Rend. Lincei Mat. Appl. 21 (2010), 193–213
DOI 10.4171/RLM/567

Partial Differential Equations — *Remarks on the H Theorem for a non involutive Boltzmann like kinetic model*, by Giulia Furioli and Elide Terraneo.

Abstract. — In this paper, we consider a one-dimensional kinetic equation of Boltzmann type in which the binary collision process is described by the linear transformation $v^* = pv + qw$, $w^* = qv + pw$, where (v, w) are the pre-collisional velocities and (v^*, w^*) the post-collisional ones and $p \geq q > 0$ are two positive parameters. This kind of model has been extensively studied by Pareschi and Toscani (in *J. Stat. Phys.*, 124(2–4):747–779, 2006) with respect to the asymptotic behavior of the solutions in a Fourier metric. In the conservative case $p^2 + q^2 = 1$, even if the transformation has Jacobian $J \neq 1$ and so it is not involutive, we remark that the H Theorem holds true. As a consequence we prove exponential convergence in L^1 of the solution to the stationary state, which is the Maxwellian.

Key words: Conservative Boltzmann equations, asymptotic behavior of solutions, kinetic equations.

2000 Mathematics Subject Classification: 76P05, 82C40, 35B40.

1. Introduction

This paper deals with the evolution of a probability density $f(v, t)$ which represents the density of a gas of one dimensional particles depending only on the velocity $v \in \mathbb{R}$ at the time $t \geq 0$. We suppose that the binary interaction between two particles obeys to the law:

$$\begin{cases} v^* = pv + qw \\ w^* = qv + pw \end{cases} \tag{1}$$

where $p \geq q > 0$ and (v, w) are the pre collisional velocities which generate (v^*, w^*) after the collision. The kinetic integro differential equation of Boltzmann type which modelizes this process is therefore as follows:

$$\partial_t f(v, t) = \int_{\mathbb{R}} \left(\frac{1}{J} f(v_*, t) f(w_*, t) - f(v, t) f(w, t) \right) \mathrm{d}w \tag{2}$$

where now (v_*, w_*) are the pre collisional velocities which generate (v, w) after the collision:

$$\begin{cases} v_* = \dfrac{1}{J}(pv - qw) \\[2mm] w_* = \dfrac{1}{J}(-qv + pw) \end{cases} \tag{3}$$

and

$$J = \left| \det \begin{pmatrix} \frac{\partial v^*}{\partial v} & \frac{\partial v^*}{\partial w} \\ \frac{\partial w^*}{\partial v} & \frac{\partial w^*}{\partial w} \end{pmatrix} \right| = p^2 - q^2.$$

We underline here that the transformation $(v, w) \mapsto (v^*, w^*)$ is such that $J \neq 1$ and so it is not involutive. The kinetic model (2) was introduced in [BBLR03] and it is worth comparing it with the Boltzmann equation and its one dimensional Kac model. We start by considering the full Boltzmann homogenous equation for Maxwellian molecules in the cut-off, elastic case which reads

$$(4) \quad \partial_t f(\mathbf{v}, t) = \int_{\mathbf{w} \in \mathbb{R}^3} \int_{\mathbf{n} \in S^2} (f(\mathbf{v}_*, t) f(\mathbf{w}_*, t) - f(\mathbf{v}, t) f(\mathbf{w}, t)) b\Big(\frac{\mathbf{v} - \mathbf{w}}{|\mathbf{v} - \mathbf{w}|} \cdot \mathbf{n}\Big) \, d\mathbf{n} \, d\mathbf{w}.$$

Here, $f(\mathbf{v}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ is the probability density of a gas which depends only on the velocity $\mathbf{v} \in \mathbb{R}^3$ at the time $t \geq 0$ and due to the physical assumptions that the gas evolves through binary, elastic collisions which are localized both in space and time, the relations between the velocities $(\mathbf{v}_*, \mathbf{w}_*)$ of two particles before the collision and $(\mathbf{v}, \mathbf{w})$ after it are the following:

$$\begin{cases} \mathbf{v}_* = \dfrac{\mathbf{v} + \mathbf{w}}{2} + \dfrac{|\mathbf{v} - \mathbf{w}|}{2} \mathbf{n} \\ \mathbf{w}_* = \dfrac{\mathbf{v} + \mathbf{w}}{2} - \dfrac{|\mathbf{v} - \mathbf{w}|}{2} \mathbf{n}, \end{cases}$$

where $\mathbf{n}$ is a vector in S^2, the unit sphere in $\mathbb{R}^3$, and parametrizes all the possible pre collisional velocities.

The collision kernel b, which is supposed to be nonnegative, is the function which selects in which way the pre collisional velocities contribute to produce particles with velocity v after the collision and is supposed (this is precisely the assumption of Maxwellian molecules) to depend only on the cosine of the deviation angle θ, namely

$$\cos \theta = \frac{\mathbf{v} - \mathbf{w}}{|\mathbf{v} - \mathbf{w}|} \cdot \mathbf{n}.$$

The cut-off assumption means that $b \in L^1(]-1, 1[)$.

