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Ordinary Differential Equations — *Recent results on the stability of time dependent sets and their application to bifurcation problems*[1], by Luigi Salvadori and Francesca Visentin.

Dedicated to the memory of Renato Caccioppoli

Abstract. — In the first part of the paper we give a short review of our recent results concerning the relationship between conditional and unconditional stability properties of time dependent sets, under smooth differential systems in $\mathbf{R}^n$. More precisely, let M be an "s-compact" invariant set in $\mathbf{R} \times \mathbf{R}^n$ and let Φ be a smooth invariant set in $\mathbf{R} \times \mathbf{R}^n$ containing M. It is assumed that M is uniformly asymptotically stable with respect to the perturbations lying on Φ. The unconditional stability properties of M depend on the stability properties of Φ "near M". This dependence has been analyzed in general, and, in the periodic case, complete characterizations are obtained. In the second part, the above results have been applied to bifurcation problems for periodic differential systems. Some our previous statements on the matter are revisited and enriched.

Key words: Invariance, first integrals, stability properties of sets, bifurcation.

AMS subject classification: 34D20, 34C75, 70K20.

1. Introduction

Let M be a time dependent set in $\mathbf{R} \times \mathbf{R}^n$. M is said to be s-compact if for any $t \in \mathbf{R}$ the section $M(t)$ is nonempty, compact, and contained in a fixed set Q in $\mathbf{R}^n$. Let S be a smooth differential system in $\mathbf{R}^n$, $\dot{x} = f(t, x)$, and let M be an s-compact set, invariant under S, and contained in a suitable invariant set Φ. The first part of the present paper is a review of some our recent results [11] concerning the unconditional stability properties of M when M is uniformly asymptotically stable on Φ (that is with respect to the initial perturbations $(t_0, x_0) \in \Phi$).

The stability problem of M involves the stability of Φ "near M" in the sense recalled in Sec. 2. The connection betweeen the stability properties of Φ near M and the (unconditional) stability properties of M is analyzed in Sec. 3. One finds that M is stable (asymptotically stable) if Φ is stable (asymptotically stable) near

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M. These statements are not completely invertible. Precisely we need that the stability or the asymptotic stability of M is *uniform* in order to obtain the same property of Φ near M.

In the case that f, M are both periodic in t for the same constant $\omega > 0$ (in particular f or M, or both t-independent), the stability and the asymptotic stability of M, Φ, when occurring, are uniform. Then we obtain in this case a complete characterization of the above stability properties of M, that is M is stable (asymptotically stable) if and only if Φ is stable (asymptotically stable) near M (Sec. 4).

It is useful to compare these latter results with some classical results (Liapunov [5], Pliss [9], Kelley [4]). In [4] $M = \mathbf{R} \times C$, where C is an equilibrium, or the orbit of a periodic solution, or a periodic surface. Moreover S is autonomous and (by a suitable modification near C) admits in $\mathbf{R}^n$ an invariant center manifold Ψ containing C and exponentially asymptotically stable near C. It was found that the unconditional stability properties of C are completely determined by the stability properties of C on Ψ: if C is stable (asymptotically stable, unstable) on Ψ, then C is stable (asymptotically stable, unstable). If C is asymptotically stable on Ψ, the unconditional asymptotic stability of C follows immediately from our results with $\Phi = \mathbf{R} \times \Psi$ and $M = \mathbf{R} \times C$. Similarly it may be treated the known result (see for instance Chow and Hale [3]) concerning the asymptotic stability problem of a ω-periodic solution $x(t)$ to a ω-periodic differential system. In this case Φ and M are ω-periodic subsets of $\mathbf{R} \times \mathbf{R}^n$ and $M = \{(t, x) : t \in \mathbf{R}, x = x(t)\}$. It has to be noticed that the exponential character in [4] of the asymptotic stability of Ψ near C does not play any role. This has been the motivation to analyze in general the influence that the stability properties of Φ near M have on the corresponding unconditional stability properties of M. We do not have discussed the extendibility to our general setup of the result in [4] relative to the case that M is nonasymptotically stable on Ψ. We only remark that for this extension the assumption that the asymptotic stability of Φ is exponential cannot be in general avoided.

The second part of the paper is devoted to revisit and enrich the results in [12] on the bifurcation problems for a smooth periodic differential system (S_μ) depending on a scalar parameter $\mu \geq 0$. The following conditions are satisfied: (i) there exists $E \in \mathbf{R}^n$ such that E is an equilibrium for any $\mu \geq 0$; (ii) (S_μ) admits a suitable invariant manifold Φ_μ containing $M_0 = \mathbf{R} \times \{E\}$ and asymptotically stable near M_0; (iii) M_0 is asymptotically stable on Φ_μ for $\mu = 0$ and completely unstable on Φ_μ for $\mu > 0$ small.

Preliminarily we have treated the case that $\Phi_\mu \equiv \mathbf{R} \times \mathbf{R}^n$ for any $\mu \geq 0$. In the case that each Φ_μ is a proper manifold of $\mathbf{R} \times \mathbf{R}^n$, by a change of the spatial variables depending on t, μ, any system (S_μ) is transformed into a new system (Σ_μ) and any Φ_μ is transformed in a unique set Φ containing M_0. The change of variables is such that for each $\mu \geq 0$ small all the properties of stability and invariance are not modified and the bifurcation problem is the same unless a homeomorphism of the bifurcating sets. One finds that $\mu = 0$ is a bifurcation value on the right and that the bifurcating sets are s-compact subsets of Φ, invariant, and asymptotically stable. Moreover each bifurcating set is the largest invariant compact set disjoint from M_0 contained in a fixed compact s-neighborhood of M_0 in $\mathbf{R} \times \mathbf{R}^n$.

The proofs are obtained by an application of the results in Sec. 4 and by using for any initial time t_0 and any small $\mu > 0$, an appropriate autonomous discrete dynamical system associated to (S_μ) or (Σ_μ) respectively.

2. Preliminaries

Denote by $\|\cdot\|$ the Euclidean norm in $\mathbf{R}^n$ and by ρ the induced distance. Denote by $\mathscr{L}(x)$ the class of functions $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $(t, x) \to f(t, x)$, which are locally Lipschitzian in x. Moreover, we will write $f \in \mathscr{L}_u(x)$ if f satisfies the condition that for every compact set $K \subset \mathbf{R}^n$ there exists a constant $L(K) > 0$ such that $\|f(t, x) - f(t, y)\| \leq L(K)\|x - y\|$ for all x, y in K and t in $\mathbf{R}$, and write $f \in \mathscr{L}_{ub}(x)$ if in addition for every compact $K \subset \mathbf{R}^n$ the function f is bounded.

Consider the system of differential equations

$$\dot{x} = f(t, x), \quad (\dot{\ }) = \frac{d}{dt} \tag{2.1}$$

where $f \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ and $f \in \mathscr{L}(x)$. For any $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$ let us denote by $x(t, t_0, x_0)$ the solution through (t_0, x_0) and by $j(t_0, x_0)$ its maximal interval of existence. Moreover we denote by $j^+(t_0, x_0)$, $j^-(t_0, x_0)$ the intersections of $j(t_0, x_0)$ with $[t_0, +\infty)$ and $(-\infty, t_0]$ respectively. The sets $\{(t, x) : t \in j(t_0, x_0),$ $x = x(t, t_0, x_0)\}$ and $\{x = x(t, t_0, x_0) : t \in j(t_0, x_0)\}$ will be called the trajectory and the orbit of $x(t, t_0, x_0)$ respectively.

We first wish to recall a resut concerning the case that: (*i*) $f(t, 0) \equiv 0$ and f is periodic in t for some constant $\omega > 0$; (*ii*) (2.1) admits a first integral $F \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^+)$, with $F(t, 0) \equiv 0$ and such that the origin is uniformly asymptotically stable with respect to the initial perturbations $(t_0, x_0) \in \ker F$. The following theorem holds.

Theorem 2.1. *Under the assumptions* (*i*), (*ii*), *the origin is* (*unconditionally*) *stable if and only if F is continuous at $x = 0$ uniformly in t on* $\mathbf{R}^- \equiv (-\infty, 0]$.

The sufficiency was proved by K. Peiffer in [8]. The proof of necessity is trivial and was given in [10]. Theorem 2.1 has been the source of our analysis, addressed to differential systems nonnecessarily periodic for which the origin and $\ker F$ are replaced respectively by two appropriate invariant sets M, Φ in $\mathbf{R} \times \mathbf{R}^n$, with Φ containing M and nonnecessarily the kernel of a first integral.

