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Calculus of Variations — *Regularity results for minimizers of integral functionals with nonstandard growth in Carnot-Carathéodory spaces*, by Flavia Giannetti and Antonia Passarelli di Napoli, presented on 15 January 2010 by Carlo Sbordone, communicated on 15 January 2010.

Abstract. — We prove regularity results for minimizers of integral functionals of the type

$$\int_\Omega f(Xu)\,dx$$

where f satisfies a nonstandard growth condition and Xu stands for the horizontal gradient of u. More precisely, we obtain regularity in the scale of Campanato spaces without assuming any restriction on the growth exponents and, under a suitable assumption on them, we get the local boundedness as well as an higher integrability result for the gradient.

Key words: Nonstandard growth conditions, Carnot Carathéodory spaces, regularity.

Mathematics Subject Classification AMS: 49N60, 49N99.

1. Introduction

Let Ω be a bounded subset in $\mathbb{R}^n$ and $X = (X_1, \dots, X_k)$ be a family of vector fields defined in a neighbourhood of Ω, with real, C^∞ smooth and globally Lipschitz coefficients satisfying the Hörmander condition. For $u : \Omega \to \mathbb{R}$, we consider the integral functional

$$\mathscr{F}(u) = \int_\Omega F(Xu)\,dx \tag{1.1}$$

where the integrand $F : \mathbb{R}^k \to \mathbb{R}$ is a continuous function satisfying

$$c\{A(|\xi|) - 1\} \le F(\xi) \le C\{A(|\xi|) + 1\} \tag{1.2}$$

$$|F(\xi + \eta)| \le C_0[F(\xi) + F(\eta)] \tag{1.3}$$

where $A : [0, \infty) \to [0, \infty)$ is an N-function, that is A is a continuous, strictly increasing and convex function satisfying

$$A(0) = 0 \quad \lim_{t\to 0} \frac{A(t)}{t} = 0 \quad \lim_{t\to\infty} \frac{A(t)}{t} = +\infty \tag{1.4}$$

We shall assume that there exist $1 < p \le q$ such that

$$\frac{A(t)}{t^p} \nearrow \quad \frac{A(t)}{t^q} \searrow \tag{1.5}$$

Definition 1.1. *A function $u \in W_X^{1,A}(\Omega)$, is a local minimizer of the integral* (1.1) *if*

$$\int_{\operatorname{supp}(u-v)} F(Xu)\,dx \le \int_{\operatorname{supp}(u-v)} F(Xv)\,dx \quad \forall v \in W_X^{1,A}(\Omega), \operatorname{supp}(u-v) \Subset \Omega$$

Combining assumptions in (1.2) and (1.5) we have that the integrand f satisfies the following bounds

$$c(|\xi|^p - 1) \le F(\xi) \le c(|\xi|^q + 1) \tag{1.6}$$

Variational integrals whose integrand satisfies growth conditions of the type (1.6) are called functionals "with non standard growth conditions" and were introduced in the Euclidean setting by Marcellini in [17]. From the very beginning, it has been clear that minimizers of functionals satisfying (1.6) can be not only irregular but also unbounded if q is too large with respect to p, see [16]. The study of the regularity of minimizers of such integrals has a long history in the Euclidean setting, see for example [1], [6], [18] and [2]. In [18], Moscariello and Nania, assuming that A and its conjugate satisfy the so called Δ_2-condition, proved that any bounded local minimizer of (1.1) is Hölder continuous in Ω. It is worth pointing out that this result was proven without any further condition on p and q. In the same paper the local boundedness of minimizers is also proved for exponents p and q opportunely close.

Here, without any assumptions on p and q, we obtain that minimizers of the integral (1.1) belong to a Campanato space and have the horizontal gradients belonging to a Morrey space. More precisely we get

Theorem 1.1. *Let u be a local minimizer of the integral functional* (1.1). *Then there exist $\sigma = \sigma(p, q, C_d)$ and $\tau = \tau(p, q, C_d)$ such that $u \in \mathscr{L}_X^{p,\sigma}(\Omega)$ and $Xu \in L_X^{p,\tau}(\Omega)$.*

With the additional assumption $p > Qq/(Q+q)$, where Q is a homogeneous dimension relative to Ω, we establish the following higher integrability result for the horizontal gradient of minimizers (Theorem 1.2) and we prove the local boundedness of the minimizers themselves (Theorem 1.3).

Theorem 1.2. *Let A be an N-function satisfying assumptions in* (1.5) *with $p > \frac{Qq}{Q+q}$ and $u \in W_X^{1,A}(\Omega)$ be a local minimizer for the functional $\mathscr{F}(u)$. There exist positive constants c and $\delta = \delta(p, q, C_d)$ such that, for any balls $B_R \subset B_{2R} \Subset \Omega$,*

$$\fint_{B_R} A^{1+\delta}(|Xu|)\,dx \le c\Big(\fint_{B_{2R}} A(|Xu|)\,dx\Big)^{1+\delta} + c \tag{1.7}$$

Theorem 1.2 is the analogous of a result contained in [6] concerning the Euclidean setting. Obviously, we need some changes due the fact that we are working in a homogeneous space.

