Differential Geometry — *A Stampacchia-type inequality for a fourth-order elliptic operator on Kähler manifolds and applications*, by Luca Lussardi.

ABSTRACT. — In this paper we will prove an integral inequality of Stampacchia-type for a fourth-order elliptic operator on complete and connected Kähler manifolds. Our inequality implies a Hodge-Kodaira orthogonal decomposition for the Sobolev-type space $W^{p,q}(X)$. In particular we will able to prove, under suitable topological conditions on the manifold X, the existence of an isomorphism between the Aeppli groups $\Lambda^{p,q}(X)$ and the groups $H^{p,q}(X)$ of all global harmonic forms of bidegree (p,q).

1. INTRODUCTION

Let X be a complex manifold, and let $p, q \geq 1$ integers. The Aeppli groups, even called $\partial\bar{\partial}$-cohomology groups, defined for the first time by Aeppli in [1] and studied, principally, by Bigolin in [10] and in [11], were introduced in order to study cycles of algebraic manifolds (see [8]). More recently the Aeppli groups are under consideration in order to investigate integral transformations (see [17]), properties of balanced manifolds (see [2], [5], [6]) and properties of 1-convex manifolds (see [3], [7], [4]). The Aeppli groups were originally defined in [1] by

$$\Lambda^{p,q}(X) = \frac{\text{Ker}\{A^{p,q}(X) \xrightarrow{d} A^{p+1,q}(X) \oplus A^{p,q+1}(X)\}}{\partial\bar{\partial}A^{p-1,q-1}(X)}$$

$$V^{p,q}(X) = \frac{\text{Ker}\{A^{p,q}(X) \xrightarrow{\partial\bar{\partial}} A^{p+1,q+1}(X)\}}{\partial A^{p-1,q}(X) + \bar{\partial}A^{p,q-1}(X)}$$

where $A^{p,q}(X)$ denotes the space af all (p,q)-differential forms with coefficients in $C^\infty(X)$ and with complex values. If X is a Stein manifolds then Aeppli, in [1], proves that the Aeppli groups are isomorphic to the complex De Rham cohomology: more precisely $\Lambda^{p,q}(X)$ and $V^{p,q}(X)$ are isomorphic, respectively, to the spaces $H^{p+q}(X)$ and $H^{p+q+1}(X)$, where $H^r(X)$ denotes the space of all global harmonic r-forms. The result of Aeppli gives a characterization of the De Rham cohomology for Stein manifolds. If the manifold X is Kähler and compact then Bigolin, in [10], proves, as a consequence of a orthogonal decomposition for the space of all $\partial\bar{\partial}$-closed forms, that both $V^{p,q}(X)$ and $\Lambda^{p,q}(X)$ are isomorphic to

$H^{p,q}(X)$, where $H^{p,q}(X)$ denotes the space of all forms in $H^{p+q}(X)$ of bidegree (p, q); moreover in the same paper some results proved by Aeppli in [1] for Stein manifolds are recovered. If we remove the compactness assumption on the manifold X then, at the moment, it is unknown the relation between Aeppli groups and $H^{p,q}(X)$. In this paper we study the non-compact case. We will able to prove, under a technical topological condition on X (see assumption (5.1)), that the Aeppli groups $\Lambda^{p,q}(X)$ are isomorphic to $H^{p,q}(X)$ whenever X is a connected and complete Kähler manifold. The main tool for the proof of our result is a suitable Hodge-Kodaira orthogonal decomposition. More precisely denoting by $D^{p,q}(X)$ the space of all forms in $A^{p,q}(X)$ with compact support in X, we can consider, on $D^{p,q}(X)$, the standard complex scalar product $(\cdot, \cdot)_X$ of L^2-type and the complex scalar product

$$(u, v)_{1, X} := (u, v)_X + (\bar{\partial} u, \bar{\partial} v)_X + (\vartheta u, \vartheta v)_X$$

where ∂ and $\bar{\partial}$ are the classical complex differential operators and $\bar{\vartheta}$ and ϑ are their adjoints, respectively. Then if we denote by $W^{p,q}(X)$ the completion of $D^{p,q}(X)$ with respect to the scalar product $(\cdot, \cdot)_{1, X}$, in §4 we will able to prove that on Kähler manifolds the following Hodge-Kodaira decomposition holds:

$$(1.1) \qquad W^{p,q}(X) = [\partial \bar{\partial} D^{p-1,q-1}(X)]_1 \oplus_\perp [\vartheta D^{p,q+1}(X) + \bar{\vartheta} D^{p+1,q}(X)]_1$$
$$\oplus_\perp \operatorname{Ker} \square \cap W^{p,q}(X)$$

where the square brackets with subscript 1 stands for the closure in $W^{p,q}(X)$ and $\oplus_\perp$ says that the direct sum is orthogonal in the sense of the scalar product $(\cdot, \cdot)_{1, X}$. The proof of (1.1), in the absence of compactness, requires an integral inequality of "Stampacchia-type" for a suitable elliptic operator, and such a inequality is the crucial point. Let us briefly recall the history of Stampacchia-type inequalities.

Let X be a complete and connected hermitian manifold. The classical Stampacchia inequality is an integral inequality which involves the complex Laplace operator $\square$; Andreotti and Vesentini proved it in [9] in order to obtain applications to the study of vanishing theorems by means of an extension of a Kodaira theorem ([13]). More precisely if $L^{p,q}(X)$ denotes the completion of $D^{p,q}(X)$ with respect to the scalar product $(\cdot, \cdot)_X$ and if B_r denotes the ball of radius r and centered in a fixed point $0 \in X$, then for any $r, R, \sigma > 0$, with $r < R$, it holds

$$(1.2) \qquad (\bar{\partial} u, \bar{\partial} u)_{B_r} + (\vartheta u, \vartheta u)_{B_r} \le \left(\frac{1}{\sigma} + \frac{c}{(R-r)^2} \right) (u, u)_{B_R} + \sigma (\square u, \square u)_{B_R}$$

for all $u \in A^{p,q}(X)$, where $c > 0$ is a constant which depends only by the complex dimension of X. In particular it descends the following characterization of the square-summable harmonic forms on X:

$$(1.3) \qquad \operatorname{Ker} \square \cap L^{p,q}(X) = \{ u \in A^{p,q}(X) \cap L^{p,q}(X) : \bar{\partial} u = \vartheta u = 0 \}.$$