We need some preliminaries. Let C be a nonempty set in $\mathbf{R}^n$ and for $a > 0$ let $B^n(C, a) = \{x \in \mathbf{R}^n : \rho(x, C) < a\}$, $B^n[C, a] = \{x \in \mathbf{R}^n : \rho(x, C) \leq a\}$. Consider a set A in $\mathbf{R} \times \mathbf{R}^n$. We say that A is s-nonempty if for any t in $\mathbf{R}$ the section $A(t) = \{x \in \mathbf{R}^n : (t, x) \in A\}$ is nonempty. If A is s-nonempty and there exists a compact set Q in $\mathbf{R}^n$ such that $A(t) \subseteq Q$ for all $t \in \mathbf{R}$, then A is said to be s-bounded. In this case the intersection of all these sets Q will be denoted by Q^*. If A is s-bounded and each $A(t)$ is compact, we say that A is s-compact. A set N

in $\mathbf{R} \times \mathbf{R}^n$ is said to be a compact (an open) s-neighborhood of an s-nonnempy set A, if for any $t \in \mathbf{R}$ the section $N(t)$ is a compact (an open) neighborhood of $A(t)$. When the mapping $t \to A(t)$ is ω-periodic for some $\omega > 0$, or in particular t-independent, we say that A is ω-periodic or t-independent respectively.

Let A be an s-nonempty positively invariant set in $\mathbf{R} \times \mathbf{R}^n$. The stability concepts of A are derived from the usual concepts concerning the stability of a single trajectory. For instance A is said to be: (i) *stable* if for any t_0 in $\mathbf{R}$ and $\varepsilon > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $\rho(x_0, A(t_0)) < \delta$ implies $j^+(t_0, x_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, x_0), A(t)) < \varepsilon$ for all $t \geq t_0$; (ii) *uniformly stable* if it is stable and δ may be chosen independent of t_0; (iii) *attracting* if for any t_0 in $\mathbf{R}$ there exists $\sigma = \sigma(t_0) > 0$ such that $\rho(x_0, A(t_0)) \leq \sigma$ implies $j^+(t_0, x_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, x_0), A(t)) \to 0$ as $t \to +\infty$; (iv) *uniformly attracting* if it is attracting, σ may be chosen independent of t_0, and $\rho(x(t, t_0, x_0), A(t)) \to 0$ as $t \to +\infty$ uniformly in $t_0 \in \mathbf{R}$, $x_0 \in B^n[A(t_0), \sigma]$; ($v$) *asymptotically stable* if it is stable and attracting. Similarly one proceeds for the concepts of the weak attractivity and the uniform asymptotic stability of A. When A is t-independent, $A(t) \equiv D$, it is customary to replace A by D and then look at the stability properties of A as stability properties of D.

Let M be a positively invariant s-compact set in $\mathbf{R} \times \mathbf{R}^n$ and let $F \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^+)$ be such that the set $\Phi = ker\, F$ is positively invariant and contains M. We need some definitions concerning properties of Φ and F "near" M.

DEFINITION 2.1. *We will say that Φ has a stability or an attractivity property near M if there exists $\gamma > 0$ such that the property is satisfied with respect to the initial perturbations (t_0, x_0) for which $t_0 \in \mathbf{R}$ and $x_0 \in B^n[M(t_0), \gamma]$.*

For instance Φ is said to be: (i) *stable near M* if there exists $\gamma > 0$ such that for any $t_0 \in \mathbf{R}$ and $\varepsilon > 0$ one may find $\delta = \delta(t_0, \varepsilon) > 0$ with the condition that $x_0 \in B^n[M(t_0), \gamma]$ and $\rho(x_0, \Phi(t_0)) < \delta$ imply $j^+(t_0, x_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, x_0), \Phi(t)) < \varepsilon$ for all $t \geq t_0$; (ii) *attracting near M* if there exists $\gamma > 0$ such that for any $t_0 \in \mathbf{R}$ one may find $\mu = \mu(t_0) > 0$ for which $x_0 \in B^n[M(t_0), \gamma]$ and $\rho(x_0, \Phi(t_0)) \leq \mu$ imply $j^+(t_0, x_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, x_0), \Phi(t)) \to 0$ as $t \to +\infty$. Similarly we may proceed for the other stability and attractivity properties near M.

REMARK 2.1. *Since M is contained in Φ, and then $\rho(x_0, \Phi(t_0)) \leq \rho(x_0, M(t_0))$ for any $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$, the uniform attractivity of Φ near M may be defined as follows: there exists a constant $\sigma > 0$ such that $t_0 \in \mathbf{R}$ and $x_0 \in B^n[M(t_0), \sigma]$ implies that $x(t, t_0, x_0)$ exists for all $t \geq t_0$ and satisfies $\rho(x(t, t_0, x_0), \Phi(t)) \to 0$ as $t \to +\infty$, uniformly in (t_0, x_0).*

We give now a concept of positive definitiveness of F in terms of M and Φ.

DEFINITION 2.2. *The function F is said to be Φ-positive definite near M if for some $\gamma > 0$ and for any $t_0 \in \mathbf{R}$, $\alpha > 0$ there exists $\beta = \beta(t_0, \alpha) > 0$ such that if $t \geq t_0$, $x \in B^n[M(t), \gamma]$, and $\rho(x, \Phi(t)) \geq \alpha$, then $F(t, x) \geq \beta$.*

We observe that because of the continuity of F we have that if the above $\beta(t_0, \alpha)$ exists for a fixed t_0, it exists for any t_0 and for $t_0' > t_0$ one may assume $\beta(t_0', \alpha) = \beta(t_0, \alpha)$. If $\Phi(t) \equiv M(t) \equiv D$, this definition reduces to the usual concept of positive definitiveness of F with respect to D. We also need a weaker property which involves the solutions of (2.1). For $\gamma > 0$, $t_0 \in \mathbf{R}$, consider the following set

$$\Pi(t_0, \gamma) = \{(t, x) : t \geq t_0, x \in B^n[M(t), \gamma], t_0 \in j^-(t, x), x(t_0, t, x) \in B^n[M(t_0), \gamma]\}.$$

Definition 2.3. *The function F is said to be weakly Φ-positive definite near M if for some $\gamma > 0$ and for any $t_0 \in \mathbf{R}$ and $\alpha > 0$, there exists $\beta = \beta(t_0, \alpha) > 0$ such that if $(t, x) \in \Pi(t_0, \gamma)$ and $\rho(x, \Phi(t)) \geq \alpha$, then $F(t, x) \geq \beta$.*

The property in Definition 2.3 is connected to the stability of Φ near M. A first connection is given by the following lemma.

Lemma 2.1. *Suppose that F is a first integral. Then the stability of Φ near M implies that F is weakly Φ-positive definite near M.*

Proof. For some fixed $\gamma > 0$ and for any $t_0 \in \mathbf{R}$, $\alpha > 0$, there exists $\eta = \eta(t_0, \alpha) > 0$ such that if x_0 is in $B^n[M(t_0), \gamma]$ and $\rho(x_0, \Phi(t_0)) < \eta$ then $j^+(t_0, x_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, x_0), \Phi(t)) < \alpha$ for all $t \geq t_0$. For fixed t_0 let us consider the function $F(t_0, \cdot)$. One has $F(t_0, x_0) > 0$ for any $x_0 \notin \Phi(t_0)$ and $F(t_0, x_0) = 0$ for $x_0 \in \Phi(t_0)$. By setting

$$\beta(t_0, \alpha) = \min\{F(t_0, x_0) : x_0 \in B^n[M(t_0), \gamma], \rho(x_0, \Phi(t_0)) \geq \eta(t_0, \alpha)\},$$

we easily obtain

$$x_0 \in B^n[M(t_0), \gamma], \quad F(t_0, x_0) < \beta(t_0, \alpha) \quad \text{imply } \rho(x(t, t_0, x_0), \Phi(t)) < \alpha \quad \forall t \geq t_0. \tag{2.2}$$

Given any $(t, x) \in \Pi(t_0, \gamma)$, let $x_0 = x(t_0, t, x)$. By definition $x_0 \in B^n[M(t_0), \gamma]$. Hence from (2.2) it follows

$$\rho(x, \Phi(t)) \geq \alpha \quad \Rightarrow \quad F(t_0, x(t_0, t, x)) \geq \beta(t_0, \alpha).$$

In conclusion, since F is a first integral and then $F(t, x) = F(t_0, x(t_0, t, x))$, we have

$$(t, x) \in \Pi(t_0, \gamma) \quad \text{and} \quad \rho(x, \Phi(t)) \geq \alpha \quad \text{imply } F(t, x) \geq \beta(t_0, \alpha).$$

The proof is complete. □

3. Conditional and unconditional stability properties

Consider again system (2.1). Let M be an invariant s-compact set in $\mathbf{R} \times \mathbf{R}^n$. We assume from now on the existence of an invariant set Φ in $\mathbf{R} \times \mathbf{R}^n$ containing M

which is the kernel of a function $F \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^+)$ and satisfies the condition that

$(\mathscr{A}\mathscr{C})$ M is uniformly asymptotically stable for perturbations $(t_0, x_0) \in \Phi$.