If one considers moreover only radially symmetric solutions and one supposes that the collision kernel $b(\theta)$ is constant (say $b(\theta) = 1$), the equation (4) simplifies into the following one dimensional Kac model ([McK66])

$$(5) \qquad \partial_t f(v, t) = \int_{w \in \mathbb{R}} \int_{\theta \in [-\pi/2, \pi/2]} (f(\tilde{v}_*, t) f(\tilde{w}_*, t) - f(v, t) f(w, t)) \, d\theta \, dw.$$

Here, $f(v, t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and the relations between the velocities $(\tilde{v}_*, \tilde{w}_*)$ of two particles before the collision and (v, w) after it are the following

$$\begin{cases} \tilde{v}_* = v\sin\theta + w\cos\theta, \\ \tilde{w}_* = v\cos\theta - w\sin\theta. \end{cases}$$

Assuming now the collision frequency to be constant, say $\theta = \bar{\theta}$, the equation simplifies once more, obtaining

$$\partial_t f(v,t) = \int_{\mathbb{R}} (f(v'_*,t)f(w'_*,t) - f(v,t)f(w,t))\,\mathrm{d}w \tag{6}$$

where (v'_*, w'_*) are now

$$\begin{cases} v'_* = pv + qw \\ w'_* = qv - pw \end{cases} \tag{7}$$

where $p = \sin\bar{\theta}$, $q = \cos\bar{\theta}$. Accordingly,

$$\begin{cases} v'^* = pv + qw \\ w'^* = qv - pw \end{cases} \tag{8}$$

and so, in this case, the transformation $(v,w) \mapsto (v'^*, w'^*)$ is involutive and so its Jacobian satisfies $J = p^2 + q^2 = 1$. This is the major difference between Equation (6) and Equation (2); we would like to investigate the consequences of this difference.

At a formal level, we can check the quantities preserved by the solution $f(t)$ of the non involutive equation (2), as it is usually done with a kinetic model. More precisely, if we consider an initial datum f_0 satisfying

$$f_0 \geq 0, \quad \int_{\mathbb{R}} f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} vf_0(v)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}} v^2 f_0(v)\,\mathrm{d}v = 1$$

we can check the evolution in time of these quantities, called respectively the mass, the momentum and the energy. It is well known that the solution of the Boltzmann equation (4) preserves mass, momentum and energy, whereas the solution of the Kac equation (5) and of the involutive Kac like equation (6) preserves mass and energy but not momentum, unless it is initially zero. At a formal level, it is easy to compute the derivatives in time of mass, momentum and energy of the solution of the non involutive model (2) getting:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} f(v,t)\,\mathrm{d}v = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} vf(v,t)\,\mathrm{d}v = (p+q-1)\int_{\mathbb{R}} vf(v,t)\,\mathrm{d}v$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} v^2 f(v,t)\,\mathrm{d}v = (p^2+q^2-1)\int_{\mathbb{R}} v^2 f(v,t)\,\mathrm{d}v$$

and so for all $t > 0$

$$
\begin{aligned}
&\int_{\mathbb{R}} f(v,t)\,\mathrm{d}v = \int_{\mathbb{R}} f_0(v)\,\mathrm{d}v, \\
&\int_{\mathbb{R}} v f(v,t)\,\mathrm{d}v = e^{(p+q-1)t}\int_{\mathbb{R}} v f_0(v)\,\mathrm{d}v, \qquad (9)\\
&\int_{\mathbb{R}} v^2 f(v,t)\,\mathrm{d}v = e^{(p^2+q^2-1)t}\int_{\mathbb{R}} v^2 f_0(v)\,\mathrm{d}v.
\end{aligned}
$$

So, mass and momentum (if initially zero) are likely to be preserved for all $p \geq q > 0$, whereas energy could be preserved only if $p^2 + q^2 = 1$ and in the other cases could grow to infinity or decrease to zero. In this paper we are going to consider only the conservative case $p^2 + q^2 = 1$.

Concerning the Cauchy problem, the rigorous results of existence, uniqueness and conservation laws are exactly the same for both the involutive and non involutive conservative models (6) and (2).

Theorem 1 (Existence, uniqueness and conservation laws). *We consider an initial datum $f_0 \geq 0$ satisfying the following assumptions:*

$$\int_{\mathbb{R}} f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} v f_0(v)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}} v^2 f_0(v)\,\mathrm{d}v = 1$$

and the following Cauchy problem:

$$
\begin{cases}
\partial_t f(v,t) = \displaystyle\int_{\mathbb{R}} \Big(\frac{1}{J} f(v_*,t) f(w_*,t) - f(v,t) f(w,t)\Big)\,\mathrm{d}w \\
f(v,0) = f_0(v)
\end{cases} \qquad (10)
$$

where (v_*, w_*) are the pre collisional velocities defined through the non involutive relations (3) or the involutive ones (7) and J is the Jacobian of the transformation $(v_*, w_*) \mapsto (v, w)$.

Then, there exists a unique non negative solution $f \in C^1([0,\infty), L^1(\mathbb{R}))$; moreover, it satisfies for all $t > 0$:

$$\int_{\mathbb{R}} f(v,t)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} v f(v,t)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}} v^2 f(v,t)\,\mathrm{d}v = 1.$$

The proof of this theorem follows the same lines as that for the Boltzmann equation for Maxwellian molecules in the cut-off case, which goes back to Morgenstern [Mor54], [Mor55] (see also [Ark72]) and so we don't recall here the method.

It is straightforward that the normalized Maxwellian

$$M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

is a stationary solution for the involutive model (6) satisfying all bounds on mass, momentum and energy; one can check indeed through some calculations that it is also a stationary solution for the non involutive model (2).