More precisely, an extension of the Maximal Theorem to the context of Orlicz spaces reveals a key tool in the proof of both results above. Moreover, a Poincaré inequality and a Caccioppoli type inequality in the setting of Orlicz-Sobolev spaces are crucial in order to prove Theorem 1.1 and Theorem 1.2 respectively. Carnot-Carathéodory spaces associated with a system of vector fields satisfying the Hörmander condition support a Poincaré inequality in Lebesgue spaces (see Proposition 2.3) and a so called A-Poincaré inequality, that is

$$\fint_B \frac{|u - u_B|}{R}\,dx \le CA^{-1}\Big(\fint_B A(|Xu|)\,dx\Big)$$

As far as we know, even it should be possible to deduce a (A, A)-Poincaré inequality (see Proposition 3.1) from a A-Poincaré inequality, there is not any explicit proof of it. Inspired by [6], we prove a (A, A)-Poincaré inequality using the Poincaré inequality in Lebesgue spaces.

In Section 3 we prove all the useful tools mentioned above.

Theorem 1.3. *Let A be an N-function satisfying assumptions in* (1.5) *with $p > \frac{Qq}{Q+q}$. Let $B_R \subset \Omega$ be a ball and $u \in W_X^{1,A}(\Omega)$ be a local minimizer for the functional $\mathcal{F}(u)$ assuming the value u_0 on ∂B_R. If $u_0 \in L^\infty(\partial B_R)$, then u is locally bounded.*

In the proof we follow an idea by Stampacchia [20] as suggested by Boccardo, Marcellini and Sbordone in the Euclidean setting, [1].

It is worth mentioning that regularity results for minimizers of integral functionals under standard growth conditions (i.e. $p = q$ in (1.6)) have been established for example in [7, 8, 3].

2. Notation and preliminary results

Carnot-Carathéodory spaces Let $X_1, \dots, X_k$ be vector fields defined in $\mathbb{R}^n$, with real, C^∞ smooth coefficients. We say that they satisfy the Hörmander's condition if there exists an integer m such that the family of commutators of $X_1, \dots, X_k$ up to length m

$$X_1, \dots, X_k, [X_{i_1}, X_{i_2}], \dots, [X_{i_1}, [X_{i_1}, \dots X_{i_m}]] \dots], \quad \forall i_j = 1, 2, \dots, k$$

spans the tangent space $T_x\mathbb{R}^n$ at every point $x \in \mathbb{R}^n$.

For any real valued Lipschitz continuous function u we define

$$X_j u(x) = \langle X_j(x), \nabla u(x)\rangle \quad j = 1, 2, \dots, k$$

and we call the horizontal gradient of u the vector $Xu = (X_1 u, \dots, X_k u)$ whose length is given by

$$|Xu| = \Big(\sum_{j=1}^{k} (X_j u)^2\Big)^{1/2}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set. For a function $u \in L^1_{\mathrm{loc}}(\Omega)$, its distributional derivative along the vector fields X_j is defined by the identity

$$\langle X_j u, \Phi \rangle = \int_\Omega u X_j^* \Phi \, dx \quad \forall \Phi \in C_0^\infty(\Omega) \tag{2.1}$$

where X_j^* denotes the formal adjoint of X_j. Throughout the paper, if u is a nonsmooth function, $X_j u$ will be meant in the distributional sense.

An absolutely continuous curve $\gamma : [a, b] \to \mathbb{R}^n$ is said to be *admissible*, if there exist functions $c_j : [a, b] \to \mathbb{R}$, $j = 1, \dots, k$ such that

$$\dot{\gamma}(t) = \sum_{j=1}^{k} c_j(t) X_j(\gamma(t)) \quad \text{and} \quad \sum_{j=1}^{k} c_j(t)^2 \le 1$$

Observe that X_j do not need to be linearly independent and therefore functions c_j do not need to be unique. Define the distance function ρ as

$$\rho(x, y) = \inf\{T > 0 : \exists \gamma : [0, T] \to \mathbb{R}^n \ \textit{admissible}, \gamma(0) = x, \gamma(T) = y\}$$

If there is not any such a curve, we set $\rho(x, y) = \infty$. The function ρ is called Carnot-Carathéodory distance and, since it is not clear whether one can connect any two points of $\mathbb{R}^n$ by an admissible curve, it's not clear whether ρ is a metric. The assumption for which the vector fields $X_1, \dots, X_k$ satisfy the Hörmander condition ensures that ρ is a metric and in this case $(\mathbb{R}^n, \rho)$ is said to be a Carnot-Carathéodory space.

The following theorem, due to Nagel, Stein and Wainger [19], shows that the metric ρ is locally Hölder continuous with respect to the Euclidean metric.

Theorem 2.1. *Let $X_1, \dots X_k$ be as above. Then for every bounded open set $\Omega \subset \mathbb{R}^n$ there are constants c_1, c_2 and $\lambda \in (0, 1]$ such that*

$$c_1|x - y| \le \rho(x, y) \le c_2|x - y|^\lambda \tag{2.2}$$

for every $x, y \in \Omega$.