We will denote by $(h)_\Phi$ the set of such functions. Moreover we will write $F \in (H)_\Phi$ if $F \in (h)_\Phi$ and F is a first integral. The present section is devoted to the analysis of the unconditional stability properties of M under further requirements. Precisely we suppose that one of the following additional conditions is satisfied:

(u) $f \in \mathscr{L}_u(x)$ (instead of $f \in \mathscr{L}(x)$) and the set $(H)_\Phi$ is nonempty;
(v) $f \in \mathscr{L}_{ub}(x)$ and $\Phi = \mathbf{R} \times ker\,\varphi$, where $\varphi \in C^1(\mathbf{R}^n, \mathbf{R}^q)$, $1 \le q \le n$, and $rank[\partial\varphi/\partial x] = q$ for any $x \in ker\,\varphi$;
(w) $f \in \mathscr{L}_{ub}(x)$ and $\Phi = \{(t, y, z) : z = g(t, y)\}$, where $(y, z) = x$, $g \in C^1(\mathbf{R} \times \mathbf{R}^m, \mathbf{R}^{n-m})$, $g \in \mathscr{L}'_{ub}(y)$. Here by $g \in \mathscr{L}'_{ub}(y)$ we want to mean that g belongs to $\mathscr{L}_{ub}(y)$ together with its partial derivatives.

Theorem 3.1. *Assume* $(\mathscr{A}\mathscr{C})$ *and* (u) *or* (v) *or* (w). *Then* M *is stable if* Φ *is stable near* M.

Proof (Outline). *Case* (u). Because of the condition $(\mathscr{A}\mathscr{C})$, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ such that if $t_0 \in \mathbf{R}$, $y_0 \in \Phi(t_0)$, and $\rho(y_0, M(t_0)) < \delta$, then $j^+(t_0, y_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, y_0), M(t)) < \varepsilon$ for all $t \ge t_0$. Moreover, given any $\gamma > 0$ there exists $\sigma \in (0, \delta(\gamma))$ satisfying the condition that for each $v > 0$ one can find a number $T = T(v) > 0$ such that if $y_0 \in \Phi(t_0)$ and $\rho(y_0, M(t_0)) \le \sigma$, then $\rho(x(t, t_0, y_0), M(t)) < v$ for all $t \ge t_0 + T$.

Let now $F \in (H)_\Phi$. Since Φ is stable near M, by virtue of Lemma 2.1, F is weakly Φ-positive definite near M. Choose the above number γ as in Definition 2.3. Moreover let $\varepsilon \in (0, \sigma)$, $\delta_1 = \delta_1(\varepsilon) = (1/2)\delta(\varepsilon/2)$, $\tau = T((1/2)\delta_1)$ and $\bar\delta = (1/2)\delta_1\, exp(-k\tau)$ where $k = L(B^n[Q^*(M), \gamma])$. By Definition 2.3 there exists $\beta = \beta(t_0, \bar\delta)$ such that for any $t_0 \in \mathbf{R}$ one has

$$(3.1)\qquad (t, x) \in \Pi(t_0, \gamma) \quad \text{and} \quad F(t, x) < \beta \quad \text{imply } \rho(x, \Phi(t)) < \bar\delta.$$

Fix t_0 and assume $x_0 \in B^n(M(t_0), \delta_1)$, $F(t_0, x_0) < \beta$. Since $\delta_1 < \gamma$ and then trivially $(t_0, x_0) \in \Pi(t_0, \gamma)$, from (3.1) it follows $\rho(x_0, \Phi(t_0)) < \bar\delta$. Hence there exists $y_0 \in \Phi(t_0)$ with $\|x_0 - y_0\| < \bar\delta$. Comparing the solutions through (t_0, x_0) and (t_0, y_0) in the interval $[t_0, t_0 + \tau]$, by Gronwall's lemma we obtain

$$\|x(t, t_0, x_0) - x(t, t_0, y_0)\| < \frac{\delta_1}{2}.$$

It easily follows:

$$(3.2)\qquad \rho(x(t, t_0, x_0), M(t)) < \varepsilon \quad \forall t \in [t_0, t_0 + \tau],$$

and

$$\rho(x(t_0+\tau, t_0, x_0), M(t_0+\tau)) < \delta_1. \tag{3.3}$$

Setting now $t_1 = t_0 + \tau$ and $x_1 = x(t_1, t_0, x_0)$, and taking into account that F is a first integral, we then recognize that $x_1 \in B^n(M(t_1), \delta_1)$ and $F(t_1, x_1) < \beta$. Since clearly $(t_1, x_1) \in \Pi(t_0, \gamma)$, by virtue of (3.1) we still have $\rho(x_1, \Phi(t_1)) < \bar{\delta}$. Therefore the result expressed by (3.2), (3.3) holds with (t_0, x_0) replaced by (t_1, x_1), and so on. In other words for any $\varepsilon \in (0, \sigma)$ and $t_0 \in \mathbf{R}$ there exist two positive numbers δ_1 and β such that if $x_0 \in B^n(M(t_0), \delta_1)$ and $F(t_0, x_0) < \beta$, then

$$\rho(x(t, t_0, x_0), M(t)) < \varepsilon, \quad \forall t \geq t_0. \tag{3.4}$$

Let now $\lambda(t_0, \varepsilon)$ be a number in the interval $(0, \delta_1)$ such that $F(t_0, x) < \beta$ for any $x \in B^n(M(t_0), \lambda)$. Then (3.4) holds for each x_0 in $B^n(M(t_0), \lambda)$ and this proves that M is stable.

Case (v). Let $\mathscr{B}$, $\mathscr{B}'$ be two bounded open sets in $\mathbf{R}^n$ with $cl\,\mathscr{B} \subset \mathscr{B}'$ and $Q^*(M) \subset \mathscr{B}$. Consider the system

$$\dot{x} = f(t, x)\alpha(x), \tag{3.5}$$

where $\alpha \in C^\infty(\mathbf{R}^n, [0, 1])$ is such that $\alpha(x) = 1$ for $x \in \mathscr{B}$ and $\alpha(x) = 0$ for $x \notin \mathscr{B}'$. Because of the local character of our stability problems near s-compact sets, system (3.5) may replace the original system (2.1). For any (t_0, x_0) we denote by $x_{(3.5)}(t, t_0, x_0)$ the solution of (3.5) through (t_0, x_0). These solutions clearly exists for all $t \in \mathbf{R}$. The proof is divided into two steps.

(a) Let us prove that Φ is invariant even for (3.5). The derivative of φ along the solutions of (3.5),

$$\frac{d\varphi}{dt}(t, x) = \alpha(x)\left\langle \frac{\partial \varphi}{\partial x}(x), f(t, x) \right\rangle,$$

satisfies the condition

$$\frac{d\varphi}{dt}(t, x) = 0 \quad \forall (t, x) \in \Phi, \tag{3.6}$$

because Φ is invariant under (2.1). To complete the proof, let

$$u = \varphi(x) \tag{3.7}$$

and consider any $x_0 \in ker\,\varphi$. Equation (3.7) is satisfied for $x = x_0$ and $u = 0$. Moreover the determinant of at least one, say s, of the $q \times q$ matrices contained in the $q \times n$ matrix $[\partial\varphi/\partial x](x_0)$ is different from zero. Suppose for instance that s is that contained in the first q columns of $[\partial\varphi/\partial x](x_0)$ and set $x = (y, z)$, $x_0 = (y_0, z_0)$, with $y = (x_1, x_2, \ldots, x_q)$, $z = (x_{q+1}, x_{q+2}, \ldots, x_n)$. Then (3.7) defines in a neighborhood $\mathcal{N}$ of $z = z_0$, $u = 0$ an implicit function $y = y(z, u)$, $y(z_0, 0) = y_0$. Hence the restriction of equation (3.5) to $\mathcal{N}$ in terms of z, u may be written as

$$\begin{aligned} \dot{z} &= Z(t, z, u) \\ \dot{u} &= U(t, z, u), \end{aligned} \tag{3.8}$$

where $U(t, z, 0) \equiv 0$ by virtue of (3.6), (3.7). For any t_0 in $\mathbf{R}$, let $(z(t), u(t))$ be the solution of (3.8) such that $z(t_0) = z_0$, $u(t_0) = 0$. As long as this solution exists in $\mathcal{N}$, one has $u(t) \equiv 0$, that is $(z(t), u(t)) \in \mathit{ker}\, \varphi$. Indeed $(3.8)_2$ is satisfied by assuming $u(t) \equiv 0$, while $(3.8)_1$ for $u = 0$ admits one and only one solution such that $z(t_0) = z_0$. Hence, since (t_0, x_0) is any point of $\Phi = \mathbf{R} \times \mathit{ker}\, \varphi$, the invariance of Φ under (3.5) is now proved.