It is interesting to consider also the differential equation satisfied by the Fourier transform of the solutions $f(t)$ of the two models, the involutive and the non involutive one. First of all, one can consider a weak form of both equation (6) and (2): for all $\varphi \in C_b(\mathbb{R})$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} f(v,t)\varphi(v)\,\mathrm{d}v = \int_{\mathbb{R}}\int_{\mathbb{R}} f(v,t)f(w,t)(\varphi(v^*) - \varphi(v))\,\mathrm{d}v\,\mathrm{d}w \tag{11}$$

(it is worth noticing that $v^* = v'^*$ in collision rules (1) and (8)). So, both solutions of the involutive and non involutive models are weak solutions of the same equation. Now, letting $\varphi = e^{-iv\xi}$ one gets the equation satisfied by the Fourier transform of both solutions, which reads

$$\frac{\partial \hat{f}}{\partial t}(\xi,t) = \hat{f}(p\xi,t)\hat{f}(q\xi,t) - \hat{f}(\xi,t)\hat{f}(0,t). \tag{12}$$

It is now completely obvious that $\hat{M}(\xi) = e^{-\xi^2/2}$ is a stationary solution of (12).

In addition to mass, momentum and energy, there is a fourth quantity which is very meaningful for the Boltzmann equation and its simplified, involutives models: the entropy, which reads, for $f(v) \geq 0$,

$$H(f) = \int_{\mathbb{R}} f(v)\log f(v)\,\mathrm{d}v. \tag{13}$$

We remark that for f satisfying $\int_{\mathbb{R}} v^2 f(v)\,\mathrm{d}v < \infty$, the entropy $H(f)$ is actually well defined since $0 \leq \int_{\mathbb{R}} f(v)\log^- f(v)\,\mathrm{d}v < \infty$ (see for instance [Tos91]). Let now

$$F_2 = \left\{ g \geq 0, \int_{\mathbb{R}} g(v)\,\mathrm{d}v = 1, \int_{\mathbb{R}} v^2 g(v)\,\mathrm{d}v \leq 1 \right\}. \tag{14}$$

It is straightforward that $M \in F_2$. A classical result called Gibbs' Lemma (see for instance [BPT88]) says that for all $g \in F_2$:

$$H(g) \geq H(M)$$

and equality holds only if $g = M$. So, since every solution $f(t)$ of the involutive equation (6) satisfies $f(t) \in F_2$ for all $t \geq 0$ we get $H(f)(t) \geq H(M)$ for all $t \geq 0$. The so called H theorem, states more precisely that the entropy of a solution $H(f)(t)$ is non increasing as a function of t (we would like to precise that in a physical framework, the entropy would be $-H(f)(t)$).

Let us recall the H Theorem in the Kac like involutive framework.

THEOREM 2 (H Theorem for the involutive model in the even case). *Let $f_0 \geq 0$ an even function satisfying the assumptions:*

$$\int_{\mathbb{R}} f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} v^2 f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} f_0(v)|\log f_0(v)|\,\mathrm{d}v < \infty.$$

Then, the solution $f \in C^1([0,\infty), L^1(\mathbb{R}))$ of the involutive model (6) *with f_0 as initial datum satisfies*

$$f(t)\log f(t) \in L^1(\mathbb{R}), \quad t \geq 0$$

and

$$H(f)(t) = \int_{\mathbb{R}} f(v,t)\log f(v,t)\,\mathrm{d}v$$

is non increasing as a function of t.

The classical proof of this theorem seems to rely strongly on the involutive character of the impact rule (6) and we will recall it later on. We would like to address in this paper the following question: is the entropy H meaningful even in the non involutive framework and in this case, is it possible to avoid involutiveness to prove the H theorem? The answer is positive and one reason for that is quite straightforward. Pareschi and Toscani in [PT06] underline that a Cauchy problem for the Fourier equation (12) has a unique solution in a suitable functional space and this implies that the solutions of the two Cauchy problems (6) and (2) are indeed the same. This argument, however simple it may be, does not seem to have been remarked so far. Another simple proof can be obtained by applying an argument introduced by Bobylev and Toscani in [BT92]. For a spatially homogenous Maxwellian gas, they proved that all convex functionals that satisfy the so called sub-additivity for convolutions are non increasing in time along the solution and the entropy functional satisfies this hypothesis. In this note we would like to stress that other explanations can be given. In particular, the proof of the H theorem does not depend indeed on the involutive nature of the transformation (8) and relies on an inequality (see inequality (16)) satisfied by any function and not only by the solutions of equations (6) or (2). This inequality is related to the well-known Shannon's entropy power inequality, and we will underline this link. Moreover, we will deduce from the H-theorem the exponential L^1 convergence of a solution of both models (6) and (2) to the stationary solution M. We will check all the details which are scattered in the literature since in the case under analysis very few is known and the powerful machinery available for the Kac equation is not allowed. Even though almost all the material contained in this note is already known, we hope that pointing out the basic ingredients needed in order to prove the strong convergence to the stationary state will be helpful with other kinetic models, dissipative for instance, where the stationary state is not the Maxwellian and other entropy functionals have to be considered.

ACKNOWLEDGEMENTS. We would like to thank G. Toscani for many stimulating and fruitful discussions.