It follows that the space $(\mathbb{R}^n, \rho)$ is homeomorphic with the Euclidean space $\mathbb{R}^n$ and therefore bounded sets in the Euclidean metric are bounded sets in the metric ρ. The inverse is not always true but it is certainly valid if $X_1, \dots X_k$ have globally Lipschitz coefficients (see [10]). In the sequel all the distances will be respect to the metric ρ, in particular all the balls will be balls with respect to the Carnot-Carathéodory metric. We shall consider in $(\mathbb{R}^n, \rho)$ the Lebesgue measure which locally satisfies the following doubling condition (see for example [19]):

Proposition 2.2. *Let Ω be an open, bounded subset of $\mathbb{R}^n$. There exists a constant $C_d \geq 1$, called doubling constant, such that*

$$|B(x_0, 2R)| \leq C_d |B(x_0, R)|$$

provided $x_0 \in \Omega$ and $R \leq 5 \operatorname{diam} \Omega$.

Let Y be a metric space and μ a Borel measure in Y. Assume μ finite on bounded sets and satisfying the doubling condition on every open, bounded subset Ω of Y. If there exists a positive constant C such that

$$\frac{\mu(B)}{\mu(B_0)} \geq C\left(\frac{R}{R_0}\right)^Q$$

for any ball B_0 having center in Ω and radius $R_0 < \operatorname{diam} \Omega$ and any ball B centered in $x \in B_0$ and having radius $R \leq R_0$, we say that Q is a *homogeneous dimension* relative to Ω.

It is well known that doubling property implies the existence of such a Q. However, Q is not unique and it may change with Ω. Obviously any $Q' \geq Q$ is also a homogeneous dimension.

For a bounded open set Ω containing a family of vector fields satisfying the Hörmander condition, the Carnot-Carathéodory space (Ω, ρ) with the Lebesgue measure has the homogeneous dimension $Q = \log_2 C_d$.

Recall that the Sobolev space $W_X^{1,p}(\Omega)$ is defined as

$$W_X^{1,p}(\Omega) = \{u \in L^p(\Omega) : X_j u \in L^p(\Omega)\ j = 1, \dots, k\}$$

and that $W_{X,0}^{1,p}(\Omega)$ denotes the closure of $C_{X,0}^{\infty}(\Omega)$ in $W_X^{1,p}(\Omega)$. The following versions of Sobolev and Poincaré type inequalities hold (see for example [5], [10]).

Proposition 2.3. *Let $X_1, \dots X_k$ be as before. Let Q be a homogeneous dimension relative to Ω. There exist constants $C_1, C_2 > 0$ such that, for every ball B_R centered in Ω and having radius $R \leq \operatorname{diam} \Omega$, the following inequalities hold*

$$\left(\fint_{B_R} |u - u_R|^{p^*}\, dx\right)^{1/p^*} \leq C_1 R \left(\fint_{B_R} |Xu|^p\, dx\right)^{1/p} \tag{2.3}$$

for $1 \leq p < Q$ and $p^* = \frac{Qp}{Q-p}$ and

$$\fint_{B_R} |u - u_R|^p\, dx \leq C_2 R^p \fint_{B_R} |Xu|^p\, dx \tag{2.4}$$

for $1 \leq p < \infty$.

We have denoted by u_R the average of the function u on B_R.

The following imbedding property holds under the previous assumptions on the vector fields $X_1, \dots X_k$.

PROPOSITION 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set with sufficiently smooth boundary and Q a homogeneous dimension relative to Ω. Let $u \in W^{1,p}_{X,0}(\Omega)$ with $1 \le p < Q$. Then there exists a constant $c > 0$ such that*

$$\|u\|_{L^{p^*}(\Omega)} \le c\|u\|_{W^{1,p}_{X,0}(\Omega)} \tag{2.5}$$

For the proof the reader can refer to [11] and [12].

For $0 < \alpha \le 1$ we say that a continuous function on Ω belongs to the Hölder class $C^{0,\alpha}_X(\Omega)$ if

$$\sup_{x \ne y, x, y \in \Omega} \frac{|u(x) - u(y)|}{\rho(x, y)^\alpha} < \infty$$

Finally $u \in L^p(\Omega)$ is said to belong to the Campanato space $\mathscr{L}^{p,\sigma}_X(\Omega)$ if

$$\frac{1}{|B_R|} \int_{B_R} |u - u_R|^p \, dx \le cR^\sigma$$

and to the Morrey space $L^{p,\tau}_X(\Omega)$ if

$$\int_{B_R} |u|^p \, dx \le cR^\tau$$

for every ball B_R centered in Ω and having radius $R < \operatorname{diam} \Omega$.