(b) Since any solution of (3.5) exists for all t in $\mathbf{R}$, we may define a function $G \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^+)$ by assuming

$$G(t, x) \equiv \| \varphi(x_{(3.5)}(0, t, x)) \|.$$

Let us prove that $\mathit{ker}\, G = \Phi$. Indeed $(t, x) \in \mathit{ker}\, G$ implies $(0, x_0) \in \Phi$, with $x_0 = x_{(3.5)}(0, t, x)$. The invariance of Φ under (3.5) then implies $(t, x) \in \Phi$. Similarly one can prove that $(t, x) \in \Phi$ implies $(t, x) \in \mathit{ker}\, G$. Since G is a first integral for (3.5), for this equation we have $G \in (H)_\Phi$. Moreover Φ is stable near M even for (3.5). Finally we observe that since $f \in \mathscr{L}_{ub}(x)$ and α is t-independent, one has that the r.h.s. of (3.5) belongs to $\mathscr{L}_u(x)$. Hence from the statement in case (u), it follows that M is stable for (3.5) and then stable for the original equation (2.1). The proof is complete.

Case (w). Letting $u = z - g(t, y)$, system (2.1) in terms of the variables y, u becomes

$$\begin{aligned} \dot{y} &= Y(t, y, u) \\ \dot{u} &= U(t, y, u), \end{aligned} \tag{3.9}$$

where Y, U are continuous functions such that $Y, U \in \mathscr{L}_u(y, u)$ and $U(t, y, 0) \equiv 0$, while Φ becomes the set $\tilde{\Phi} = \{(t, y, u) : u = 0\}$ and M becomes a set $\tilde{M}$. It is easy to see that since g belongs to $\mathscr{L}_u(y, u)$, the stability problems of M and Φ near M for (2.1) are respectively equivalent to the stability problems of $\tilde{M}$ and $\tilde{\Phi}$ near $\tilde{M}$ for (3.9). Setting $\varphi(y, u) \equiv u$ we have $\tilde{\Phi} = \mathbf{R} \times \mathit{ker}\, \varphi$ and clearly φ satisfies for (3.9) the conditions in the case (v) with $q = n - m$. Since $\tilde{\Phi}$ is stable near $\tilde{M}$, the result follows from the statement relative to the case (v). The proof is complete. $\square$

Theorem 3.1 does not appear in general invertible. However it may be proven that the uniform stability of M implies the stability of Φ near M. This implication does not require any of the restrictions on Φ in $(u), (v), (w)$. Precisely the following theorem holds.

THEOREM 3.2. *Assume* $(\mathscr{AC})$. *If* $f \in \mathscr{L}_u(x)$ *and* M *is uniformly stable, then* Φ *is uniformly stable near* M.

PROOF. For any $\varepsilon > 0$ let $\delta(\varepsilon) > 0$ be the number associated with ε in the definition of the uniform stability of M. Let $\sigma > 0$ be such that $\rho(x(t, t_0, y_0), M(t)) \to 0$ as $t \to +\infty$, uniformly in $\{(t_0, y_0) : t_0 \in \mathbf{R}, y_0 \in \Phi(t_0) \cap B^n[M(t_0), \sigma]\}$. Thus if $t_0 \in \mathbf{R}$ and $y_0 \in \Phi(t_0) \cap B^n[M(t_0), \delta(\sigma)]$ we have: (i) $\rho(x(t, t_0, y_0), M(t)) < \sigma$ for all $t \geq t_0$; (ii) for any $v > 0$, there exists $T = T(v) > 0$ such that $\rho(x(t, t_0, y_0), M(t)) < v$ for all $t \geq t_0 + T$. Let $\gamma \in (0, \delta(\sigma)/2)$. Fixing now $\varepsilon \in (0, \gamma)$ and $v \in (0, \delta(\varepsilon))$, choose a number $\eta = \eta(\varepsilon) > 0$ with the condition

$$0 < \eta < \frac{\delta(\varepsilon) - v}{exp(kT)}, \quad k = L(B^n[Q^*(M), \sigma]).$$

Let $t_0 \in \mathbf{R}$. Assume $x_0 \in B^n[M(t_0), \gamma]$ and $y_0 \in \Phi(t_0)$ such that $\rho(x_0, \Phi(t_0)) < \eta$ and $\|x_0 - y_0\| < \eta$. Since

$$\|x(t, t_0, x_0) - x(t, t_0, y_0)\| < \eta\, exp(kT) < \delta(\varepsilon) - v < \varepsilon \quad \forall t \in [t_0, t_0 + T], \tag{3.10}$$

and Φ is an invariant set, one has

$$\rho(x(t, t_0, x_0), \Phi(t)) < \varepsilon \quad \forall t \in [t_0, t_0 + T]. \tag{3.11}$$

We also have

$$\rho(y_0, M(t_0)) \leq \|x_0 - y_0\| + \rho(x_0, M(t_0)) < \eta + \gamma < 2\gamma < \delta(\sigma)$$

from which it follows by virtue of (ii)

$$\rho(x(t, t_0, y_0), M(t)) < v \quad \forall t \geq t_0 + T. \tag{3.12}$$

Consequently by virtue of (3.10), (3.12), it easily follows (for details see [11])

$$\rho(x(t, t_0, x_0), M(t)) < \varepsilon \quad \forall t \geq t_0 + T.$$

Hence, since $M(t) \subseteq \Phi(t)$ for every t, the inequality (3.11) is satisfied even for $t > t_0 + T$. In conclusion for each $\varepsilon \in (0, \gamma)$ there exists $\eta > 0$ such that if $(t_0, x_0) \in \mathbf{R} \times B^n[M(t_0), \gamma]$ and $\rho(x_0, \Phi(t_0)) < \eta$ then

$$\rho(x(t, t_0, x_0), \Phi(t)) < \varepsilon \quad \forall t \geq t_0.$$

The proof is complete. □

Analogous theorems are obtained for asymptotic stability. The proofs are similar and they will be completely omitted (for details see [11]). Precisely the following theorems hold.

THEOREM 3.3. *Assume* $(\mathscr{AC})$ *and* (u) *or* (v) *or* (w). *Then* M *is asymptotically stable if* Φ *is asymptotically stable near* M.

THEOREM 3.4. *Assume* $(\mathscr{AC})$. *If* $f \in \mathscr{L}_u(x)$ *and* M *is uniformly asymptotically stable, then* Φ *is uniformly asymptotically stable near* M.

4. THE PERIODIC CASE

The case in which f and M are both ω-periodic in t for the same constant $\omega > 0$ will be specified as the periodic case. In this case the properties $f \in \mathscr{L}(x)$, $f \in \mathscr{L}_u(x)$, $f \in \mathscr{L}_{ub}(x)$ are equivalent. Moreover in the periodic case, the stability and the asymptotic stability of M when occurring are uniform. This is obtained by the same arguments as those used in [13] (Theorems 7.3 and 7.4) to prove the statements when $M = \mathbf{R} \times \{E\}$ and E is an equilibrium. Under the conditions (u) or (v) or (w) (which are now only reduced to restrictions on Φ) even the stability and the asymptotic stability of Φ near M when occurring are uniform. Indeed if Φ is stable near M, M is (uniformly) stable by virtue of Theorem 3.1 and then Φ is uniformly stable near M by virtue of Theorem 3.2. Similarly one may proceed for asymptotic stability by the aid of Theorems 3.3, 3.4. Then in the periodic case Theorems 3.1 and 3.3 are invertible. In other words the following theorem holds.

THEOREM 4.1. *Suppose that* f *and* M *are both* ω*-periodic in* t *for the same constant* $\omega > 0$. *Then, under condition* $(\mathscr{AC})$ *and the restrictions on* Φ *in* (u) *or* (v) *or* (w), M *is stable (asymptotically stable) if and only if* Φ *is stable (asymptotically stable) near* M.