A real version of inequality (1.2) was proved by Vesentini, with the same technique, in [18]: if M denotes a complete and connected riemannian manifold then for any $r, R, \sigma > 0$, with $r < R$, it holds

$$(1.4) \qquad (du, du)_{B_r} + (\delta u, \delta u)_{B_r} \leq \left(\frac{1}{\sigma} + \frac{c}{(R-r)^2} \right) (u, u)_{B_R} + \sigma (\Delta u, \Delta u)_{B_R}$$

for any $u \in A^p(M)$, where $c > 0$ is a constant which depends only by the dimension of M. The Stampacchia-type inequality (1.4) implies that

$$\operatorname{Ker} \Delta \cap L^p(M) = \{ u \in A^p(M) \cap L^p(M) : du = \delta u = 0 \}$$

from which it follows the Hodge-Kodaira decomposition of $L^p(M)$:

$$(1.5) \quad L^p(M) = [dD^{p-1}(M)]_{L^p(M)} \oplus_{\perp} [\delta D^{p+1}(M)]_{L^p(M)} \oplus_{\perp} \operatorname{Ker} \Delta \cap L^p(M).$$

In this paper we will prove a Stampacchia-type inequality like (1.2) for the fourth-order elliptic operator $\mathscr{D}$ given by

$$\mathscr{D} = \partial \bar{\partial} \vartheta \bar{\vartheta} + \vartheta \bar{\vartheta} \partial \bar{\partial} + \bar{\vartheta} \bar{\partial} \vartheta \partial + \vartheta \partial \bar{\vartheta} \bar{\partial} + \bar{\vartheta} \partial + \vartheta \bar{\partial}$$

which was first considered by Kodaira and Spencer in [15] for the study of the stability of Kähler manifolds under small deformations (see moreover the important book of Morrow and Kodaira [16], Ch. 4, §4). Such a operator $\mathscr{D}$ was also considered by Bigolin [10] in the compact case. More precisely in §3, following the same technique of Andreotti and Vesentini, we will prove that there exist four positive constants c_1, c_2, c_3, c_4, eventually depending only by the complex dimension of X, such that for any $r, R, \sigma > 0$, with $r < R$, it holds

$$(1.6) \quad (\square u, \square u)_{B_r} + (\vartheta \bar{\vartheta} u, \vartheta \bar{\vartheta} u)_{B_r} + (\partial \bar{\partial} u, \partial \bar{\partial} u)_{B_r} + (\vartheta \partial u, \vartheta \partial u)_{B_r} + (\bar{\vartheta} \bar{\partial} u, \bar{\vartheta} \bar{\partial} u)_{B_r}$$

$$+ (\partial u, \partial u)_{B_r} + (\bar{\partial} u, \bar{\partial} u)_{B_r} \leq \left(\frac{c_1}{(R-r)^2} + \frac{c_2}{(R-r)^4} + \frac{1}{\sigma} \right) (u, u)_{B_R}$$

$$+ \frac{c_3}{(R-r)^2} ((\bar{\partial} u, \bar{\partial} u)_{B_R} + (\vartheta u, \vartheta u)_{B_R}) + c_4 \sigma (\mathscr{D} u, \mathscr{D} u)_{B_R}$$

for any $u \in A^{p,q}(X)$. By means of inequality (1.6) we will able to prove the decomposition (1.1) and then, in the last section, we will apply such a decomposition in order to study a relation between the Aeppli cohomology and classical De Rham cohomology.

2. Riemannian and hermitian manifolds

2.1. Riemannian Manifolds

For a thorough treatment of the argument we refer the reader to [12]. Let M be a n-dimensional orientable complete riemannian manifold. Let $g_{\alpha\beta}$ be the metric tensor on M and let $g^{\alpha\beta}$ be the inverse of $g_{\alpha\beta}$; we also denote by $g = \det g_{\alpha\beta}$. For any positive integer p, with $p \leq n$, we will denote by $K^p(M)$ the space of all currents on M of degree p; the subspace $A^p(M)$ will denote the space of all p-differential forms with C^∞-coefficients and real values. In this setting it is well defined the volume form $e_{\alpha_1 \ldots \alpha_n} dx^1 \wedge \cdots \wedge dx^n$. Given $u \in A^p(M)$ the adjoint of u is the form given, in local coordinates, by $*u_{\beta_1 \ldots \beta_{n-p}} = e_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_{n-p}} u^{\alpha_1 \ldots \alpha_p}$. The operator $* : A^p(M) \to A^{n-p}(M)$ can be extended to a unique operator $* : K^p(M) \to K^{n-p}(M)$. On the subspace $D^p(M)$ given by all forms in $A^p(M)$ which have compact support in M the operator $*$ permits us to define the real scalar product given by

$$(u, v)_M := \int_M u \wedge *v.$$

We will denote by $L^p(M)$ the completion of the space $D^p(M)$ with respect to the scalar product $(\cdot, \cdot)_M$. It turns out that $L^p(M)$ is an Hilbert space. Let $d : K^p(M) \to K^{p+1}(M)$ be the exterior differential and let $\delta : K^p(M) \to K^{p-1}(M)$ its formal adjoint, i.e. $\delta = (-1)^{np+n+1} * d*$; it is well known that $d^2 = \delta^2 = 0$. The laplacian of a current $T \in K^p(M)$ is given by $\Delta T = d\delta T + \delta dT$; the currents belong to $\operatorname{Ker}\Delta$ are called harmonic currents, and the forms belong to $\operatorname{Ker}\Delta$ are called harmonic forms. By ellipticity it turns out that if $T \in \operatorname{Ker}\Delta$ then actually $T \in A^p(M)$.

2.2. Hermitian Manifolds

For a thorough treatment of the subject we refer the reader to [16] and [19]. Let X be a complete hermitian manifold of complex dimension n, let $g_{\alpha\beta}$ be the hermitian metric on X, and let $g^{\alpha\beta}$ be its inverse; as in the real case we denote by $g = \det g_{\alpha\beta}$. For any positive integers p, q, with $p, q \leq n$, we will denote by $K^{p,q}(X)$ the space of all currents on X of bidegree (p, q); the subspace $A^{p,q}(X)$ will denote the space of all (p, q)-differential forms with C^∞-coefficients and complex values. Associated to an hermitian metric we have the fundamental real form $\omega = ig_{\alpha\beta} dz^\alpha d\bar{z}^\beta$; X is a Kähler manifold if $d\omega = 0$. Let, in local coordinates, $e_{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_n} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_n} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_n}$ be the volume form on X. Given $u \in A^{p,q}(X)$ the adjoint of u is the form given, in local coordinates, by $*u_{\mu_1 \ldots \mu_{n-q} v_1 \ldots v_{n-p}} = e_{\mu_1 \ldots \mu_{n-q} \alpha_1 \ldots \alpha_q v_1 \ldots v_{n-p} \beta_1 \ldots \beta_p} u^{\alpha_1 \ldots \alpha_q \beta_1 \ldots \beta_p}$. The operator $* : A^{p,q}(X) \to A^{n-q,n-p}(X)$ can be extended to a unique operator $* : K^{p,q}(X) \to K^{n-q,n-p}(X)$. As in the riemannian case on the subspace $D^{p,q}(X)$ given by all forms in