Note that the following Theorem holds (see for example [14])

THEOREM 2.5. *If $\gamma < 0$, the Campanato space $\mathscr{L}^{p,\gamma}_X(\Omega)$ is isomorphic to the Morrey space $L^{p,\gamma}_X(\Omega)$. If $0 < \gamma < p$, the Campanato space $\mathscr{L}^{p,\gamma}_X(\Omega)$ is isomorphic to $C^{0,\alpha}_X(\Omega)$ with $\alpha = \frac{\gamma}{p}$.*

Orlicz and Orlicz-Sobolev spaces Let $A : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing and convex function satisfying (1.4). We shall assume that there exist $1 < p \le q$ such that

$$pA(t) \le tA'(t) \le qA(t) \quad \forall t \ge 0 \tag{2.6}$$

It is easy to verify that the second inequality in (2.6) is equivalent to say that there exists a constant $k > 1$ such that

$$A(2t) \le kA(t) \quad \forall t \ge 0 \tag{2.7}$$

that is the so called Δ_2-condition on A, while both the inequalities in (2.6) are equivalent to

$$\frac{A(t)}{t^p} \nearrow \quad \frac{A(t)}{t^q} \searrow \tag{2.8}$$

and to the following conditions

$$\frac{A^*(t)}{t^{p'}} \searrow \quad \frac{A^*(t)}{t^{q'}} \nearrow \tag{2.9}$$

where p' and q' denote the Hölder conjugate exponents of p and q respectively and A^* is the conjugate N-function of A defined by

$$A^*(s) = \sup_{t \geq 0}\{st - A(t)\} \tag{2.10}$$

It follows that

$$c_1(t^p - 1) \leq A(t) \leq c_2(t^q + 1) \tag{2.11}$$

for some constants c_1, c_2.

Note that (2.9) implies that A^* also satisfies a Δ_2-condition.

There are many functions A which behave as above. For example, it is easy to verify that the function

$$A(t) = t^p \log(1 + t) \quad p > 1$$

satisfies conditions in (2.8) with $p = p - \varepsilon$ and $q = p + \varepsilon$ for all $\varepsilon > 0$.

Let $\Omega \subset \mathbb{R}^n$ be an open set, the Orlicz class $L^A(\Omega)$ defined by

$$L^A(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable, } \int_\Omega A(|u(x)|)\, dx < \infty \right\}$$

is a Banach space equipped with the Luxemburg norm

$$\|u\|_{L^A(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega A\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}$$

The space generated by A^* is the dual of L^A and the two following fundamental inequalities hold

$$A^*\left(\frac{A(t)}{t}\right) \leq A(t) \tag{2.12}$$

$$st \leq A(t) + A^*(s) \quad \text{(Young's inequality)} \tag{2.13}$$

The Orlicz-Sobolev space $W_X^{1,A}(\Omega)$ is the subspace of $L^A(\Omega)$ of functions u such that the horizontal gradient Xu belongs to $L^A(\Omega)$. It is equipped with the norm

$$\|u\|_{W_X^{1,A}(\Omega)} = \|u\|_{L^A(\Omega)} + \|Xu\|_{L^A(\Omega)}$$

Maximal functions For $f \in L^1_{\mathrm{loc}}(\Omega)$, we define the maximal function by

$$M_R^{\Omega} f(x) := \sup_{0<r<R,\, B(x,r)\subset\Omega} \fint_{B(x,r)} |f|\, dy$$

In order to simplify the notations we will write M_R and M_{2R} in place of $M_R^{B_R}$ and $M_{2R}^{B_{2R}}$ respectively. The following proposition contains a metric version of a "weak type" inequality for the maximal function whose proof can be found in [12].

Proposition 2.6. *Assume X be a metric space equipped with a doubling measure μ on an open set $\Omega \subset X$. Let h be a locally integrable function in Ω. Then*

$$\mu(\{x \in \Omega : M_R^{\Omega} h(x) > t\}) \le \frac{c}{t} \int_{\Omega} |h|\, d\mu \tag{2.14}$$

for $t > 0$, where the constant c depends only on the doubling constant C_d.

From now on we shall denote by Ω a bounded open set in $\mathbb{R}^n$ and by Q a homogeneous dimension relative to Ω. Let us conclude this section with a useful inequality due to Hajlasz and Strzelecki [13].

Proposition 2.7. *Let $u \in W_X^{1,p}(\Omega)$. Then*

$$\frac{|u(x) - u_R|}{R} \le c M_{2R}^{\Omega} |Xu|(x)$$

for almost every $x \in \Omega'$ with $\Omega' \Subset \Omega$.

3. Crucial inequalities

In this section we prove some propositions that reveal crucial in the sequel. We start with the following (A, A)-Poincaré inequality

Proposition 3.1. *Let A be an N-function satisfying* (1.5) *and Q a homogeneous dimension relative to Ω. If $u \in W_X^{1,A}(\Omega)$, then there exists a positive constant $C = C(p, q, Q)$ such that*

$$\fint_{B_R} A\Big(\frac{|u - u_R|}{R}\Big)\, dx \le C \fint_{B_R} A(|Xu|)\, dx \tag{3.1}$$

for each ball B_R well contained in Ω.

Proof. Define the function

$$K(t) = \int_0^t A(s^{1/q})/s\, ds \tag{3.2}$$

It is obviously increasing and, in virtue of (2.6), it is easy to verify that is concave and satisfies the following inequalities

$$A(t^{1/q}) \le K(t) \le \frac{q}{p} A(t^{1/q}) \tag{3.3}$$

Denoting by $H(t) = A(t^{1/q})$, we have

$$\begin{aligned}
(3.4)\qquad \fint_{B_R} A\Big(\frac{|u-u_R|}{R}\Big)\,dx &= \fint_{B_R} H\Big(\frac{|u-u_R|^q}{R^q}\Big)\,dx \\
&\le \fint_{B_R} K\Big(\frac{|u-u_R|^q}{R^q}\Big)\,dx \le K\Big(\fint_{B_R} \frac{|u-u_R|^q}{R^q}\,dx\Big) \\
&\le K\Big(c\fint_{B_R} |Xu|^q\,dx\Big) \le \frac{q}{p} A\Big[c\Big(\fint_{B_R} |Xu|^q\,dx\Big)^{1/q}\Big] \\
&\le cA\Big[\Big(\fint_{B_R} |Xu|^q\,dx\Big)^{1/q}\Big]
\end{aligned}$$

where we used (3.3), Jensen's inequality, Poincaré inequality in (2.4) and Δ_2-condition.