In the periodic case, under condition $(\mathscr{AC})$, Lemma 2.1 is invertible. Indeed assume $F \in (H)_\Phi$ and F is weakly Φ-positive definite near M. From the proof of Theorem 3.1, in case (u), it follows that M is stable. Hence, by virtue of Theorem 4.1, Φ is stable near M.

Moreover, from Theorem 4.1 we derive a statement which is equivalent to Theorem 2.1. Precisely the following corollary holds.

COROLLARY 4.1. *Under the conditions* (i), (ii) *of Theorem* 2.1, *the origin is stable if and only if* $\ker F$ *is stable near* $\mathbf{R} \times \{0\}$.

Thus, if the conditions (i), (ii) are satisfied, the property in Theorem 2.1 that F is continuous at $x = 0$ uniformly on $\mathbf{R}^-$, is equivalent to the property that $\Phi = \ker F$ is stable near $\mathbf{R} \times \{0\}$ (or also to the property that F is weakly Φ-positive definite near $\mathbf{R} \times \{0\}$).

The asymptotic stability problems considered in [4] may be obtained by using Theorem 4.1. For simplicity we consider the case that the origin 0 is an equilibrium and $M = \mathbf{R} \times \{0\}$. Consider the autonomous system

$$\begin{aligned} \dot{y} &= Ay + u(y, z), \\ \dot{z} &= Bz + v(y, z), \end{aligned} \tag{4.1}$$

$y \in \mathbf{R}^m$, $z \in \mathbf{R}^{n-m}$. Here A and B are square matrices, the eigenvalues of A have zero real parts and the eigenvalues of B have negative real parts. Finally u and v are C^2 functions which vanish together with their derivatives at the origin. It is known (see for instance [2], [3]) the existence of a differential system $\mathscr{S}$ associated to (4.1) having the same regularity of (4.1) and such that: (1) $\mathscr{S}$ coincides with (4.1) for $\|y\| < \delta$, $\delta > 0$ small; (2) $\mathscr{S}$ admits an invariant manifold in $\mathbf{R} \times \mathbf{R}^n$, $\Phi = \{(t, y, z) : t \in \mathbf{R}, y \in \mathbf{R}^m, z = g(y)\}$ $g \in C^2$, $g(0) = 0$. Moreover Φ is exponentially asymptotically stable for $\mathscr{S}$ near $M = \mathbf{R} \times \{0\}$. The set $\Phi^* = \{(t, y, z) : t \in \mathbf{R}, \|y\| < \delta, z = g(y)\}$ is locally invariant for (4.1). Clearly the unconditional stability properties of M and the stability properties of M on Φ^* are preserved when the original system (4.1) is replaced by $\mathscr{S}$ and Φ^* is replaced by Φ. Thus the result in [4] relative to the asymptotic stability of equilibrium (expressed in terms of Φ^* and system (4.1)) may be stated in terms of the invariant manifold Φ and system $\mathscr{S}$, by saying that for $\mathscr{S}$ the asymptotic stability of M on Φ implies the asymptotic stability of M. Therefore the result is an immediate consequence of Theorem 4.1.

Similarly it may be treated the asymptotic stability problem of a ω-periodic solution $x(t)$ of a ω-periodic differential equation. In this case Φ and M are ω-periodic subsets of $\mathbf{R} \times \mathbf{R}^n$ and $M = \{(t, x) : t \in \mathbf{R}, x = x(t)\}$.

In Section 3 and in the present one, condition $(\mathscr{AC})$ has always been assumed. It is natural to consider the problem of weaken condition $(\mathscr{AC})$. For example for system (4.1) if $M = \mathbf{R} \times \{0\}$ is stable on Φ, then M is unconditionally stable [4]. This is due to the pecularity of system (4.1), in particular to the property that the asymptotic stability of Φ near M is of exponential type. This last property cannot be in general avoided. Indeed consider the system:

$$\begin{aligned} \dot{y} &= yz^2 \\ \dot{z} &= -z^3, \end{aligned} \tag{4.2}$$

with $y, z \in \mathbf{R}$. The set $\Phi = \{(t, y, z) : z = 0\}$ is invariant and asymptotically stable. With respect to the solutions lying on Φ the origin $(0, 0)$ is nonasymptotically stable. In contrast, $(0, 0)$ is unstable. Indeed $(4.2)_1$ by means of $(4.2)_2$ may be written as

$$\dot{y} = \frac{yz_0^2}{1 + 2z_0^2(t - t_0)}$$

from which one has

$$y(t, t_0, y_0, z_0) = y_0[1 + 2z_0^2(t - t_0)]^{1/2}.$$

Hence our assert follows.

In a forthcoming paper, still in progress, we are analyzing in the periodic case the possibility to transfer the total stability properties from an invariant manifold Φ to the whole space, provided that Φ is asymptotically stable near M.

5. Bifurcation results for periodic differential equations

Up to now we have considered the problem of stability of s-compact sets by its reduction to the problem of stability on an invariant manifold. Since the bifurcation phenomena are normally connected to drastic changes of the stability properties under perturbations, it appears clearly the possibility to use the stability results in Sections 3.4 in order to reduce even problems of existence and stability of bifurcating sets to analogous problems on spaces with a smaller number of dimensions. In the present section we restrict the analysis to cases in which the unperturbed system as well as the perturbed one are all periodic with the same period.

We need some preliminaries. Consider the differential system

$$\dot{x} = f(t, x), \tag{5.1}$$

with $f \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$, $f \in \mathscr{L}(x)$, and periodic in t for some constant $\omega > 0$. Without any loss of generality in the treatment of our local problems we may and do assume the existence of the solutions for every t in $\mathbf{R}$ (see the proof of Theorem 3.1 in the case (v)).

Let $\mathbf{Z}$ be the set of all integers. For any fixed $t_0 \in \mathbf{R}$ consider the map $\Pi_{t_0} : \mathbf{Z} \times \mathbf{R}^n \to \mathbf{R}^n$ defined by $\Pi_{t_0}(i, x) = x(t_0 + i\omega, t_0, x)$. Clearly $\Pi_{t_0}(0, x) = x$ and $\Pi_{t_0}(i_1 + i_2, x) = \Pi_{t_0}(i_1, \Pi_{t_0}(i_2, x))$ for any $i_1, i_2 \in \mathbf{Z}$ and $x \in \mathbf{R}^n$. Hence Π_{t_0} defines an autonomous discrete dynamical system.

These maps may be fruitfully used in the analysis of invariance, attractivity and stability properties of ω-periodic sets. In this line we give here some lemmas which are preliminary to our treatment of the bifurcation problem for periodic differential systems from an equilibrium $\{E\}$. These lemmas do not appear in [12]. Besides the interest in themselves, they are here employed in order to make more clear the proofs of the bifurcations theorems in [12] and, mainly, to obtain for the perturbed differential systems an additional statement on the asymptotic behavior of all the trajectories starting from a fixed s-neighborhood of $\mathbf{R} \times \{E\}$. From now on, if P is any property which occurs with respect to Π_{t_0}, the property will be denoted by $\Pi_{t_0} - P$.

Let M be an s-compact ω-periodic set in in $\mathbf{R} \times \mathbf{R}^n$ and let N be an s-compact, ω-periodic, positively invariant s-neighborhood of M.

Lemma 5.1. *The set M is the largest invariant set contained in N if and only if for every $t_0 \in \mathbf{R}$ the section $M(t_0)$ is the largest Π_{t_0}-invariant set contained in $N(t_0)$.*

Proof. The necessity is trivial. In order to prove the sufficiency, we only need to prove that, given any fixed t^* in $\mathbf{R}$ and any $t_0 > t^*$, $M(t_0)$ is the image of $M(t^*)$ under (5.1). Assume for instance $t^* = 0$. Letting $G = x(t_0, 0, M(0))$, we have

$$\begin{aligned}
\Pi_{t_0}(i, G) &= x(t_0 + i\omega, t_0, G) = x(t_0 + i\omega, t_0, x(t_0, 0, M(0))) \\
&= x(t_0 + i\omega, i\omega, x(i\omega, 0, M(0))) = x(t_0 + i\omega, i\omega, M(i\omega)) \\
&= x(t_0, 0, M(i\omega)) = x(t_0, 0, M(0)) = G.
\end{aligned}$$

Thus G is Π_{t_0}-positively invariant and $G \subseteq N(t_0)$. Therefore $G \subseteq M(t_0)$. Consider the set $W = x(0, t_0, M(t_0))$ and replace Π_{t_0} by Π_0. We obtain

$$\begin{aligned}\Pi_0(i, W) &= x(i\omega, 0, W) = x(i\omega, 0, x(0, t_0, M(t_0)))\\ &= x(i\omega, t_0 + i\omega, x(t_0 + i\omega, t_0, M(t_0))) = x(i\omega, t_0 + i\omega, M(t_0 + i\omega))\\ &= x(0, t_0, M(t_0)) = W.\end{aligned}$$

Hence $W \subseteq M(0) \subseteq N(0)$. Since $x(t_0, 0, W) = M(t_0)$, it follows $M(t_0) \subseteq G$. Hence $G = M(t_0)$ and the proof is complete. □

Even the attractivity or stability properties of M for system (5.1) may be reduced to the same properties for each section $M(t_0)$ under Π_{t_0}.