$A^{p,q}(X)$ which have compact support in X the operator $*$ permits us to define a complex scalar product given by

$$(u, v)_X := \int_X u \wedge \overline{*v}.$$

We will denote by $L^{p,q}(X)$ the completion of $D^{p,q}(X)$ with respect to the scalar product $(\cdot, \cdot)_X$. It turns out that $L^{p,q}(X)$ is an Hilbert space. Let $\partial : K^{p,q}(X) \to K^{p+1,q}(X)$ and $\bar{\partial} : K^{p,q}(X) \to K^{p,q+1}(X)$ be the classical complex differential operators. It is well known that $\partial^2 = \bar{\partial}^2 = 0$ and $d = \partial + \bar{\partial}$. The operators $\vartheta : K^{p,q}(X) \to K^{p,q-1}(X)$ and $\bar{\vartheta} : K^{p,q}(X) \to K^{p-1,q}(X)$ can be defined by setting $\vartheta = - * \partial *$ and $\bar{\vartheta} = - * \bar{\partial} *$, and we get $\vartheta^2 = \bar{\vartheta}^2 = 0$. Let us now recall the following useful formulas: If at least one form between u and v belong to $D^{p,q}(X)$ then

$$(2.1) \qquad (\bar{\partial}u, v)_X = (u, \vartheta v)_X \quad \text{and} \quad (\partial u, v)_X = (u, \bar{\vartheta}v)_X;$$

moreover it holds $\partial\bar{\partial} = -\bar{\partial}\partial$ and $\vartheta\bar{\vartheta} = -\bar{\vartheta}\vartheta$. If X is a Kähler manifold then it is well known that

$$(2.2) \qquad \partial\vartheta + \vartheta\partial = 0, \quad \bar{\partial}\bar{\vartheta} + \bar{\vartheta}\bar{\partial} = 0, \quad \bar{\partial}\vartheta + \vartheta\bar{\partial} = \partial\bar{\vartheta} + \bar{\vartheta}\partial.$$

We recall that the complex laplacian $\square : K^{p,q}(X) \to K^{p,q}(X)$ is defined by $\square = \bar{\partial}\bar{\vartheta} + \bar{\vartheta}\bar{\partial}$; the currents belong to $\operatorname{Ker}\square$ are called harmonic currents, and the forms belong to $\operatorname{Ker}\square$ are called harmonic forms. On Kähler manifolds by (2.2) it descends

$$\square = \overline{\square} := \partial\vartheta + \vartheta\partial.$$

By ellipticity it turns out that if $T \in \operatorname{Ker}\square$ then actually $T \in A^{p,q}(X)$. On Kähler manifolds it holds $\square = \frac{1}{2}\Delta$. Finally we will denote by $W^{p,q}(X)$ the Sobolev-type space given by the completion of $D^{p,q}(X)$ with respect to the scalar product

$$(u, u)_{1,X} := (u, u)_X + (\bar{\partial}u, \bar{\partial}u)_X + (\vartheta u, \vartheta u)_X.$$

It turns out that $W^{p,q}(X)$ is an Hilbert space.

3. A STAMPACCHIA-TYPE INEQUALITY FOR THE OPERATOR $\mathscr{D}$

In the rest of the paper X will denote a complete and connected Kähler manifold of complex dimension n. Let $p, q \le n$ be positive integers. Consider the fourth-order operator $\mathscr{D} : K^{p,q}(X) \to K^{p,q}(X)$ given by

$$(3.1) \qquad \mathscr{D} = \vartheta\bar{\partial}\vartheta\bar{\partial} + \bar{\partial}\vartheta\bar{\partial}\vartheta + \bar{\vartheta}\partial + \vartheta\bar{\partial}$$

REMARK 3.1. *An easy application of formulas (2.2) shows that*

$$(3.2) \qquad \Box^2 = \vartheta\bar\partial\vartheta\bar\partial + \bar\partial\vartheta\bar\partial\vartheta$$

and

$$(3.3) \qquad \vartheta\bar\partial\vartheta\bar\partial + \bar\partial\vartheta\bar\partial\vartheta = \partial\bar\partial\vartheta\bar\vartheta + \vartheta\bar\vartheta\partial\bar\partial + \bar\vartheta\bar\partial\vartheta\partial + \vartheta\partial\bar\vartheta\bar\partial.$$

In [15] Kodaira and Spencer show that $\mathscr{D}$ (they used, for the principal part of $\mathscr{D}$, the form given by the right-hand side of (3.3)) is an elliptic operator, since its principal part is given by

$$\sum_{\alpha\beta\gamma\delta} g^{\beta\alpha} g^{\delta\gamma} \frac{\partial^4}{\partial z^\alpha \partial \bar z^\beta \partial z^\gamma \partial \bar z^\delta}$$

in any local coordinates system. For any $u \in A^{p,q}(X)$ let

$$(u,u)_{2,X} := (\vartheta\bar\vartheta u, \vartheta\bar\vartheta)_X + (\partial\bar\partial u, \partial\bar\partial u)_X + (\vartheta\partial u, \vartheta\partial u)_X + (\bar\vartheta\bar\partial u, \bar\vartheta\bar\partial u)_X.$$

Let $0 \in X$ be a fixed point; for any $r > 0$ we will denote by B_r the ball centered in 0 with radius r. For the sake of simplicity we will use the notation $(\cdot,\cdot)_r$ and $(\cdot,\cdot)_{2,r}$ respectively for the quantities $(\cdot,\cdot)_{B_r}$ and $(\cdot,\cdot)_{2,B_r}$. Notice that the completeness of X ensures that the generic ball B_r is relatively compact in X, by Hopf-Rinow theorem; in particular all quantities $(u,u)_r$ and $(u,u)_{2,r}$ are finite. The fundamental result of this section is an integral inequality of Stampacchia-type for the operator $\mathscr{D}$.

THEOREM 3.2 (Stampacchia-type inequality). *For every $R, r, \sigma > 0$ with $r < R$ it holds*

$$(3.4) \qquad (\Box u, \Box u)_r + (u,u)_{2,r} + (\partial u, \partial u)_r + (\bar\partial u, \bar\partial u)_r$$

$$\leq \left(\frac{c_1}{(R-r)^4} + \frac{c_2}{(R-r)^2} + \frac{1}{\sigma} \right)(u,u)_R$$

$$+ \frac{c_3}{(R-r)^2}\left((\bar\partial u, \bar\partial u)_R + (\vartheta u, \vartheta u)_R\right) + c_4\sigma(\mathscr{D}u, \mathscr{D}u)_R$$

for any $u \in A^{p,q}(X)$, with c_1, c_2, c_3, c_4 positive constants eventually depending only by the complex dimension n.