Now denoting by $\Psi(t) = \int_0^t \frac{A(\sigma)}{\sigma}\,d\sigma$, a simple change of variable gives

$$\Psi(t) = q\int_0^{t^{1/q}} \frac{A(s^q)}{s}\,ds =: \tilde{\Psi}(t^{1/q}) \tag{3.5}$$

Using conditions in (2.6), we can easily prove that the function $\Psi(t)$ is convex and that the function $\tilde{\Psi}(t^{1/q})$ defined in (3.5) satisfies the following inequalities

$$\frac{1}{q} A(t^{1/q}) \le \tilde{\Psi}(t^{1/q}) \le A(t^{1/q}) \tag{3.6}$$

Therefore, by Jensen's inequality,

$$\begin{aligned}
(3.7)\qquad cA\Big[\Big(\fint_{B_R} |Xu|^q\,dx\Big)^{1/q}\Big] &\le c\tilde{\Psi}\Big[\Big(\fint_{B_R} |Xu|^q\,dx\Big)^{1/q}\Big] \\
&= c\Psi\Big(\fint_{B_R} |Xu|^q\,dx\Big) \le c\fint_{B_R} \Psi(|Xu|^q)\,dx \\
&= c\fint_{B_R} \tilde{\Psi}(|Xu|)\,dx \le c\fint_{B_R} A(|Xu|)\,dx
\end{aligned}$$

The conclusion follows combining inequalities in (3.4) and (3.7). □

Next Proposition contains an extension of the Maximal Theorem to the context of Orlicz spaces.

Proposition 3.2. *Let B_R be a ball well contained in Ω. If A is an N-function satisfying conditions* (1.5) *and $f \in L^A(B_R)$ is a non negative function, then there exists a positive constant c depending on p, q and on the doubling constant C_d, such that*

$$\int_{B_R} A(M_R f)\,dx \le c\int_{B_R} A(f)\,dx \tag{3.8}$$

Proof. Defining, for any $t > 0$,

$$\lambda(t) = |\{x \in B_R : M_R f(x) > t\}| \tag{3.9}$$

we have

$$\begin{aligned} \int_{B_R} A(M_R f)\,dx &= \int_{B_R} dx \int_0^{M_R f} A'(t)\,dt \\ &= \int_0^\infty A'(t)\lambda(t)\,dt \end{aligned} \tag{3.10}$$

Now choosing

$$h(x) = \begin{cases} f(x) & \text{if } f(x) > \frac{t}{2} \\ 0 & \text{otherwise} \end{cases}$$

we have that $M_R f \le \frac{t}{2} + M_R h$ and therefore

$$\{x \in B_R : M_R f(x) > t\} \subset \left\{x \in B_R : M_R h(x) > \frac{t}{2}\right\}$$

It follows that

$$\begin{aligned} \int_{B_R} A(M_R f)\,dx &\le \int_0^\infty A'(t)\left|\left\{x \in B_R : M_R h(x) > \frac{t}{2}\right\}\right| dt \\ &\le c(C_d)\int_0^\infty \frac{A'(t)}{t}\,dt \int_{f>t/2} f\,dx. \end{aligned} \tag{3.11}$$

where, in the last inequality, we have used Proposition 2.6. By Fubini's theorem, integration by parts and assumptions on A we get

$$\begin{aligned}
(3.12)\qquad \int_{B_R} A(M_R f)\,dx &\le c(C_d)\int_{B_R} f(x)\,dx \int_0^{2f(x)} [A'(t)/t]\,dt \\
&= c(C_d)\int_{B_R} A(2f(x))/2\,dx \\
&\quad + c(C_d)\int_{B_R} f(x)\,dx \int_0^{2f(x)} [A(t)/t^2]\,dt \\
&\le \frac{1}{2}c(k, C_d)\int_{B_R} A(f(x))\,dx \\
&\quad + c(C_d)\int_{B_R} f(x)\,dx \int_0^{2} [A(sf(x))/s^2 f(x)]\,ds \\
&= \frac{1}{2}c(k, C_d)\int_{B_R} A(f(x))\,dx \\
&\quad + c(C_d)\int_{B_R} dx \int_0^{2} [A(sf(x))/s^2]\,ds
\end{aligned}$$

Note that in the last equality we have used the change of variable $t = sf(x)$. Splitting the last integral, by using (1.5) and the Δ_2-condition, we have

$$\begin{aligned}
(3.13)\qquad \int_0^2 [A(sf(x))/s^2]\,ds &= \int_0^1 [A(sf(x))/s^2]\,ds + \int_1^2 [A(sf(x))/s^2]\,ds \\
&\le A(f(x))\int_0^1 s^{p-2}\,ds + kA(f(x))\int_1^2 s^{-2}\,ds
\end{aligned}$$

Inserting (3.13) in (3.12) we conclude that

$$\int_{B_R} A(M_R f)\,dx \le c(C_d, p, q)\int_{B_R} A(f(x))\,dx \qquad \square$$

Arguing as in [6], we can easily deduce from Proposition 3.2 the following extension of Gehring's lemma for N-functions A satisfying (1.5).