2. H THEOREM FOR THE NON INVOLUTIVE MODEL

The rigourous proof of Theorem 2 is quite technical and follows once again the same lines as for the Boltzmann equation ([Ark72]). Let us make nevertheless some remarks to understand how it works. At a formal level, let us compute the derivative in time of H. We will drop the time variable t which does not have any role in the computation. Thanks to the conservation of the mass, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} f(v)\log f(v)\,\mathrm{d}v &= \int_{\mathbb{R}}(\log f(v)+1)\partial_t f(v)\,\mathrm{d}v = \int_{\mathbb{R}}\log f(v)\partial_t f(v)\,\mathrm{d}v \\ &\int_{\mathbb{R}}\int_{\mathbb{R}}\log f(v)\Big(\frac{1}{J}f(v'_*)f(w'_*) - f(v)f(w)\Big)\,\mathrm{d}v\,\mathrm{d}w \end{aligned} \tag{15}$$

where $J = 1$ and v'_*, w'_* are like in (7). We stress that this expression makes sense also for the non involutive model, with (v_*, w_*) instead of (v'_*, w'_*). Now performing the change of variables $(v, w) \mapsto (v'_*, w'_*)$ we get

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}}\log f(v)\Big(\frac{1}{J}f(v'_*)f(w'_*) - f(v)f(w)\Big)\,\mathrm{d}v\,\mathrm{d}w \\ &\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}}(\log f(v'^*) - \log f(v))f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w. \end{aligned}$$

So formally, the H function will be a non increasing function of t as soon as

$$\int_{\mathbb{R}}\int_{\mathbb{R}}(\log f(v'^*) - \log f(v))f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w \le 0$$

or more simply

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\log f(v'^*)f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w \le \int_{\mathbb{R}} f(v)\log f(v)\,\mathrm{d}v. \tag{16}$$

By performing one after the other the two changes of variables $(v, w) \mapsto (w, v)$ and $(v, w) \mapsto (v, -w)$ and remembering that the solution issued from an even initial datum is even itself (as one can check as a byproduct of the proof of existence) we get

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}(\log f(v'^*) - \log f(v))f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}(\log f(v'^*) + \log f(w'^*) - \log f(v) - \log f(w))f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\log\Big(\frac{f(v'^*)f(w'^*)}{f(v)f(w)}\Big)f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w.
\end{aligned}
$$

By the conservation of the mass, which reads

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}(f(v'_*)f(w'_*) - f(v)f(w))\,\mathrm{d}v\,\mathrm{d}w \\
&\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\frac{f(v'_*)f(w'_*)}{f(v)f(w)} - 1\Big)f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w = 0
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\log\Big(\frac{f(v'^*)f(w'^*)}{f(v)f(w)}\Big)f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[\log\Big(\frac{f(v'^*)f(w'^*)}{f(v)f(w)}\Big) - \frac{f(v'_*)f(w'_*)}{f(v)f(w)} + 1\Big]f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w.
\end{aligned}
\tag{17}
$$

We finally exploit the involutive property of the collisions (7), which means $(v'_*, w'_*) = (v'^*, w'^*)$ and the elementary inequality $\log x - x + 1 \le 0$ so that we can get $\frac{\mathrm{d}H}{\mathrm{d}t}(t) \le 0$. Once this formal calculation is performed, it is possible to make it rigourous by an approximation procedure which proves in the end that the H function is non increasing along the solution. What we actually proved is something stronger: for all $f(v) \ge 0$, even, independent of the time t, the inequality (16) holds true.

Let us come back to the non involutive model (2) and the corresponding Cauchy problem. As we have already recalled, the stationary solution of (2) is again the Maxwellian $M(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2}$ (given the same bounds on the initial mass, momentum and energy). One could wonder whether in this case, the functional H still has the physical meaning of an entropy, or there exists some other functional which would be more adapted to this situation and which would be proved to be minimized on the stationary state M. The answer of this question lies in a paper by G. Toscani ([Tos99]), where he proved that if one looks for a strongly convex functional on the class of probability densities with fixed mass and energy and which assumes its minimum on a given function, there is only one possible choice, depending strongly on the minimizer. So, in this case, the choice is the H functional (13).

In order to prove that the H function is again decreasing along the solution of the non involutive equation (2), we get

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) \leq 0 \Leftrightarrow \int_{\mathbb{R}} \int_{\mathbb{R}} \log f(v^*) f(v) f(w) \,\mathrm{d}v \,\mathrm{d}w \leq \int_{\mathbb{R}} f(v) \log f(v) \,\mathrm{d}v$$

Since $v'^* = v^*$ as we already noticed, this is again inequality (16). Now, the collision rule (1) is not involutive and it wouldn't be possible to repeat the same argument as before with the corresponding changes of variables. Nevertheless, from inequality (16) onward we did not exploit the fact that $f(t)$ was a solution of the involutive model so inequality (16) have been proved to be true also for even solutions of the non involutive model (2) and we got the H Theorem in this framework.

Theorem 3 (H Theorem for the non involutive model in the even case). *Let $f_0 \geq 0$ an even function satisfying the assumptions:*

$$\int_{\mathbb{R}} f_0(v) \,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} v^2 f_0(v) \,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} f_0(v) |\log f_0(v)| \,\mathrm{d}v < \infty.$$

Then, the solution $f \in C^1([0,\infty), L^1(\mathbb{R}))$ of the non involutive model (2) *with f_0 as initial datum satisfies*

$$f(t) \log f(t) \in L^1(\mathbb{R}), \quad t \geq 0$$

and

$$H(f)(t) = \int_{\mathbb{R}} f(v,t) \log f(v,t) \,\mathrm{d}v$$

is non increasing as a function of t.