PROOF. Using the same argument of Lemma 6 in [9] we can construct a function $\varphi : X \to [0,1]$ with $\varphi = 1$ on B_r, $\varphi = 0$ on $X \setminus B_R$ such that there exist two positive constants M_1 and M_2, depending only by n, with

$$(3.5) \qquad (L\varphi \wedge u, L\varphi \wedge u)_R \le \frac{M_1}{(R-r)^2}(u,u)_R,$$

$$(N\varphi \wedge u, N\varphi \wedge u)_R \le \frac{M_2}{(R-r)^4}(u,u)_R$$

for any $u \in \mathrm{A}^{p,q}(X)$, whenever $L \in \{\partial, \bar{\partial}, \vartheta, \bar{\vartheta}\}$ and $N \in \{\partial\bar{\partial}, \vartheta\bar{\vartheta}, \vartheta\partial, \bar{\vartheta}\bar{\partial}\}$. Let $u \in \mathrm{A}^{p,q}(X)$; then $\varphi^m u$ has support in B_R for any positive integer m. Now we divide the proof in two steps; first we collect some useful estimates for the first and the second order terms that appears in $\mathscr{D}$, and then we will prove (3.4).

Step 1. Let us consider the first order terms. We have

$$\partial(\varphi^4 u) = 4\varphi^3 \partial\varphi \wedge u + \varphi^4 \partial u$$

and then

$$(\partial u, \partial(\varphi^4 u))_R = (\varphi^2 \partial u, 4\varphi\partial\varphi \wedge u + \varphi^2 \partial u)_R = 4(\varphi^2 \partial u, \varphi\partial\varphi \wedge u)_R + (\varphi^2 \partial u, \varphi^2 \partial u)_R.$$

Taking into account formulas (2.1) we deduce that

$$(3.6) \qquad (\varphi^2 \partial u, \varphi^2 \partial u)_R = (\bar{\vartheta}\partial u, \varphi^4 u)_R - 4(\varphi^2 \partial u, \varphi\partial\varphi \wedge u)_R.$$

By applying the same argument we get

$$(3.7) \qquad (\varphi^2 \bar{\partial} u, \varphi^2 \bar{\partial} u)_R = (\vartheta\bar{\partial} u, \varphi^4 u)_R - 4(\varphi^2 \bar{\partial} u, \varphi\bar{\partial}\varphi \wedge u)_R.$$

Let us now consider the second order terms. We easily have

$$\begin{aligned}
\vartheta\bar{\partial}(\varphi^4 u) &= \vartheta(2\varphi^3 \bar{\partial}\varphi \wedge u + \varphi^2 \bar{\partial}(\varphi^2 u)) \\
&= 6\varphi^2 \vartheta\varphi \wedge \bar{\partial}\varphi \wedge u + 2\varphi^3 \vartheta\bar{\partial}\varphi \wedge u + (-1)^{p+q}2\varphi^3 \bar{\partial}\varphi \wedge \vartheta u \\
&\quad + 2\varphi\vartheta\varphi \wedge \bar{\partial}(\varphi^2 u) + \varphi^2 \vartheta\bar{\partial}(\varphi^2 u) \\
&= 10\varphi^2 \vartheta\varphi \wedge \bar{\partial}\varphi \wedge u + 2\varphi^3 \vartheta\bar{\partial}\varphi \wedge u + (-1)^{p+q}2\varphi^3 \bar{\partial}\varphi \wedge \vartheta u \\
&\quad + 2\varphi^3 \vartheta\varphi \wedge \bar{\partial}u + \varphi^2 \vartheta\bar{\partial}(\varphi^2 u)
\end{aligned}$$

from which we obtain

$$\begin{aligned}
(\vartheta\bar{\partial}u, \vartheta\bar{\partial}(\varphi^4 u))_R &= (\varphi^2 \vartheta\bar{\partial}u, 10\vartheta\varphi \wedge \bar{\partial}\varphi \wedge u + 2\varphi\vartheta\bar{\partial}\varphi \wedge u + (-1)^{p+q}2\varphi\bar{\partial}\varphi \wedge \vartheta u \\
&\quad + 2\varphi\vartheta\varphi \wedge \bar{\partial}u + \vartheta\bar{\partial}(\varphi^2 u))_R \\
&= (\vartheta\bar{\partial}(\varphi^2 u) - 2\vartheta\varphi \wedge \bar{\partial}\varphi \wedge u - 2\varphi\vartheta\bar{\partial}\varphi \wedge u - (-1)^{p+q}2\varphi\bar{\partial}\varphi \wedge \vartheta u \\
&\quad - 2\varphi\vartheta\varphi \wedge \bar{\partial}u, 10\vartheta\varphi \wedge \bar{\partial}\varphi \wedge u + 2\varphi\vartheta\bar{\partial}\varphi \wedge u + (-1)^{p+q}2\varphi\bar{\partial}\varphi \wedge \vartheta u \\
&\quad + 2\varphi\vartheta\varphi \wedge \bar{\partial}u + \vartheta\bar{\partial}(\varphi^2 u))_R.
\end{aligned}$$

Then taking into account (2.1) we get

$$(3.8) \quad (\vartheta\bar{\partial}(\varphi^2 u), \vartheta\bar{\partial}(\varphi^2 u))_R$$

$$= (\vartheta\bar{\partial}\vartheta\bar{\partial}u, \varphi^4 u)_R - 10(\vartheta\bar{\partial}(\varphi^2 u), \vartheta\varphi \wedge \bar{\partial}\varphi \wedge u)_R$$

$$+ 20(\vartheta\varphi \wedge \bar{\partial}\varphi \wedge u, \vartheta\varphi \wedge \bar{\partial}\varphi \wedge u)_R + 20(\varphi\vartheta\bar{\partial}\varphi \wedge u, \vartheta\varphi \wedge \bar{\partial}\varphi \wedge u)_R$$

$$+ 20(-1)^{p+q}(\varphi\bar{\partial}\varphi \wedge \vartheta u, \vartheta\varphi \wedge \bar{\partial}\varphi \wedge u)_R + 20(\varphi\vartheta\varphi \wedge \bar{\partial}u, \vartheta\varphi \wedge \bar{\partial}\varphi \wedge u)_R$$

$$- 2(\vartheta\bar{\partial}(\varphi^2 u), \varphi\vartheta\bar{\partial}\varphi \wedge u)_R + 4(\vartheta\varphi \wedge \bar{\partial}\varphi \wedge u, \varphi\vartheta\bar{\partial}\varphi \wedge u)_R$$