PROPOSITION 3.3. *Let A be an N-function satisfying conditions* (1.5), *and let $f \in L^1_{\mathrm{loc}}(\Omega)$ a non negative function such that, for any ball $B_R \Subset \Omega$,*

$$(3.14)\qquad \fint_{B_{R/2}} A(f)\,dx \le b_1 A\Big(\fint_{B_R} f\Big) + b_2.$$

Then there exist c_1, c_2, $\delta > 0$ depending on b_1, b_2, p, q, Q such that

$$(3.15)\qquad \fint_{B_{R/2}} A^{1+\delta}(f)\,dx \le c_1 A^{1+\delta}\Big(\fint_{B_R} f\Big) + c_2.$$

Let us conclude this section with the following Caccioppoli type inequality

THEOREM 3.4. *Let A be an N-function satisfying conditions* (1.5) *and $u \in W_X^{1,A}(\Omega)$ be a minimizer for the functional $\mathscr{F}(u)$. Then, for $R \le s < t \le 2R$,*

$$\int_{B_s} A(|Xu|)\,dx \le c\left[\int_{B_t\setminus B_s} A\left(\frac{|u-u_R|}{t-s}\right)dx + \int_{B_t\setminus B_s} A(|Xu|)\,dx + R^Q\right] \tag{3.16}$$

where c is a constant depending on q and on the Δ_2-constant of A.

PROOF. Let $\eta \in C_0^\infty(B_t)$ be a cut-off function such that $\eta \equiv 1$ on B_s, $|X\eta| \le \frac{c}{t-s}$. The proof of the existence of a such function can be found, for example, in [4]. Since $\varphi = (u-u_R)\eta$ belongs to the space $W_X^{1,A}(B_t)$, it can be used as a test function in Definition (1.1). The assumption on (1.2) and (1.3), the monotonicity of the function A and the Δ_2-condition give us

$$\begin{aligned}\int_{B_t} F(Xu)\,dx &\le \int_{B_t} F(X(u-\varphi))\,dx \\ &= \int_{B_t} F((1-\eta)Xu - X\eta(u-u_R))\,dx \\ &\le c\left[\int_{B_t\setminus B_s} A(|1-\eta|\,|Xu|)\,dx + \int_{B_t\setminus B_s} A(|X\eta|\,|u-u_R|)\,dx + R^Q\right] \\ &\le c\left[\int_{B_t\setminus B_s} A(|1-\eta|\,|Xu|)\,dx + \int_{B_t\setminus B_s} A\left(\frac{|u-u_R|}{t-s}\right)dx + R^Q\right]\end{aligned}$$

Therefore assumption on (1.2) and the monotonicity of A imply

$$\begin{aligned}\int_{B_t} A(|Xu|)\,dx &\le c\left[\int_{B_t} F(Xu)\,dx + R^Q\right] \\ &\le c\left[\int_{B_t\setminus B_s} A(|Xu|)\,dx + \int_{B_t\setminus B_s} A\left(\frac{|u-u_R|}{t-s}\right)dx + R^Q\right]\end{aligned}$$

hence the conclusion. □

4. THE REGULARITY RESULT

This section is devoted to the proof of Theorem 1.1.

PROOF. Let B_{2R} be a ball in Ω. Combining the Maximal inequality proved in Proposition 3.2, the Caccioppoli type inequality in (3.16) for $s = R$ and $t = 2R$ and the pointwise inequality $|Xu| \le M_{2R}(|Xu|)$, we easily get

$$\begin{aligned}\int_{B_R} & A(M_{2R}(|Xu|))\,dx \\ & \le \int_{B_{2R}} A(M_{2R}(|Xu|))\,dx \le c\int_{B_{2R}} A(|Xu|)\,dx \\ & \le c\left[\int_{B_{2R}\setminus B_R} A(M_{2R}(|Xu|))\,dx + \int_{B_{2R}\setminus B_R} A\Big(\frac{|u-u_R|}{R}\Big)\,dx + R^Q\right]\end{aligned}$$

Now, since A is increasing and Proposition 2.7 holds, we get

$$\int_{B_R} A(M_{2R}(|Xu|))\,dx \le c\left[\int_{B_{2R}\setminus B_R} A(M_{2R}(|Xu|))\,dx + R^Q\right]$$

Now we fill the hole adding $c\int_{B_R} A(M_{2R}(|Xu|))\,dx$ to both sides of the obtained inequality having

$$\int_{B_R} A(M_{2R}(|Xu|))\,dx \le \theta\int_{B_{2R}} A(M_{2R}(|Xu|))\,dx + cR^Q$$

for $\theta \in (0,1)$. A standard iteration argument implies the existence of a constant τ such that the following decay estimate holds

$$\int_{B_R} A(M_{2R}(|Xu|))\,dx \le cR^\tau \tag{4.1}$$

and observing that

$$\int_{B_R} A(|Xu|)\,dx \le \int_{B_R} A(M_R(|Xu|))\,dx \le \int_{B_R} A(M_{2R}(|Xu|))\,dx \tag{4.2}$$

we get

$$\int_{B_R} |Xu|^p\,dx \le cR^\tau \tag{4.3}$$

that means $Xu \in L_X^{p,\tau}(\Omega)$.