Let us concentrate again on inequality (16), which we rewrite here explicitly

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \log f(pv + qw) f(v) f(w) \,\mathrm{d}v \,\mathrm{d}w \leq \int_{\mathbb{R}} f(v) \log f(v) \,\mathrm{d}v.$$

The change of variables $(v,w) \mapsto (pv + qw, w)$ allows us to recognize in this inequality a convolution operator. Denoting $f_p(v) = \frac{1}{p} f\left(\frac{v}{p}\right)$, we actually get

$$\int_{\mathbb{R}} f_p * f_q(v) \log f(v) \,\mathrm{d}v \leq \int_{\mathbb{R}} f(v) \log f(v) \,\mathrm{d}v. \tag{18}$$

We would like now to remark that inequality (18) might be proved in a completely different way. We begin by noticing that for all $f \geq 0$ such that $\int_{\mathbb{R}} f(v) \,\mathrm{d}v = 1$ the following inequality holds true:

$$\int_{\mathbb{R}} f_p * f_q(v) \log f(v) \,\mathrm{d}v \leq \int_{\mathbb{R}} f_p * f_q(v) \log(f_p * f_q)(v) \,\mathrm{d}v; \tag{19}$$

this is a simple consequence of the convexity of the function $z \log z$, which implies for $z \geq 0$ $z \log z \geq z - 1$ and so

$$\begin{aligned}
&\int_{\mathbb{R}} f_p * f_q(v) \log f(v)\, \mathrm{d}v - \int_{\mathbb{R}} f_p * f_q(v) \log(f_p * f_q(v))\, \mathrm{d}v \\
&\quad = -\int_{\mathbb{R}} \Big(\frac{f_p * f_q(v)}{f(v)} \log \frac{f_p * f_q(v)}{f(v)} \Big) f(v)\, \mathrm{d}v \\
&\quad \leq \int_{\mathbb{R}} \Big(1 - \frac{f_p * f_q(v)}{f(v)} \Big) f(v)\, \mathrm{d}v \\
&\quad = \int_{\mathbb{R}} (f(v) - f_p * f_q(v))\, \mathrm{d}v = 0.
\end{aligned}$$

So, in order to prove inequality (16), it is enough to prove the stronger inequality

$$\int_{\mathbb{R}} f_p * f_q(v) \log(f_p * f_q)(v)\, \mathrm{d}v \leq \int_{\mathbb{R}} f(v) \log f(v)\, \mathrm{d}v. \tag{20}$$

But this inequality is a particular case of the well-known Shannon's entropy power inequality ([Sha48], [Bla65], [Sta59], [Tos91]) which reads as follows.

Theorem 4 (Shannon's inequality). *Let $f \geq 0$, $g \geq 0$ such that $\int_{\mathbb{R}} f(v)\, \mathrm{d}v = \int_{\mathbb{R}} g(v)\, \mathrm{d}v = 1$ and that*

$$\int_{\mathbb{R}} f(v)(v^2 + \log f(v))\, \mathrm{d}v < \infty, \quad \int_{\mathbb{R}} g(v)(v^2 + \log g(v))\, \mathrm{d}v < \infty.$$

The following inequality holds true

$$e^{-2\int_{\mathbb{R}} f*g(v) \log(f*g)(v)\, \mathrm{d}v} \geq e^{-2\int_{\mathbb{R}} f(v) \log f(v)\, \mathrm{d}v} + e^{-2\int_{\mathbb{R}} g(v) \log g(v)\, \mathrm{d}v} \tag{21}$$

with equality if and only if f and g are suitable Gaussian functions.

Letting $f = f_p$, $g = f_q$ and remembering that $\int_{\mathbb{R}} f_p(v)\, \mathrm{d}v = \int_{\mathbb{R}} f_q(v)\, \mathrm{d}v = 1$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} f_p(v) \log f_p(v)\, \mathrm{d}v &= \log \frac{1}{p} + \int_{\mathbb{R}} f(v) \log f(v)\, \mathrm{d}v \\
\int_{\mathbb{R}} f_q(v) \log f_q(v)\, \mathrm{d}v &= \log \frac{1}{q} + \int_{\mathbb{R}} f(v) \log f(v)\, \mathrm{d}v
\end{aligned}$$

and so, by (21) and $p^2 + q^2 = 1$ we get

$$
\begin{aligned}
-2 \int_{\mathbb{R}} & f_p * f_q(v) \log(f_p * f_q)(v) \, \mathrm{d}v \\
& \geq \log(e^{\log(p^2) - 2\int_{\mathbb{R}} f(v) \log f(v) \, \mathrm{d}v} + e^{\log(q^2) - 2\int_{\mathbb{R}} f(v) \log f(v) \, \mathrm{d}v}) \\
& = \log(e^{-2\int_{\mathbb{R}} f(v) \log f(v) \, \mathrm{d}v}) = -2 \int_{\mathbb{R}} f(v) \log f(v) \, \mathrm{d}v
\end{aligned}
$$

which is nothing but inequality (20).

2.1. The Fisher Information

We would like to recall here another functional: the Fisher information, which reads

$$I(f) = \int_{\mathbb{R}} \frac{(f'(v))^2}{f(v)} \mathrm{d}v = 4 \int_{\mathbb{R}} \left(\frac{\mathrm{d}}{\mathrm{d}v} \sqrt{f(v)} \right)^2 \mathrm{d}v.$$

In the classical case of the Kac equation, Mc Kean proved in [McK66] that the Fisher information is a decreasing functional. His proof depends on the form of the Kac equation and does not apply unmodified to the Kac-like models. Nevertheless, the result still holds true for any solution $f(t)$ of the Kac-like involutive model (6) or the non involutive one (2).