$$+ 4(\varphi\vartheta\bar{\partial}\varphi \wedge u, \varphi\vartheta\bar{\partial}\varphi \wedge u)_R + 2(-1)^{p+q}(\varphi\bar{\partial}\varphi \wedge \vartheta u, \varphi\vartheta\bar{\partial}\varphi \wedge u)_R$$

$$+ 4(\varphi\vartheta\varphi \wedge \bar{\partial}u, \varphi\vartheta\bar{\partial}\varphi \wedge u)_R - 2(-1)^{p+q}(\vartheta\bar{\partial}(\varphi^2 u), \varphi\bar{\partial}\varphi \wedge \vartheta u)_R$$

$$+ 4(-1)^{p+q}(\vartheta\varphi \wedge \bar{\partial}\varphi \wedge u, \varphi\bar{\partial}\varphi \wedge \vartheta u)_R$$

$$+ 4(-1)^{p+q}(\varphi\vartheta\bar{\partial}\varphi \wedge u, \varphi\bar{\partial}\varphi \wedge \vartheta u)_R + 4(\varphi\bar{\partial}\varphi \wedge \vartheta u, \varphi\bar{\partial}\varphi \wedge \vartheta u)_R$$

$$+ 4(-1)^{p+q}(\varphi\vartheta\varphi \wedge \bar{\partial}u, \varphi\bar{\partial}\varphi \wedge \vartheta u)_R - 2(\vartheta\bar{\partial}(\varphi^2 u), \varphi\vartheta\varphi \wedge \bar{\partial}u)_R$$

$$+ 4(\vartheta\varphi \wedge \bar{\partial}\varphi \wedge u, \varphi\vartheta\varphi \wedge \bar{\partial}u)_R + 4(\varphi\vartheta\bar{\partial}\varphi \wedge u, \varphi\vartheta\varphi \wedge \bar{\partial}u)_R$$

$$+ 4(-1)^{p+q}(\varphi\bar{\partial}\varphi \wedge \vartheta u, \varphi\vartheta\varphi \wedge \bar{\partial}u)_R + 4(\varphi\vartheta\varphi \wedge \bar{\partial}u, \varphi\vartheta\varphi \wedge \bar{\partial}u)_R$$

$$+ 2(\vartheta\varphi \wedge \bar{\partial}\varphi \wedge u, \vartheta\bar{\partial}(\varphi^2 u))_R + 2(\varphi\vartheta\bar{\partial}\varphi \wedge u, \vartheta\bar{\partial}(\varphi^2 u))_R$$

$$+ 2(-1)^{p+q}(\varphi\bar{\partial}\varphi \wedge \vartheta u, \vartheta\bar{\partial}(\varphi^2 u))_R + 2(\varphi\vartheta\varphi \wedge \bar{\partial}u, \vartheta\bar{\partial}(\varphi^2 u))_R.$$

After the same computation we can obtain a similar identity for the term $(\bar{\partial}\vartheta(\varphi^2 u), \bar{\partial}\vartheta(\varphi^2 u))_R$.

Step 2. Now we will prove (3.4). By taking the sum of (3.6), (3.7), (3.8) and the similar identity for the term $(\bar{\partial}\vartheta(\varphi^2 u), \bar{\partial}\vartheta(\varphi^2 u))_R$, taking into account the very definition of $\mathscr{D}$, Young inequality and (3.5) we easily obtain

$$(3.9) \quad (\vartheta\bar{\partial}(\varphi^2 u), \vartheta\bar{\partial}(\varphi^2 u))_R + (\bar{\partial}\vartheta(\varphi^2 u), \bar{\partial}\vartheta(\varphi^2 u))_R$$

$$+ (\varphi^2 \partial u, \varphi^2 \partial u)_R + (\varphi^2 \bar{\partial}u, \varphi^2 \bar{\partial}u)_R$$

$$\leq |(\mathscr{D}u, \varphi^4 u)_R| + \frac{1}{2}[(\vartheta\bar{\partial}(\varphi^2 u), \vartheta\bar{\partial}(\varphi^2 u))_R + (\bar{\partial}\vartheta(\varphi^2 u), \bar{\partial}\vartheta(\varphi^2 u))_R$$

$$+ (\varphi^2 \partial u, \varphi^2 \partial u)_R + (\varphi^2 \bar{\partial}u, \varphi^2 \bar{\partial}u)_R] + \left(\frac{\alpha}{(R-r)^2} + \frac{\beta}{(R-r)^4}\right)(u, u)_R$$

$$+ \frac{\gamma}{(R-r)^2}((\bar{\partial}u, \bar{\partial}u)_R + (\vartheta u, \vartheta u)_R)$$

for some positive constants α, β, γ depending only on the complex dimension n. Then

$$(\vartheta\bar{\partial}(\varphi^2 u), \vartheta\bar{\partial}(\varphi^2 u))_R + (\bar{\partial}\vartheta(\varphi^2 u), \bar{\partial}\vartheta(\varphi^2 u))_R + (\varphi^2\partial u, \varphi^2\partial u)_R + (\varphi^2\bar{\partial}u, \varphi^2\bar{\partial}u)_R$$

$$\leq 2|(\mathscr{D}u, \varphi^4 u)_R| + \left(\frac{2\alpha}{(R-r)^2} + \frac{2\beta}{(R-r)^4}\right)(u, u)_R$$

$$+ \frac{2\gamma}{(R-r)^2}((\bar{\partial}u, \bar{\partial}u)_R + (\vartheta u, \vartheta u)_R).$$

Now observe that

$$(\vartheta\bar{\partial}(\varphi^2 u), \vartheta\bar{\partial}(\varphi^2 u))_R + (\bar{\partial}\vartheta(\varphi^2 u), \bar{\partial}\vartheta(\varphi^2 u))_R = (\Box(\varphi^2 u), \Box(\varphi^2 u))_R$$

and, at the same time, applying (3.3),

$$(\vartheta\bar{\partial}(\varphi^2 u), \vartheta\bar{\partial}(\varphi^2 u))_R + (\bar{\partial}\vartheta(\varphi^2 u), \bar{\partial}\vartheta(\varphi^2 u))_R = (\varphi^2 u, \varphi^2 u)_{2,R}.$$

Thus, since $\varphi = 1$ on B_r, we deduce that

$$(\Box u, \Box u)_r + (u, u)_{2,r} + (\partial u, \partial u)_r + (\bar{\partial}u, \bar{\partial}u)_r$$

$$\leq \left(\frac{4\alpha}{(R-r)^2} + \frac{4\beta}{(R-r)^4}\right)(u, u)_R$$

$$+ \frac{4\gamma}{(R-r)^2}((\bar{\partial}u, \bar{\partial}u)_R + (\vartheta u, \vartheta u)_R) + 4|(\mathscr{D}u, \varphi^4 u)_R|.$$

Finally, applying again Young inequality, we obtain, for any $\eta > 0$,

$$(\Box u, \Box u)_r + (u, u)_{2,r} + (\partial u, \partial u)_r + (\bar{\partial}u, \bar{\partial}u)_r$$

$$\leq \left(\frac{4\alpha}{(R-r)^2} + \frac{4\beta}{(R-r)^4} + \frac{4}{\eta}\right)(u, u)_R$$

$$+ \frac{4\gamma}{(R-r)^2}((\bar{\partial}u, \bar{\partial}u)_R + (\vartheta u, \vartheta u)_R) + 4\eta|(\mathscr{D}u, \mathscr{D}u)_R|$$

which is, up to constants, inequality (3.4). $\qquad\square$

4. A HODGE-KODAIRA DECOMPOSITION FOR THE SPACE $W^{p,q}(X)$

This section is devoted to the proof of a Hodge-Kodaira orthogonal decomposition for the space $W^{p,q}(X)$.