LEMMA 5.2. *Suppose that M is invariant. Let $(t_0, x_0) \in N$, be such that $\rho(\Pi_{t_0}(i, x_0), M(t_0)) \to 0$ as $i \to +\infty$. Then*

$$\rho(x(t, t_0, x_0), M(t)) \to 0 \quad \text{as } t \to +\infty.$$

PROOF. It is easy to see that given any $\beta > 0$ we can find $\delta(\beta) \in (0, \beta)$ such that $\rho(x_0, M(t_0)) < \delta(\beta)$ implies $\rho(x(t, t_0, x_0), M(t)) < \beta$ for any t_0 and for any $t \in [t_0, t_0 + \omega]$. This follows from the s-compactness and invariance of M, and the uniform Lipschitz condition on f. Let $k = k(\beta) \geq 0$ be such that $\rho(\Pi_{t_0}(i, x_0), M(t_0)) < \delta(\beta)$ for any $i \geq k$. Consider $x(t, t_0, x_0)$ for $t \geq t_0 + k\omega$. In particular for $t \in [t_0 + k\omega, t_0 + (k+1)\omega]$ we have:

$$\begin{aligned}&x(t, t_0, x_0) = x(t, t_0 + k\omega, x(t_0 + k\omega, t_0, x_0)) = x(t - k\omega, t_0, x^*),\\ &x^* = x(t_0 + k\omega, t_0, x_0) = \Pi_{t_0}(k, x_0),\end{aligned}$$

with $t - k\omega \in [t_0, t_0 + \omega]$ and $x^* \in B^n[M(t_0), \delta(\beta)]$. Hence

$$\rho(x(t, t_0, x_0), M(t)) < \beta \quad \forall t \in [t_0 + k\omega, t_0 + (k+1)\omega].$$

Moreover $x(t_0 + (k+1)\omega, t_0, x_0) = \Pi_{t_0}(k+1, x_0) \in B^n[M(t_0), \delta(\beta)]$. Then we may proceed as before in any interval $[t_0 + i\omega, t_0 + (ik+1)\omega]$, $i \geq k$. In conclusion we obtain

$$\rho(x(t, t_0, x_0), M(t)) < \beta \quad \forall t \geq t_0 + k\omega.$$

Since $\beta > 0$ is arbitrary, then $\rho(x(t, t_0, x_0), M(t)) \to 0$ as $t \to +\infty$. □

LEMMA 5.3. *Suppose that M is invariant. Then M is asymptotically stable under* (5.1) *if and only if for every t_0 the section $M(t_0)$ is Π_{t_0}-asymptotically stable.*

PROOF. The necessity is trivial because the application $t \to M(t)$ is ω-periodic. Prove the sufficiency. As in the proof of Lemma 5.2 we may associate with each $\beta > 0$ a number $\delta(\beta) \in (0, \beta)$ such that

$$\rho(x, M(t_0)) < \delta(\beta) \quad \text{implies } \rho(x(t, t_0, x), M(t)) < \beta \quad \forall t \in [t_0, t_0 + \omega]. \tag{5.2}$$

Fix t_0 and let $\sigma = \sigma(t_0)$ denote any positive number such that the set $\{x : \rho(x, M(t_0)) \leq \sigma\}$ is contained in $N(t_0)$. Since the section $M(t_0)$ is a Π_{t_0}-uniform attractor we have that relatively to $\delta(\beta)$ there exists an integer $h(t_0, \beta) > 0$ such that $\rho(x_0, M(t_0)) \leq \sigma$ implies $\rho(x(t_0 + j\omega, t_0, x_0), M(t)) < \beta$ for all $\beta \in (0, \sigma)$, and any integer $j > h(t_0, \beta)$. Consequently by virtue of (5.2) it follows

$$(5.3) \qquad \rho(x_0, M(t_0)) \leq \sigma \quad \text{implies}\ \rho(x(t, t_0, x), M(t)) < \beta \quad \forall t \geq h(t_0, \beta).$$

Since $\beta > 0$ is arbitrary, then $\rho(x(t, t_0, x), M(t)) \to 0$ as $t \to +\infty$. Therefore M is attractive. Moreover there exists a number $\delta^*(\beta) \in (0, \delta(\beta))$ such that

$$\rho(x_0, M(t_0)) < \delta^*(\beta) \quad \text{implies}\ \rho(x(t, t_0, x), M(t)) < \beta \quad \forall t \in [t_0, h(t_0, \beta)]$$

By virtue of (5.3) we conclude that M is even stable and then asymptotically stable. The proof is complete. □

Consider now the family $\mathscr{S}$ of differential systems, $\{(S_\mu) : \mu \geq 0\}$, defined by

$$(S_\mu) \qquad \dot{x} = f(t, x, \mu),$$

with $f \in C^1(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^+, \mathbf{R}^n)$ and periodic in t for some constant $\omega > 0$. We assume $f(t, 0, \mu) \equiv 0$ so that (S_μ) admits the null solution for every $\mu \geq 0$. As for equation (5.1) we assume the existence of the solutions for every t in $\mathbf{R}$. System (S_μ) will be specified as the unperturbed system if $\mu = 0$ and as a perturbed one if $\mu > 0$. We denote by M_0 the so-called null set, $M_0 = \mathbf{R} \times \{0\}$, and by $x(t, t_0, x_0, \mu)$ the solution through (t_0, x_0). The case that we examine concerns the bifurcation from M_0 into invariant, ω-periodic, s-compact sets in $\mathbf{R} \times \mathbf{R}^n$, through the value $\mu = 0$ of the parameter. Exactly the following definition will be assumed.

Definition 5.1. *We say that $\mu = 0$ is a bifurcation value (on the right) for the family $\mathscr{S}$ at $x = 0$ if there exist $\mu^* > 0$ and a family $\{M_\mu\}$, $\mu \in (0, \mu^*)$, of s-compact and ω-periodic subsets of $(\mathbf{R} \times \mathbf{R}^n) - M_0$ having the following properties:*

(*a*) *for each $\mu \in (0, \mu^*)$, M_μ is invariant under (S_μ);*
(*b*) *$M_\mu(t) \to \{0\}$ as $\mu \to 0$ uniformly in t.*

Firstly we prove the following theorem.

Theorem 5.1. *Suppose that the origin $x = 0$ is asymptotically stable for $\mu = 0$ and completely unstable (i.e. asymptotically stable in the past) for $\mu > 0$. Then $\mu = 0$ is a bifurcation value on the right. Precisely there exist $\mu^* > 0$ and a compact s-neighborhood H of M_0 such that for each $\mu \in (0, \mu^*)$ the largest s-compact invariant set of (S_μ) contained in $H - M_0$, say M_μ, is nonempty, ω-periodic, and the family $\{M_\mu\}$ satisfies (b) in Definition 5.1. Moreover each M_μ is asymptotically stable.*

Proof. (*i*) Since the origin is asymptotically stable for $\mu = 0$, there exist a number $\gamma > 0$ and a function $V \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$, ω-periodic in t, such that

$$a(\|x\|) \le V(t,x) \le b(\|x\|), \tag{5.4}$$

$$\dot{V}_{(S_0)}(t,x) \le -c(\|x\|), \tag{5.5}$$

for all $t \in \mathbf{R}$ and $x \in B^n(\gamma)$ [6, 7]. Here a, b, c are continuous strictly increasing functions from $\mathbf{R}^+$ into $\mathbf{R}^+$ with $a(0) = b(0) = c(0) = 0$, and the left hand side of (5.5) is the derivative of V along the solutions of the unperturbed system. Choose $\lambda \in (0, a(\gamma))$. From (5.5) by using continuity arguments it follows the existence of $\mu^* > 0$ such that for any $\mu \in (0, \mu^*)$ the derivative of V along the solutions of the perturbed system (S_μ) satisfies the condition

$$\dot{V}_{(S_\mu)}(t,x) \le -\frac{c(b^{-1}(\lambda))}{2} \quad \forall t \in \mathbf{R}, \quad \forall x \in B^n(\gamma) - B^n(b^{-1}(\lambda)). \tag{5.6}$$

By (5.4), (5.5), (5.6), we easily see that the set

$$H = \{(t,x) : \|x\| \le \gamma, V(t,x) \le \lambda\}$$

has the following properties: (1) each section $H(t)$ is a compact neighborhood of $x = 0$ and is contained in the open ball $B^n(\gamma)$; (2) for any $\mu \in (0, \mu^*)$ H is under (S_μ) asymptotically stable and invariant only in the future; (3) the region of attraction of H contains a fixed s-neighborhood H^* of H and we will choose $H^* = \{(t,x) : \|x\| \le \gamma, V(t,x) \le \lambda^*\}$, for some $\lambda^* > \lambda$; (4) H is ω-periodic, that is $H(t) = H(t + \omega)$.