Proposition 5. *Let $f_0 \geq 0$ an even function satisfying the assumptions:*

$$\int_{\mathbb{R}} f_0(v) \, \mathrm{d}v = 1, \quad \int_{\mathbb{R}} v^2 f_0(v) \, \mathrm{d}v = 1,$$
$$\int_{\mathbb{R}} f_0(v) \, |\log f_0(v)| \, \mathrm{d}v < \infty, \quad \int_{\mathbb{R}} \left(\frac{\mathrm{d}}{\mathrm{d}v} \sqrt{f_0(v)} \right)^2 \mathrm{d}v < \infty.$$

Then, the solutions $f \in C^1([0, \infty), L^1(\mathbb{R}))$ of both Kac-like models (2) *and* (6) *with f_0 as initial datum satisfy*

$$\int_{\mathbb{R}} \left(\frac{\mathrm{d}}{\mathrm{d}v} \sqrt{f(v,t)} \right)^2 \mathrm{d}v < \infty, \quad t \geq 0$$

and

$$I(f)(t) = \int_{\mathbb{R}} \left(\frac{\mathrm{d}}{\mathrm{d}v} \sqrt{f(v,t)} \right)^2 \mathrm{d}v$$

is non increasing as a function of t.

The proof relies on the following result, relating the H functional to the Fisher information for a function $f(v)$, established by Mc Kean in [McK66] (we underline that in the original paper by Mc Kean, the H functional is $-\int_{\mathbb{R}} f(v)\log f(v)\,\mathrm{d}v$).

Proposition 6 ([McK66]). *Let $f \geq 0$ such that $\int_{\mathbb{R}} f(v)\,\mathrm{d}v = 1$, $\int_{\mathbb{R}} v^2 f(v)\,\mathrm{d}v < \infty$. Let*

$$H(f) = \int_{\mathbb{R}} f(v)\log f(v)\,\mathrm{d}v, \quad I(f) = \int_{\mathbb{R}} \frac{(f'(v))^2}{f(v)}\,\mathrm{d}v$$

and $f_\delta(v) = f * \varphi_\delta(v)$, where $\varphi(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$ and $\varphi_\delta = \frac{1}{\sqrt{\delta}}\varphi\big(\frac{v}{\sqrt{\delta}}\big)$. Then, $H(f_\delta)$ is a decreasing convex function of $\delta \geq 0$ with slope $-\frac{1}{2}I(f)$:

$$\lim_{\delta \downarrow 0} \frac{1}{\delta}(H(f_\delta) - H(f)) = -\frac{1}{2}I(f)$$

and $I(f) < \infty$ only if $H(f) < \infty$.

Proof of Proposition 5. We are going to explain how to modify the classical proof by Mc Kean. As we have already stressed, the following equality (17) holds true for both models:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f)(t) = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{f(v'_*)f(w'_*)}{f(v)f(w)} - \log\Big(\frac{f(v'^*)f(w'^*)}{f(v)f(w)}\Big) - 1\right] f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w$$

where, following the notations already used in the Introduction,

$$\begin{cases} v'^* = v'_* = pv + qw \\ w'^* = w'_* = qv - pw \end{cases}$$

and we dropped the t variable just in order to simplify the expression. In what follows we will write v_* instead of $v'^* = v'_*$ and w_* instead of $w'^* = w'_*$. We are going to prove that for $\delta > 0$ fixed

$$\frac{\mathrm{d}}{\mathrm{d}t}(H(f)(t) - H(f_\delta)(t)) \leq 0, \quad t > 0$$

which reads precisely

$$\begin{aligned} &\int_{v\in\mathbb{R}}\int_{w\in\mathbb{R}}\Big(\frac{f_\delta(v_*)f_\delta(w_*)}{f_\delta(v)f_\delta(w)} - \log\Big(\frac{f_\delta(v_*)f_\delta(w_*)}{f_\delta(v)f_\delta(w)}\Big) - 1\Big) f_\delta(v)f_\delta(w)\,\mathrm{d}v\,\mathrm{d}w \\ &\quad \leq \int_{v\in\mathbb{R}}\int_{w\in\mathbb{R}}\Big(\frac{f(v_*)f(w_*)}{f(v)f(w)} - \log\Big(\frac{f(v_*)f(w_*)}{f(v)f(w)}\Big) - 1\Big) f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w. \end{aligned} \tag{22}$$

Assuming therefore that (22) holds true, we get

$$(H(f)(t_1) - H(f_\delta)(t_1)) - (H(f)(t_2) - H(f_\delta)(t_2)) \leq 0, \quad t_1 \geq t_2$$

and so, by Proposition 6,

$$\lim_{\delta \downarrow 0} \frac{1}{\delta}((H(f)(t_1) - H(f_\delta)(t_1)) - (H(f)(t_2) - H(f_\delta)(t_2)))$$
$$= \frac{1}{2}(I(f)(t_1) - I(f)(t_2)) \leq 0.$$

That means precisely that $I(f)(t)$ is decreasing in time.

In order to prove inequality (22), we remark first that for v, w, ξ and η in $\mathbb{R}$ we have

$$(v_* - \xi_*)^2 + (w_* - \eta_*)^2 = (v - \xi)^2 + (w - \eta)^2. \tag{23}$$

Since the function $g(s) = s - \log s - 1$ is convex on $s > 0$ we would like to apply Jensen's inequality. We write for fixed $v, w \in \mathbb{R}$:

$$\frac{f_\delta(v_*)f_\delta(w_*)}{f_\delta(v)f_\delta(w)} = \int_{\xi \in \mathbb{R}} \int_{\eta \in \mathbb{R}} \frac{e^{-(v_*-\xi)^2/2\delta}}{\sqrt{2\pi\delta}} \frac{e^{-(w_*-\eta)^2/2\delta}}{\sqrt{2\pi\delta}} \frac{f(\xi)f(\eta)}{f_\delta(v)f_\delta(w)} \mathrm{d}\xi\,\mathrm{d}\eta = A.$$

