichlet condition, i.e. for the problem

$$\begin{cases} -\Delta\varphi + |\nabla\varphi|^q = f(x) & \text{in } \Omega, \\ \varphi = g & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

where $g \in C(\partial\Omega)$. In such a situation, it seems suitable to consider viscosity solutions (since g may not be the trace of a H^1 function). It follows from [8] that this problem has a viscosity solution if and only if $c_0 > 0$ (see also Corollary 2.2). Observe that the constant c_0 is independent from the Dirichlet data. Actually, the same result of Theorem 1.1 holds true provided one considers simply viscosity solutions. We obtain then the following result

THEOREM 2.1. *Assume that* $1 < q \le 2$, *and* $f \in L^\infty(\Omega)$. *For* $\lambda > 0$, *let* u_λ *be the viscosity solution of*

$$\begin{cases} \lambda u_\lambda - \Delta u_\lambda + |\nabla u_\lambda|^q = f(x) & \textit{in } \Omega, \\ u_\lambda = g & \textit{on } \partial\Omega. \end{cases} \tag{2.21}$$

Then we have

(i) *If problem* (2.20) *has a viscosity solution* φ, *then* $u_\lambda \to \varphi$ *uniformly as* $\lambda \to 0$.
(ii) *If problem* (2.20) *has no viscosity solution, then we have, as* $\lambda \to 0$,

$$u_\lambda(x) \to -\infty \quad \textit{for every } x \in \Omega,$$
$$\lambda u_\lambda \to c_0 \quad \textit{locally uniformly in } \Omega,$$
$$u_\lambda + \|u_\lambda\|_\infty \to v_0 \quad \textit{locally uniformly in } \Omega,$$

where c_0 *is the unique constant such that problem* (1.4) *has a solution and* v_0 *is the unique solution of* (1.4) (*in* $W^{2,p}_{loc}(\Omega)$ $\forall p < \infty$) *such that* $\min_\Omega v_0(x) = 0$.

PROOF. The proof follows the same steps as Theorem 1.1. Since $\lambda\|u_\lambda\|_\infty \le \|f\|_\infty + \lambda\|g\|_\infty$, the interior gradient bound of Lemma 2.1 holds true; note in particular that the sequence u_λ locally belongs to $W^{2,p}$ for any $p < \infty$. If problem (2.20) admits a viscosity solution (or even merely a subsolution), using the comparison principle for viscosity solutions one has that u_λ remains uniformly bounded. Thanks to standard stability results, one can conclude that (a subsequence of) u_λ converges towards a viscosity solution of (2.20). In addition u_λ will be locally bounded in $W^{2,p}$ by elliptic regularity. The uniqueness of viscosity solutions of (2.20) implies the convergence of the whole sequence.

If problem (2.20) does not have a viscosity solution, then $\|u_\lambda\|_\infty \to +\infty$. Proposition 2.1 continues to hold with minor changes, in particular (2.1) is replaced by

$$u_\lambda(x) \ge -d(x)^\theta - \|g\|_\infty - \sup_{\{d(x)=\delta_0\}} u_\lambda^- \quad \forall x : d(x) \le \delta_0, \tag{2.22}$$

which holds true for any δ_0 small enough (only depending on θ, f, q, Ω). As in Theorem 1.1 $u_\lambda + \|u_\lambda\|_\infty$ can be proved to be locally compact and converges to a solution v_0 of (1.4). In order to prove that $\min_\Omega v_0 = 0$, one proceeds as in Theorem 1.1 using now (2.22) instead of (2.1) to deduce that the minimum points x_λ of u_λ cannot go to the boundary. $\square$

Finally, let us observe that the error estimates of Theorem 1.2 hold for the solutions of (2.21) as well, and can be proved in exactly the same way.

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