Moreover, applying the Poincaré inequality of Proposition 3.1 to the left hand side of (4.2) and using (4.1), we have

$$\begin{aligned}\int_{B_R} A\Big(\frac{|u-u_R|}{R}\Big)\,dx & \le \int_{B_R} A(|Xu|)\,dx \\ & \le \int_{B_R} A(M_{2R}(|Xu|))\,dx \le cR^\tau\end{aligned}$$

and then

$$\frac{1}{|B_R|}\int_{B_R}\frac{|u-u_R|^p}{R^p}\,dx \le cR^{\tau-Q}$$

that is $u \in \mathscr{L}_X^{p,p+\tau-Q}(\Omega)$, i.e. the conclusion. □

Remark 4.1. Since $\mathscr{L}_X^{p,p+\tau-Q}(\Omega)$ is isomorphic to $C_X^{0,\alpha}(\Omega)$ for $\alpha = 1+\frac{\tau-Q}{p}$, provided $\tau > Q-p$, in the particular case $p = Q$ the minimizers of the integral (1.1) belong to $C_X^{0,\alpha}(\Omega)$ for $\alpha = \frac{\tau}{Q}$.

5. The higher integrability

The Caccioppoli type inequality in (3.16) combined with the Gehring's lemma 3.3 will give Theorem 1.2.

Proof (of Theorem 1.2). Fix B_{2R} an arbitrary ball well contained in Ω and $R < s < t \le 2R$. By Theorem 3.4 we have

$$\int_{B_s} A(|Xu|)\,dx \le c\left[\int_{B_t\setminus B_s} A\left(\frac{|u-u_R|}{t-s}\right)dx + \int_{B_t\setminus B_s} A(|Xu|)\,dx + R^Q\right]$$

and therefore, filling the hole adding to both sides of the inequality the integral $c\int_{B_s} A(|Xu|)\,dx$, we get

$$\int_{B_s} A(|Xu|)\,dx \le \theta\int_{B_t} A(|Xu|)\,dx + c\left[\int_{B_t} A\left(\frac{|u-u_R|}{t-s}\right)dx + R^Q\right] \tag{5.1}$$

for $\theta \in (0,1)$. It follows that

$$\int_{B_R} A(|Xu|)\,dx \le c\left[\int_{B_{2R}} A\left(\frac{|u-u_R|}{R}\right)dx + R^Q\right] \tag{5.2}$$

hence by Hölder's inequality, we deduce that

$$\begin{aligned}
&\fint_{B_R} A(|Xu|)\,dx \\
&\quad\le c\fint_{B_{2R}} \frac{A(|u-u_R|/R)}{|(u-u_R)/R|^{Qq/(Q+q)}}\left|\frac{u-u_R}{R}\right|^{Qq/(Q+q)} + c \\
&\quad\le c\left[\fint_{B_{2R}} \frac{A^{(Q+q)/q}(|u-u_R|/R)}{|(u-u_R)/R|^{Q}}\right]^{q/(Q+q)}\left[\fint_{B_{2R}} |(u-u_R)/R|^q\,dx\right]^{Q/(Q+q)} + c
\end{aligned}$$

Define

$$K(t) = \int_0^t [A(s^{1/q})/s]^{(Q+q)/q}\,ds, \quad H(t) = \frac{[A(t^{1/q})]^{(Q+q)/q}}{t^{Q/q}}, \tag{5.3}$$

In virtue of (2.6), it is possible to prove that $K(t)$ is concave and that there exists a constant c such that

$$H(t) \leq K(t) \leq cH(t) \quad \forall t > 0 \tag{5.4}$$

Therefore, for $q_* = \frac{Qq}{Q+q}$, using Proposition 2.3, we have

$$\begin{aligned} \fint_{B_R} A(|Xu|)\,dx &\leq c\left[\fint_{B_{2R}} K(|(u-u_R)/R|^q)\right]^{q/(Q+q)} \fint_{B_{2R}} |Xu|^{q_*}\,dx + c \\ &\leq cK^{q/(Q+q)}\Big(\fint_{B_{2R}} |(u-u_R)/R|^q\,dx\Big)\fint_{B_{2R}} |Xu|^{q_*}\,dx + c \\ &\leq cH^{q/(Q+q)}\left(\left[\fint_{B_{2R}} |Xu|^{q_*}\right]^{q/q_*}\right)\fint_{B_{2R}} |Xu|^{q_*}\,dx + c \\ &= c\frac{A\big(\big[\fint_{B_{2R}} |Xu|^{q_*}\,dx\big]^{1/q_*}\big)}{\big(\fint_{B_{2R}} |Xu|^{q_*}\,dx\big)}\fint_{B_{2R}} |Xu|^{q_*}\,dx + c \\ &= cA\left(\left[\fint_{B_{2R}} |Xu|^{q_*}\,dx\right]^{1/q_*}\right) + c \end{aligned} \tag{5.5}$$