Performing the change of variables $(v, w) \mapsto (v_*, w_*)$ which is involutive and using equality (23), we get

$$A = \int_{\xi \in \mathbb{R}} \int_{\eta \in \mathbb{R}} \frac{e^{-(v_*-\xi_*)^2/2\delta}}{\sqrt{2\pi\delta}} \frac{e^{-(w_*-\eta_*)^2/2\delta}}{\sqrt{2\pi\delta}} \frac{f(\xi_*)f(\eta_*)}{f_\delta(v)f_\delta(w)} \mathrm{d}\xi\,\mathrm{d}\eta$$
$$= \int_{\xi \in \mathbb{R}} \int_{\eta \in \mathbb{R}} \frac{f(\xi_*)f(\eta_*)}{f(\xi)f(\eta)} \frac{e^{-(v-\xi)^2/2\delta}}{\sqrt{2\pi\delta}} \frac{e^{-(w-\eta)^2/2\delta}}{\sqrt{2\pi\delta}} \frac{f(\xi)f(\eta)}{f_\delta(v)f_\delta(w)} \mathrm{d}\xi\,\mathrm{d}\eta.$$

Now, for all $v, w \in \mathbb{R}$ fixed we denote

$$\mathrm{d}\mu(\xi, \eta)_{v,w} = \frac{e^{-(v-\xi)^2/2\delta}}{\sqrt{2\pi\delta}} \frac{e^{-(w-\eta)^2/2\delta}}{\sqrt{2\pi\delta}} \frac{f(\xi)f(\eta)}{f_\delta(v)f_\delta(w)} \mathrm{d}\xi\,\mathrm{d}\eta$$

and so

$$A = \int_{\xi \in \mathbb{R}} \int_{\eta \in \mathbb{R}} \frac{f(\xi_*)f(\eta_*)}{f(\xi)f(\eta)} \mathrm{d}\mu(\xi, \eta)_{v,w}$$

with

$$\int_{\xi \in \mathbb{R}} \int_{\eta \in \mathbb{R}} \mathrm{d}\mu(\xi, \eta)_{v,w} = 1.$$

Since g is convex, by Jensen's inequality we get

$$g\Big(\frac{f_\delta(v_*)f_\delta(w_*)}{f_\delta(v)f_\delta(w)}\Big) \le \int_{\xi\in\mathbb{R}}\int_{\eta\in\mathbb{R}} g\Big(\frac{f(\xi_*)f(\eta_*)}{f(\xi)f(\eta)}\Big)\,\mathrm{d}\mu(\xi,\eta)_{v,w}$$

and so

$$\begin{aligned}
&f_\delta(v)f_\delta(w)g\Big(\frac{f_\delta(v_*)f_\delta(w_*)}{f_\delta(v)f_\delta(w)}\Big)\\
&\quad\le \int_{\xi\in\mathbb{R}}\int_{\eta\in\mathbb{R}} f_\delta(v)f_\delta(w)\Big(\frac{f(\xi_*)f(\eta_*)}{f(\xi)f(\eta)} - \log\Big(\frac{f(\xi_*)f(\eta_*)}{f(\xi)f(\eta)}\Big) - 1\Big)\,\mathrm{d}\mu(\xi,\eta)_{v,w}\\
&\quad= \int_{\xi\in\mathbb{R}}\int_{\eta\in\mathbb{R}} \Big(\frac{f(\xi_*)f(\eta_*)}{f(\xi)f(\eta)} - \log\Big(\frac{f(\xi_*)f(\eta_*)}{f(\xi)f(\eta)}\Big) - 1\Big)f(\xi)f(\eta)\\
&\qquad\times\frac{e^{-(v-\xi)^2/2\delta}}{\sqrt{2\pi\delta}}\frac{e^{-(w-\eta)^2/2\delta}}{\sqrt{2\pi\delta}}\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}$$

Integrating in v and w we get inequality (22) (with ξ, η instead of v and w).

REMARK. A simpler proof of the non increasing property of the Fisher information can be obtained as for the entropy by applying the general result by Bobylev and Toscani [BT92].

3. AN APPLICATION: EXPONENTIAL CONVERGENCE IN L^1 TO THE STATIONARY STATE

We are interested now in the long time behavior of the solution f with respect to the stationary solution. We would like to show that the solution $f(t)$ of the non involutive equation (2) satisfies the following convergence result.

THEOREM 7. *Let $p \ge q > 0$ such that $p^2 + q^2 = 1$. If f_0 is an even, positive function satisfying the following bounds: for $s \in (2,3]$ and $\rho > 0$ suitably chosen,*

$$\int_{\mathbb{R}} f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} v^2 f_0(v)\,\mathrm{d}v = 1,$$

$$\int_{\mathbb{R}} |v|^s f_0(v)\,\mathrm{d}v < \infty,$$

$$\|\sqrt{f_0}\|_{H^1} < \infty, \quad \|f_0\|_{H^\rho} < \infty,$$

then the solution $f(t)$ of the Cauchy problem for the non involutive model (2) *converges exponentially fast to the stationary solution in L^1. More precisely, there exist two positive constants C_1, C_2 such that*

$$\|f(t) - M\|_{L^1} \leq C_1 e^{C_2(p^s+q^s-1)t}, \quad t \geq 0.$$

In [PT06], L. Pareschi and G. Toscani have proved that both the solutions of (6) and (2) converge exponentially fast to the stationary solution M in a Fourier based distance. Let us recall their framework. Let $s > 0$, $A_s = \{k \in \mathbb{N} : 0 \leq k \leq \lfloor s \rfloor\}$ (or $A_s = \{k \in \mathbb{N} : 0 \leq k \leq s-1\}$ if $s \in \mathbb{N}\backslash\{0\}$) and I_k be fixed non negative numbers for $k \in A_s$. We introduce

$$M_s = \left\{ f \geq 0 : \int_{\mathbb{R}} (1 + |v|^s) f(v)\,\mathrm{d}v < \infty, \int_{\mathbb{R}} v^k f(v)\,\mathrm{d}v = I_k, k \in A_k \right\}$$

and let

$$d_s(f,g) := \sup_{\xi \in \mathbb{R}} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^s}.$$