(*ii*) For any fixed $t_0 \in \mathbf{R}$ and $\mu \in (0, \mu^*)$ consider the above autonomous discrete dynamical system $\Pi_{t_0} = \Pi_{t_0\mu} : \mathbf{Z} \times \mathbf{R}^n \to \mathbf{R}^n$ defined by $\Pi_{t_0}(i,x) = x(t_0 + i\omega, t_0, x, \mu)$. Let $\Lambda^+_{t_0}(x) = \Lambda^+_{t_0\mu}(x)$, $\Lambda^-_{t_0}(x) = \Lambda^-_{t_0\mu}(x)$ be the the positive and the negative limit sets of x under Π_{t_0}. Precisely:

$$\begin{aligned}
\Lambda^+_{t_0}(x) = \{\xi \in \mathbf{R}^n &: \text{there exists a sequence } (i_n), i_n \to +\infty, \\
&\quad \text{with } \Pi_{t_0}(i_n, x) \to \xi\}, \\
\Lambda^-_{t_0}(x) = \{\xi \in \mathbf{R}^n &: \text{there exists a sequence } \{i_n\}, i_n \to -\infty, \\
&\quad \text{such that } \Pi_{t_0}(i_n, x) \to \xi\}.
\end{aligned}$$

Moreover let $J^+_{t_0}(x) = J^+_{t_0\mu}(x)$, $J^-_{t_0}(x) = J^-_{t_0\mu}(x)$ be the positive and the negative prolongational limit set of x under Π_{t_0}. Precisely:

$$\begin{aligned}
J^+_{t_0}(x) = \{\xi \in \mathbf{R}^n &: \text{there exist two sequences } \{i_n\}, i_n \to +\infty, \{x_n\}, x_n \to x, \\
&\quad \text{such that } \pi(i_n, x_n) \to \xi\}, \\
J^-_{t_0}(x) = \{\xi \in \mathbf{R}^n &: \text{there exist two sequences } \{i_n\}, i_n \to -\infty, \{x_n\}, x_n \to x, \\
&\quad \text{such that } \pi(i_n, x_n) \to \xi\}.
\end{aligned}$$

The Π_{t_0}-attractivity and the Π_{t_0}-uniform attractivity for compact sets in $\mathbf{R}^n$ may be characterized by conditions on the above limit and prolongational limit sets, see [1]. In [1] the proofs are given for ordinary autonomous dynamical systems, but all the statements we employ here are valid even for autonomous discrete dynamical systems and are obtained by the same arguments. The set $H(t_0)$ is a Π_{t_0}-uniform attractor and its region of uniform attraction contains $H^*(t_0)$. Therefore $x \in H^*(t_0)$ implies $J^+_{t_0}(x) \neq \emptyset$ and $J^+_{t_0}(x) \subseteq H(t_0)$. Let $\phi_{t_0} = \phi_{t_0\mu}$ be the largest Π_{t_0}-invariant subset of $H(t_0)$. Since $J^+_{t_0}(x)$ is Π_{t_0}-invariant, then $J^+_{t_0}(x) \subseteq \phi_{t_0}$. Moreover ϕ_{t_0} contains the region $A^-_{t_0}(0) = A^-_{t_0\mu}(0)$ of negative attraction of the origin of $\mathbf{R}^n$ under Π_{t_0}. Indeed even $A^-_{t_0}(0)$ is Π_{t_0}-invariant and is contained in $H(t_0)$. Clearly the set $M_\mu(t_0) := \phi_{t_0} - A^-_{t_0}(0)$ is the largest Π_{t_0}-invariant compact set contained in $H(t_0) - \{0\}$. We prove that $M_\mu(t_0)$ is a Π_{t_0}-uniform attractor and the region of uniform attractivity contains $H^*(t_0) - \{0\}$. Assume $x \in H^*(t_0) - \{0\}$. Since $J^+_{t_0}(x) \subseteq \phi_{t_0}$, it remains only to show that $y \in J^+_{t_0}(x)$ implies $y \notin A^-_{t_0}(0)$. We have $x \in J^-_{t_0}(y)$. Therefore if $y \in A^-_{t_0}(0)$, then $J^-_{t_0}(y) = \{0\}$ and consequently $x = 0$. This is a contradiction and the assert is proved. Since $H(t_0) = H(t_0 + \omega)$ and Π_{t_0} remains unchanged when t_0 is replaced by $t_0 + \omega$, we have $M_\mu(t_0) = M_\mu(t_0 + \omega)$.

For each $\mu \in (0, \mu^*)$ define now the set M_μ with the condition that its section at any time $t_0 \in \mathbf{R}$ is $M_\mu(t_0) = \phi_{t_0} - A^-_{t_0}(0)$. Then M_μ is a ω-periodic set in $\mathbf{R} \times \mathbf{R}^n$. It is immediate to recognize that M_μ is s-compact and that $M_\mu(t) \to \{0\}$ as $\mu \to 0$ uniformly in t.

(*iii*) Since for any t_0 in $\mathbf{R}$, $M_\mu(t_0)$ is the largest Π_{t_0}-invariant set, contained in $H(t_0) - \{0\}$, by virtue of Lemma 5.1 it follows that M_μ is the largest s-compact invariant set under (S_μ) contained in $H - M_0$. Finally, since for any t_0 in $\mathbf{R}$, $M_\mu(t_0)$ is a Π_{t_0}-uniform attractor, and then Π_{t_0}-asymptotically stable, by Lemma 5.3 we recognize that M_μ is asymptotically stable. Thus M_μ satisfies all the conditions in Definition 5.1. The proof is complete. □

In the autonomous case the sets M_μ are t-independent. Precisely one has:

Proposition 5.1. *Let us assume that (S_μ) is autonomous for each $\mu > 0$. Then the bifurcating sets M_μ are t-independent, that is $M_\mu = \mathbf{R} \times C_\mu$, where C_μ is the largest compact invariant subset of $\mathbf{R}^n$ disjoint from the origin and contained in a fixed positively invariant neighborhood of the origin.*

Clearly, since in Proposition 5.1 $M_\mu(t) \equiv C_\mu$ for any t, as observed before, we may consider all the properties associated with M_μ as properties of the sets C_μ of $\mathbf{R}^n$. Precisely we may say that the sets C_μ are asymptotically stable and that $C_\mu \to \{0\}$ as $\mu \to 0$.

We treat now the case that the bifurcating sets lie on an invariant manifold. Consider again the above family of differential systems and assume in addition that each (S_μ) admits a ν-dimentional invariant manifold $(0 < \nu < n)$

$$\Phi_\mu = \{(t, y, z) : t \in \mathbf{R}, y \in \mathbf{R}^\nu, z = g(t, y, \mu)\},$$

where $g \in C^1(\mathbf{R} \times \mathbf{R}^\nu \times \mathbf{R}^+, \mathbf{R}^{n-\nu})$, g is ω-periodic in t, its partial derivatives are locally Lipschitzian in y, and $g(t, 0, \mu) \equiv 0$. We notice that the above conditions ensure $g \in \mathscr{L}'_{ub}(y)$. Let $u = z - g(t, y, \mu)$. In terms of y, u each (S_μ) may be written as

$$(\Sigma_\mu) \qquad \begin{aligned} \dot{y} &= Y(t, y, u, \mu), \\ \dot{u} &= U(t, y, u, \mu), \end{aligned}$$

where Y, U are continuous and locally Lipschitzian in (y, u), $Y(t, 0, 0, \mu) \equiv 0$, $U(t, y, 0, \mu) \equiv 0$. Moreover in the (t, y, u)-space the manifolds Φ_μ coincide all with the manifold $\Phi = \mathbf{R} \times \Psi$ with

$$\Psi = \{(y, u) : u = 0\}. \tag{5.7}$$

The bifurcating sets of the family $\Sigma = \{(\Sigma_\mu), \mu \geq 0\}$ are homeomorphic to those of the original family $\mathscr{S}$ while the stability properties involved are clearly the same. Letting now $M_0 = \mathbf{R} \times \{(0, 0)\}$ and considering the differential system of the solutions of (Σ_μ) lying on Φ,

$$(\Sigma_{y\mu}) \qquad \dot{y} = Y(t, y, 0, \mu).$$

we are at last in position to state the main theorem of the section.