PROPOSITION 4.1. *It holds*

(4.1) $\operatorname{Ker}\square \cap \mathrm{W}^{p,q}(X) = \operatorname{Ker}\mathscr{D} \cap \mathrm{W}^{p,q}(X)$
$$= \{u \in \mathrm{A}^{p,q}(X) \cap \mathrm{W}^{p,q}(X) : \vartheta\bar\vartheta u = \partial u = \bar\partial u = 0\}.$$

PROOF. Let $u \in \mathrm{W}^{p,q}(X)$ with $\mathscr{D}u = 0$. Then inequality (3.4) implies that

$$(\vartheta\bar\vartheta u, \vartheta\bar\vartheta u)_r + (\partial u, \partial u)_r + (\bar\partial u, \bar\partial u)_r \le \left(\frac{c_1}{(R-r)^4} + \frac{c_2}{(R-r)^2} + \frac{1}{\sigma}\right)(u,u)_X$$
$$+ \frac{c_3}{(R-r)^2}\left((\bar\partial u, \bar\partial u)_X + (\vartheta u, \vartheta u)_X\right)$$

for any $R, r, \sigma > 0$ with $r < R$. Observe that since X is connected we get

$$(\vartheta\bar\vartheta u, \vartheta\bar\vartheta u)_r + (\partial u, \partial u)_r + (\bar\partial u, \bar\partial u)_r \to (\vartheta\bar\vartheta u, \vartheta\bar\vartheta u)_X + (\partial u, \partial u)_X + (\bar\partial u, \bar\partial u)_X$$

as $r \to +\infty$. Choosing $r = R/2$ and by taking the lim sup as $R, \sigma \to +\infty$ we deduce that $(\vartheta\bar\vartheta u, \vartheta\bar\vartheta u)_X = (\partial u, \partial u)_X = (\bar\partial u, \bar\partial u)_X = 0$. Then $\vartheta\bar\vartheta u = \partial u = \bar\partial u = 0$. Conversely if $u \in \mathrm{A}^{p,q}(X)$ and if $\vartheta\bar\vartheta u = \partial u = \bar\partial u = 0$ then recalling (3.3) we immediately have $\mathscr{D}u = 0$. Then

$$\operatorname{Ker}\mathscr{D} \cap \mathrm{W}^{p,q}(X) = \{u \in \mathrm{A}^{p,q}(X) \cap \mathrm{W}^{p,q}(X) : \vartheta\bar\vartheta u = \partial u = \bar\partial u = 0\}.$$

Now if $u \in \mathrm{A}^{p,q}(X) \cap \mathrm{W}^{p,q}(X)$ and $\square u = 0$ then applying (1.2) and the same for $\bar\square$ we get $\partial u = \bar\partial u = \vartheta u = \bar\vartheta u = 0$, and thus $\mathscr{D}u = 0$. Conversely if $u \in \mathrm{A}^{p,q}(X) \cap \mathrm{W}^{p,q}(X)$ and $\mathscr{D}u = 0$ then by (3.4) we have

$$(\square u, \square u)_r \le \left(\frac{c_1}{(R-r)^4} + \frac{c_2}{(R-r)^2} + \frac{1}{\sigma}\right)(u,u)_X$$
$$+ \frac{c_3}{(R-r)^2}\left((\bar\partial u, \bar\partial u)_X + (\vartheta u, \vartheta u)_X\right).$$

Reasoning as before we conclude. □

LEMMA 4.2. *If at least one form between u and v belong to $\mathrm{D}^{p,q}(X)$ then*

(4.2) $(\partial u, v)_{1,X} = (u, \bar\vartheta v)_{1,X}$ *and* $(\bar\partial u, v)_{1,X} = (u, \vartheta v)_{1,X}.$

PROOF. By direct computation we have, since (2.1) and (2.2) hold,

$$(\partial u, v)_{1,X} = (\partial u, v)_X + (\bar\partial\partial u, \bar\partial v)_X + (\vartheta\partial u, \vartheta v)_X$$
$$= (u, \bar\vartheta v)_X + (\vartheta\bar\partial\partial u, v)_X + (\bar\partial\vartheta\partial u, v)_X$$
$$= (u, \bar\vartheta v)_X + ((\vartheta\bar\partial + \bar\partial\vartheta)\partial u, v)_X = (u, \bar\vartheta v)_X + ((\partial\bar\vartheta + \bar\vartheta\partial)\partial u, v)_X$$
$$= (u, \bar\vartheta v)_X + (\partial\bar\vartheta\partial u, v)_X = (u, \bar\vartheta v)_X + (\bar\vartheta\partial u, \bar\vartheta v)_X$$

and

$$(u, \bar{\vartheta}v)_{1,X} = (u, \bar{\vartheta}v)_X + (\bar{\partial}u, \bar{\partial}\bar{\vartheta}v)_X + (\vartheta u, \vartheta\bar{\vartheta}v)_X$$
$$= (u, \bar{\vartheta}v)_X + (\vartheta\bar{\partial}u, \bar{\vartheta}v)_X + (\bar{\partial}\vartheta u, \bar{\vartheta}v)_X = (u, \bar{\vartheta}v)_X + (\vartheta\bar{\partial}u + \bar{\partial}u, \bar{\vartheta}v)_X$$
$$= (u, \bar{\vartheta}v)_X + (\partial\bar{\vartheta}u + \bar{\vartheta}\partial u, \bar{\vartheta}v)_X = (u, \bar{\vartheta}v)_X + (\bar{\vartheta}\partial u, \bar{\vartheta}v)_X.$$