Setting $\Phi(t) = A(t^{1/q_*})$, we have

$$\Phi(2t) \leq k\Phi(t) \quad \text{and} \quad \Phi'(t) \geq \frac{p}{q_*}\frac{\Phi(t)}{t}$$

where, by assumption, $\frac{p}{q_*} > 1$. Hence inequality (5.5) can be written as

$$\fint_{B_R} \Phi(|Xu|^{q_*})\,dx \leq c\Phi\Big(\fint_{B_{2R}} |Xu|^{q_*}\,dx\Big) + c$$

Using now Proposition 3.3, we deduce that there exists $\delta > 0$ such that

$$\fint_{B_R} \Phi^{1+\delta}(|Xu|^{q_*})\,dx \leq c\Phi^{1+\delta}\Big(\fint_{B_{2R}} |Xu|^{q_*}\,dx\Big) + c$$

that is

$$\fint_{B_R} A^{1+\delta}(|Xu|)\,dx \leq cA^{1+\delta}\left(\left[\fint_{B_{2R}} |Xu|^{q_*}\,dx\right]^{1/q_*}\right) + c \tag{5.6}$$

Setting

$$\Psi(t) = \int_0^t \frac{A(s)}{s}\,ds, \tag{5.7}$$

it is easy to prove that

$$\frac{1}{q}A(t) \le \Psi(t) \le A(t) \tag{5.8}$$

and that $\Psi(t)$ and $\Psi(t^{1/p})$ are both convex. It follows that

$$\left[\fint_{B_{2R}} |Xu|^p \, dx \right]^{1/p} \le \Psi^{-1}\left(\fint_{B_{2R}} \Psi(|Xu|) \, dx \right) + c \tag{5.9}$$

see [15]. Finally, since $p > q_*$, we have from (5.8) and (5.9) that

$$\begin{aligned} \frac{1}{q}A\left(\left[\fint_{B_{2R}} |Xu|^{q_*} \, dx \right]^{1/q_*} \right) &\le \frac{1}{q}A\left(\left[\fint_{B_{2R}} |Xu|^p \, dx \right]^{1/p} \right) + c \\ &\le c \fint_{B_{2R}} A(|Xu|) \, dx \end{aligned} \tag{5.10}$$

The conclusion follows from (5.6) and (5.10). □

For the case of spherical Quasi-minima, compare with the proof given in [7].

6. The local boundedness

In this section we prove the boundedness of the local minimizers of the functional (1.1) with a fixed boundary value.

Proof (of Theorem 1.3). For a positive constant $\lambda \ge \|u_0\|_\infty$, let us consider the function

$$w = \operatorname{sign}(u) \max\{|u| - \lambda, 0\} \tag{6.1}$$

and use $v = u - w$ as test function in Definition (1.1), that is

$$\int_{\operatorname{supp} w} F(Xu) \, dx \le \int_{\operatorname{supp} w} F(Xv) \, dx$$

Since $Xu = Xw$ on the set $E_\lambda = \{x \in B_R : |u(x)| > \lambda\}$, it follows that

$$\int_{E_\lambda} F(Xu) \, dx \le \int_{E_\lambda} F(0) \, dx$$

thus, by assumptions in (1.2), we get

$$\int_{E_\lambda} A(|Xu|) \, dx \le c|E_\lambda| \tag{6.2}$$

By a simple use of the Sobolev embedding in (2.5) and the hypotheses on A, we have

$$(6.3)\qquad \Big(\int_{B_R} |w|^{p^*}\,dx\Big)^{p/p^*} \le c\int_{B_R} |Xw|^p\,dx \le c\int_{B_R} A(|Xw|)\,dx$$
$$= c\left[\int_{B_R\setminus E_\lambda} A(|Xw|)\,dx + \int_{E_\lambda} A(|Xw|)\,dx\right]$$
$$= c\int_{E_\lambda} A(|Xu|)\,dx$$

and combining (6.2) and (6.3) we obtain

$$(6.4)\qquad \Big(\int_{B_R} |w|^{p^*}dx\Big)^{p/p^*} \le c|E_\lambda|$$

Recalling the definition of the function w, we have for $\delta > \lambda$,

$$(6.5)\qquad \int_{B_R} |w|^{p^*}\,dx = \int_{E_\lambda} |\,|u| - \lambda|^{p^*}\,dx \ge \int_{E_\delta} |\,|u| - \lambda|^{p^*}\,dx$$
$$\ge \int_{E_\delta} |\delta - \lambda|^{p^*}\,dx = |\delta - \lambda|^{p^*}|E_\delta|$$

and therefore, from (6.4) and (6.5), we have

$$|E_\delta| \le c\frac{|E_\lambda|^{p^*/p}}{|\delta - \lambda|^{p^*}}$$

Applying Lemma 4.1 of [20], we obtain that

$$|E_\tau| = 0 \quad \text{where } \tau = c|B_R|^{1/Q} = cR$$

that implies

$$\sup_{B_R} |u| \le \|u_0\|_\infty + cR$$

i.e. the conclusion. □

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