Theorem 5.2. *Suppose that:* (1) *the solution* $y = 0$ *of* $(\Sigma_{y\mu})$ *is asymptotically stable if* $\mu = 0$ *and completely unstable if* $\mu > 0$ *small;* (2) Φ *is asymptotically stable near* M_0 *for all* $\mu \geq 0$ *small. Then* $\mu = 0$ *is a bifurcation value on the right for the family* Σ. *Precisely there exist* $\mu^* > 0$ *and a compact s-neighborhood* H *of* M_0 *such that for each* $\mu \in (0, \mu^*)$ *the largest s-compact invariant set of* (Σ_μ) *contained in* $H - M_0$, *say* M_μ, *is nonempty, lies on* Φ, *is* ω*-periodic, asymptotically stable, and the family* $\{M_\mu\}$ *satisfies* (*b*) *in Definition* 5.1.

Proof. Assumption (1) for the part relative to $\mu = 0$ is equivalent to say that the null set M_0 is for $\mu = 0$ asymptotically stable on Φ. Taking into account assumption (2), we recognize then by virtue of Theorem 4.1 that M_0 is for $\mu = 0$ (unconditionally) asymptotically stable. Hence, as we have seen in the proof of Theorem 5.1, if $\gamma > 0$ and $\mu^* > 0$ are small, there exists for any $\mu \in (0, \mu^*)$ a compact s-neighborhood H of M_0 which is ω-periodic and asymptotically stable. Moreover each section $H(t)$ is contained in $B^n(\gamma)$. In the following we choose γ smaller than the number σ in Remark 2.1.

By virtue of Theorem 5.1 applied to the restriction of system (Σ_μ) to Φ (that is to the subspace $u = 0$), we recognize that if $\mu^* > 0$ is sufficiently small then for each $\mu \in (0, \mu^*)$ there exists for system (Σ_μ) a set M_μ which has the following properties: (*i*) M_μ is the largest s-compact invariant subset of Φ contained in $[H \cap \Phi] - M_0$; (*ii*) M_μ is ω-periodic, asymptotically stable with respect to the initial perturbations lying on Φ; (*iii*) $M_\mu(t) \to \{(0, 0)\}$ as $\mu \to 0$ uniformly in t. Moreover, since for every t in $\mathbf{R}$ the section $H(t)$ and then the section $M_\mu(t)$ are contained in $B^n(\gamma)$, we see that for our choice of γ the manifold Φ is asymptotically stable near each M_μ. By virtue of Theorem 4.1 it follows that the sets M_μ are all unconditionally asymptotically stable.

In order to complete the proof, it only remains to prove that for any $\mu \in (0, \mu^*)$, M_μ is the largest invariant s-compact subset of $H - M_0$. Let c_μ be any trajectory of (Σ_μ) entirely contained in H and let t_0, x_0, $x_0 = (y_0, u_0)$, be any point of c_μ. Clearly by virtue of the above condition (i) it is sufficient to prove that $x_0 \in \Psi$. For this consider as in the proof of Theorem 5.1 the autonomous discrete dynamical system $\Pi_{t_0} = \Pi_{t_0\mu} : \mathbf{Z} \times \mathbf{R}^n \to \mathbf{R}^n$ relative now to system (Σ_μ). Because of the ω-periodicity of H, it follows that the orbit of x_0 under Π_{t_0} is entirely contained in $H(t_0)$. Suppose then $x_0 \notin \Psi$. The negative limit set of x_0 under Π_{t_0}, $\Lambda^-_{t_0}(x_0)$, is nonempty, contained in $H(t_0)$, Π_{t_0}-invariant, and compact. One has $\Lambda^-(x_0) \cap [\Psi \cap H(t_0)] = \emptyset$, otherwise x_0 would be for (Σ_μ) weakly attracted to Ψ in the past and then Φ could not be stable near M_0. Let $\delta > 0$ be the distance between the two compact sets $\Lambda^-_{t_0}(x_0)$ and $\Psi \cap H(t_0)$, and let ξ be any point in $\Lambda^-_{t_0}(x_0)$. Because of the Π_{t_0}-invariance of $\Lambda^-_{t_0}(x_0)$, we have

$$\rho(\Pi_{t_0}(i, \xi), \Psi \cap H(t_0)) > \delta \quad \text{for every integer } i \geq 0. \tag{5.8}$$

On the other hand ξ is attracted to Ψ under Π_{t_0} and then $\Lambda^+_{t_0}(\xi) \subseteq \Psi \cap H(t_0)$. Thus we get a contradiction and the proof is complete. □

We conclude by the statement announced before which completes, under the assumptions of Theorem 5.2, the analysis, only partially carried out in [12], of the asymptotic behavior in the future of the solutions $x(t, t_0, x_0, \mu)$, with $x_0 = (y_0, u_0)$, $(t_0, x_0) \in H$ and $\mu > 0$ small. Incidentally we notice that this further result has allowed us to recognize and correct an overview in the proof of Theorem 3.2 in [12].

THEOREM 5.3. *Let $(t_0, x_0) \in H$, that is $t_0 \in \mathbf{R}$ and $x_0 \in H(t_0)$. Assume that, for a given $\mu \in (0, \mu^*)$, $\rho(x(t, t_0, x_0, \mu), M_\mu(t)) \nrightarrow 0$ as $t \to +\infty$, then $\|x(t, t_0, x_0, \mu)\| \to 0$ as $t \to +\infty$.*

PROOF. Let $\Gamma = \{(t, x) : t \in \mathbf{R}, x \in H(t) - A(M_\mu(t)) \cap H(t)\}$, where $A(M_\mu(t))$ is the region of attraction of $M_\mu(t)$ under Π_t. We observe that $A(M_\mu(t))$ is open and Π_t-invariant. Hence Γ is s-compact and Π_t-positively invariant, and $\{(0, 0)\}$ is the largest Π_t-invariant set contained in $\Gamma(t)$. From our assumptions and by virtue of Lemma 5.2, then it follows that $\rho(\Pi_{t_0}(i, x_0), M_\mu(t_0)) \nrightarrow 0$ as $t \to +\infty$. Thus $x_0 \notin A(M_\mu(t_0))$ and then $x_0 \in \Gamma(t_0)$. It follows that x_0 is attracted to the origin $(0, 0)$ under Π_{t_0}. Indeed $\Lambda^-_{t_0}(x_0)$ is nonempty. Moreover it is Π_{t_0}-invariant and then it is contained in the largest Π_{t_0}-invariant subset of $\Gamma(t_0)$), that is it coincides with $\{(0, 0)\}$. By virtue again of Lemma 5.2, now applied to the null set M_0 and for $N = \Gamma$, we have $\rho(x(t, t_0, x_0, \mu), M_0(t)) \to 0$ as $t \to +\infty$. Since $M_0(t) \equiv \{(0, 0)\}$, the proof is complete. □

More detailed information on the structure of the bifurcating sets M_μ may be found in the cases $v = 1$ and $v = 2$ [12]. In the first case, for any t, $\Phi(t)$ is homeomorphic to a straigthline $\{(y, u) : u = 0\}$ passing through the origin. Each section

$M_\mu(t)$ is homeomorphic to the union of two segments located in the regions $y > 0$ and $y < 0$ respectively. The end points are fixed with respect to the discrete dynamical system induced on Φ, while their motion with respect to the differential system is periodic with the same period ω of the system. If $v = 2$, in the autonomous case we find results already known in the usual treatment of Hopf bifurcation although now the asymptotic stability of Φ near the origin is not necessarily exponential. In the general periodic case instead, under some additional assumption we find that the sections $M_\mu(t)$ are homeomorphic to Jordan curves and then the sets M_μ are homeomorphic to tori by interpreting t as an angular variable.

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