The other one is similar. $\qquad\qquad\square$

THEOREM 4.3 (Hodge-Kodaira decomposition). *The following Hodge-Kodaira orthogonal decomposition holds:*

$$(4.3) \qquad \mathrm{W}^{p,q}(X) = [\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1 \oplus_\perp [\vartheta\mathrm{D}^{p,q+1}(X) + \bar{\vartheta}\mathrm{D}^{p+1,q}(X)]_1$$
$$\oplus_\perp \mathrm{Ker}\,\square \cap \mathrm{W}^{p,q}(X).$$

PROOF. Taking into account (4.1) it is sufficient to show that

$$\mathrm{W}^{p,q}(X) = [\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1 \oplus_\perp [\vartheta\mathrm{D}^{p,q+1}(X) + \bar{\vartheta}\mathrm{D}^{p+1,q}(X)]_1$$
$$\oplus_\perp \{u \in \mathrm{A}^{p,q}(X) \cap \mathrm{W}^{p,q}(X) : \vartheta\bar{\vartheta}u = \partial u = \bar{\partial}u = 0\}.$$

STEP 1. First we prove that the subspaces

$$[\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1 \quad \text{and} \quad [\vartheta\mathrm{D}^{p,q+1}(X) + \bar{\vartheta}\mathrm{D}^{p+1,q}(X)]_1$$

are orthogonal in the space $\mathrm{W}^{p,q}(X)$. Let $u = \partial\bar{\partial}\tilde{u}$ for some $\tilde{u} \in \mathrm{D}^{p-1,q-1}(X)$ and let $v = \vartheta\tilde{v}_1 + \bar{\vartheta}\tilde{v}_2$ for some $\tilde{v}_1 \in \mathrm{D}^{p,q+1}(X)$ and $\tilde{v}_2 \in \mathrm{D}^{p+1,q}(X)$. Then taking into account (4.2) we get

$$(u, v)_{1,X} = (\partial\bar{\partial}\tilde{u}, \vartheta\tilde{v}_1)_{1,X} + (\partial\bar{\partial}\tilde{u}, \bar{\vartheta}\tilde{v}_2)_{1,X} = -(\bar{\partial}\partial\tilde{u}, \vartheta\tilde{v}_1)_{1,X} + (\partial\bar{\partial}\tilde{u}, \bar{\vartheta}\tilde{v}_2)_{1,X}$$
$$= -(\partial\tilde{u}, \vartheta^2\tilde{v}_1)_{1,X} + (\bar{\partial}\tilde{u}, \bar{\vartheta}^2\tilde{v}_2)_{1,X} = 0.$$

Passing to the closures in $\mathrm{W}^{p,q}(X)$ we conclude.

STEP 2. Taking into account *Step 1* and applying the projection theorem in an Hilbert space we obtain the orthogonal decomposition

$$\mathrm{W}^{p,q}(X) = [\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1 \oplus_\perp [\vartheta\mathrm{D}^{p,q+1}(X) + \bar{\vartheta}\mathrm{D}^{p+1,q}(X)]_1$$
$$\oplus_\perp [\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1^\perp \cap [\vartheta\mathrm{D}^{p,q+1}(X) + \bar{\vartheta}\mathrm{D}^{p+1,q}(X)]_1^\perp.$$

Using the same argument as before we easily get

$$\{u \in \mathrm{A}^{p,q}(X) \cap \mathrm{W}^{p,q}(X) : \vartheta\bar{\vartheta}u = \partial u = \bar{\partial}u = 0\}$$
$$\subseteq [\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1^\perp \cap [\vartheta\mathrm{D}^{p,q+1}(X) + \bar{\vartheta}\mathrm{D}^{p+1,q}(X)]_1^\perp.$$

Now if $u \in [\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1^\perp \cap [\vartheta\mathrm{D}^{p,q+1}(X) + \bar{\vartheta}\mathrm{D}^{p+1,q}(X)]_1^\perp$ then for each $v \in \mathrm{D}^{p-1,q-1}(X)$, $w \in \mathrm{D}^{p,q+1}(X)$ and $z \in \mathrm{D}^{p+1,q}(X)$ we have

(4.4) $$(\partial\bar{\partial}v, u)_{1,X} = (\vartheta w + \bar{\vartheta}z, u)_{1,X} = 0.$$

Let $\bar{u} \in D^{p,q}(X)$; then considering (3.3) we have

$$(\mathscr{D}\bar{u}, u)_{1,X} = (\partial\bar{\partial}\omega_1, u)_{1,X} + (\vartheta\omega_2 + \bar{\vartheta}\omega_3, u)_{1,X}$$

where

$$\omega_1 = \vartheta\bar{\vartheta}\bar{u} \in D^{p-1,q-1}(X), \quad \omega_2 = \bar{\partial}\vartheta\partial\bar{u} \in D^{p,q+1}(X),$$
$$\omega_3 = \bar{\vartheta}\partial\bar{\partial}\bar{u} + \partial\bar{\vartheta}\partial\bar{u} \in D^{p+1,q}(X)$$

and thus from (4.4) we deduce that $(\mathscr{D}\bar{u}, u)_X = 0$. Then u is a weak solution of the equation $\mathscr{D}u = 0$; since $\mathscr{D}$ is an elliptic operator we get $u \in A^{p,q}(X)$. By (4.2) we finally obtain

$$(v, \vartheta\bar{\vartheta}u)_{1,X} = (w, \bar{\partial}u)_{1,X} = (z, \partial u)_{1,X} = 0$$

for all $v \in D^{p-1,q-1}(X)$, $w \in D^{p,q+1}(X)$ and $z \in D^{p+1,q}(X)$. Therefore $\vartheta\bar{\vartheta}u = \partial u = \bar{\partial}u = 0$, and this concludes the proof of (4.3). $\qquad\square$

5. APPLICATIONS TO THE STUDY OF AEPPLI GROUPS

Let $p, q \geq 1$ integers. As recalled in the Introduction, the Aeppli groups $\Lambda^{p,q}$ were originally defined by

$$\Lambda^{p,q} = \frac{\mathrm{Ker}\{A^{p,q}(X) \xrightarrow{d} A^{p+1,q}(X) \oplus A^{p,q+1}(X)\}}{\partial\bar{\partial}A^{p-1,q-1}(X)}.$$

Bigolin, in [11], proves, using certain resolutions of the sheaf of germs of $\partial\bar{\partial}$-closed functions, that there exists an algebraic isomorphism between $\Lambda^{p,q}$ and

$$\tilde{\Lambda}^{p,q} := \frac{\mathrm{Ker}\{K^{p,q}(X) \xrightarrow{d} K^{p+1,q}(X) \oplus K^{p,q+1}(X)\}}{\partial\bar{\partial}K^{p-1,q-1}(X)}.$$

First we prove the following lemma.

LEMMA 5.1. *The natural map*

$$\frac{[\mathrm{Ker}\{D^{p,q}(X) \xrightarrow{d} D^{p+1,q}(X) \oplus D^{p,q+1}(X)\}]_1}{[\partial\bar{\partial}D^{p-1,q-1}(X)]_1}$$

$$\rightarrow \frac{[\mathrm{Ker}\{D^{p+q}(X) \xrightarrow{d} D^{p+q+1}(X)\}]_{L^{p+q}}}{[dD^{p+q-1}(X)]_{L^{p+q}}}.$$

is injective.

PROOF. Using the same argument of the proof of theorem 4.3 it is possible to show that

$$[\text{Ker}\{D^{p,q}(X) \xrightarrow{d} D^{p+1,q}(X) \oplus D^{p,q+1}(X)\}]_1 = [\vartheta D^{p,q+1}(X) + \bar\vartheta D^{p+1,q}(X)]_1^\perp.$$

Taking into account the Hodge-Kodaira decomposition (4.3) we deduce that

$$\frac{[\text{Ker}\{D^{p,q}(X) \xrightarrow{d} D^{p+1,q}(X) \oplus D^{p,q+1}(X)\}]_1}{[\partial\bar\partial D^{p-1,q-1}(X)]_1} \quad \text{and} \quad \text{Ker}\,\Box \cap W^{p,q}(X)$$

are isomorphic. Now $\text{Ker}\,\Box \cap W^{p,q}(X) \subseteq \text{Ker}\,\Delta \cap L^{p+q}(X)$. Since

$$[\text{Ker}\{D^{p+q}(X) \xrightarrow{d} D^{p+q+1}(X)\}]_{L^{p+q}(X)} = [\delta D^{p+1}(X)]_{L^p(X)}^\perp$$

then, by the classical Hodge-Kodaira decomposition (1.5),

$$\text{Ker}\,\Delta \cap L^{p+q}(X) \quad \text{and} \quad \frac{[\text{Ker}\{D^{p+q}(X) \xrightarrow{d} D^{p+q+1}(X)\}]_{L^{p+q}(X)}}{[dD^{p+q-1}(X)]_{L^{p+q}(X)}}$$

are isomorphic, and this concludes the proof. $\qquad\Box$

In order to prove the main theorem of this section, i.e. a characterization of the Aeppli groups $\Lambda^{p,q}(X)$, we have to assume a technical topological condition on the manifold X. More precisely we will assume that

(5.1) $\partial\bar\partial K^{p-1,q-1}(X)$ is weakly closed in $K^{p,q}(X)$

where the weak topology on $K^{p,q}(X)$ is the usual weak topology of distributions (recall that $K^{p,q}(X)$ is the dual space of $D^{p,q}(X)$). It is well known that compact manifolds and Stein manifolds are examples of manifolds satisfying condition (5.1), so that our result extends the results contained in [1] and [10]. Moreover we point out that condition (5.1) is a necessary condition in order to prove only the next theorem: all the rest of the paper holds independently from this assumption; in particular the Stampacchia-type inequality (3.4) and the Hodge-Kodaira decomposition (4.3) hold for any connected and complete Kähler manifold.

THEOREM 5.2. *Let us assume* (5.1). *Then the Aeppli group* $\Lambda^{p,q}(X)$ *is isomorphic to the group* $H^{p,q}(X)$, *where we recall that* $H^{p,q}(X)$ *denotes the space of all global harmonic* $(p+q)$-*forms of bidegree* (p,q).

PROOF. Since

$$H^{p+q}(X) \simeq \frac{\text{Ker}\{K^{p+q}(X) \xrightarrow{d} K^{p+q+1}(X)\}}{dK^{p+q-1}(X)}$$

and since the image of the natural map

$$i : \frac{\mathrm{Ker}\{\mathrm{K}^{p,q}(X) \xrightarrow{d} \mathrm{K}^{p+1,q}(X) \oplus \mathrm{K}^{p,q+1}(X)\}}{\partial\bar{\partial}\mathrm{K}^{p-1,q-1}(X)} \to \frac{\mathrm{Ker}\{\mathrm{K}^{p+q}(X) \xrightarrow{d} \mathrm{K}^{p+q+1}(X)\}}{d\mathrm{K}^{p+q-1}(X)}$$

is exactly $\mathrm{H}^{p,q}(X)$, then it is sufficient to show that i is injective. Let $T \in \mathrm{K}^{p,q}(X)$ with $T = dS$ for some $S \in \mathrm{K}^{p+q-1}(X)$. Then we have to show that there exists $R \in \mathrm{K}^{p-1,q-1}(X)$ such that $T = \partial\bar{\partial}R$. Since $\mathrm{D}^{p+q-1}(X)$ is dense in $\mathrm{K}^{p+q-1}(X)$ then there exists a sequence $(S_h)_{h\in\mathbb{N}} \subseteq \mathrm{D}^{p+q-1}(X)$ with $S_h \to S$ as $h \to +\infty$. Then $dS_h \to T$ and we can suppose, without loss of generality, that $dS_h \in \mathrm{D}^{p,q}(X)$. Let $T_h = dS_h$. Taking into account lemma 5.1 we get

$$T_h \in [\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1$$

so that

$$T_h = \lim_{k\to+\infty} T_h^k$$

with $T_h^k = \partial\bar{\partial}U_h^k$ for some $U_h^k \in \mathrm{D}^{p-1,q-1}(X)$. Since we are assuming $\partial\bar{\partial}\mathrm{K}^{p-1,q-1}(X)$ weakly closed in $\mathrm{K}^{p,q}(X)$ then

$$T_h = \partial\bar{\partial}R_h$$

for some $R_h \in \mathrm{K}^{p-1,q-1}(X)$, and then $T = \partial\bar{\partial}R$ for some $R \in \mathrm{K}^{p-1,q-1}(X)$, which ends the proof. □

REMARK 5.3. *One can repeat all the considerations on the operator*

$$\mathscr{D}^* = \vartheta\bar{\partial}\vartheta\bar{\partial} + \bar{\partial}\vartheta\bar{\partial}\vartheta + \bar{\partial}\vartheta + \partial\bar{\vartheta}$$

and in particular we get the Hodge-Kodaira orthogonal decomposition

$$\mathrm{W}^{p,q}(X) = [\vartheta\bar{\vartheta}\mathrm{D}^{p+1,q+1}(X)]_1 \oplus_\perp [\partial\mathrm{D}^{p-1,q}(X) + \bar{\partial}\mathrm{D}^{p,q-1}(X)]_1$$
$$\oplus_\perp \mathrm{Ker}\,\square \cap \mathrm{W}^{p,q}(X)$$

which permits to study the Aeppli groups $\mathrm{V}^{p,q}(X)$ *reasoning as in lemma* 5.1 *and in theorem* 5.2.

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