It is easy to show that if $f, g \in M_s$, then $d_s(f,g) < \infty$. So, if f_0 is the initial datum of the Cauchy problem (10), $\int_{\mathbb{R}} |v|^s f_0(v)\,\mathrm{d}v < \infty$ for $2 < s \leq 3$ and $I_0 = 1$, $I_1 = 0$, $I_2 = 1$, then we get $d_s(f_0, M) < \infty$. It is worth remembering that Pareschi and Toscani in [PT06] proved also that the s-th moment $\int_{\mathbb{R}} |v|^s f(v,t)\,\mathrm{d}v$ is uniformly bounded in time along the solution $f(t)$.

The convergence result by Pareschi and Toscani is as follows.

Theorem 8 (Weak Fourier-based convergence [PT06]). *Let $2 < s \leq 3$; we consider an initial datum $f_0 \geq 0$ satisfying the assumptions:*

$$\int_{\mathbb{R}} f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} v f_0(v)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}} v^2 f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}} |v|^s f_0(v)\,\mathrm{d}v < \infty$$

and the solutions $f \in C^1([0,\infty), L^1(\mathbb{R}))$ of the Cauchy problems (10) *for both the non involutive and involutive models and $M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$. Then, we get*

$$d_s(f(t), M) \leq e^{(p^s+q^s-1)t} d_s(f_0, M), \quad t \geq 0.$$

In order to convert this weak Fourier based convergence of f to the stationary state into a strong L^1, one possible method, which has been successfully performed in several papers ([CGT99], [BCT05], [CT06]) is the following: starting from a smooth initial datum (belonging for instance to some Sobolev space) one proves first the uniform propagation of the smoothness along the solution and then, by interpolating between the very weak Fourier based distance decreasing exponentially fast and the strong uniform bound, one gets the L^1 distance decreasing exponentially fast at a rate depending on the spaces involved.

PROOF OF THEOREM 7. The proof of Theorem 7 is classical and is detailed for the Kac equation in [CGT99]. Nevertheless we go here through this proof for the reader's convenience, because there are a few points which are scattered in the literature and we found it not trivial to track every detail. First of all, it is easy to prove the two following interpolation bounds (see Theorems 4.1 and 4.2 in [CGT99]): for $s \in (2, 3]$, there exists a positive constant C such that

$$\|h\|_{L^1} \leq C\|h\|_{L^2}^{2s/(1+2s)} \| \, |v|^s h\|_{L^1}^{1/(1+2s)}$$

and there exist positive constants M, N, α and β such that

$$\|h\|_{L^2} \leq C\Big(\sup_{\mathbb{R}} \frac{|\hat{h}(\xi)|}{|\xi|^s} \Big)^{\alpha} (\|h\|_{H^M} + \|h\|_{H^N})^{\beta}.$$

So, letting $h = f(t) - M$, we get

$$\begin{aligned} \|f(t) - M\|_{L^1} \leq C d_s(f(t), M)^{\tilde{\alpha}} (\|f(t)\|_{H^M} + \|f(t)\|_{H^N} + \|M\|_{H^M} \\ + \|M\|_{H^N})^{\tilde{\beta}} (\| \, |v|^s f(t)\|_{L^1} + \| \, |v|^s M\|_{L^1})^{\tilde{\gamma}} \end{aligned}$$

for suitable exponents $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$. Therefore, knowing the exponential convergence of f to M in the d_s distance as in Theorem 8 and the uniform boundedness of the s-th moment ([PT06]), we will get the L^1 exponential convergence if we are able to prove the uniform boundedness of f in a suitable Sobolev space H^ρ. Of course, we need to assume $f_0 \in H^\rho$.

Let us recall how to prove the uniform boundedness of f in a Sobolev space H^ρ. First of all it is easy to prove that if $f_0 \in H^\rho$, than $f(t) \in H^\rho$ for all $t > 0$, without any uniformity in time. The goal is therefore to get the following differential inequality: for two positive constants H and K

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f(t)\|_{H^\rho}^2 \leq -H\|f(t)\|_{H^\rho}^2 + K, \quad t \geq t_0 \tag{24}$$

so that

$$\|f(t)\|_{H^\rho}^2 \leq C \max(\|f(t_0)\|_{H^\rho}^2, 1), \quad t \geq t_0.$$

In order to get inequality (24), we consider Equation (2) in the Fourier variable

$$\frac{\partial \hat{f}}{\partial t}(\xi, t) = \hat{f}(p\xi, t)\hat{f}(q\xi, t) - \hat{f}(\xi, t)$$

where $\hat{f}$ is a real (because f is even) and write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{H^\rho}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(\xi,t)^2\,\mathrm{d}\xi \\
&= 2\int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(\xi,t)\partial_t\hat{f}(\xi,t)\,\mathrm{d}\xi \\
&= 2\int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(\xi,t)(\hat{f}(p\xi,t)\hat{f}(q\xi,t)-\hat{f}(\xi,t))\,\mathrm{d}\xi \\
&= 2\int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(\xi,t)\hat{f}(p\xi,t)\hat{f}(q\xi,t)\,\mathrm{d}\xi - 2\int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(\xi,t)^2\,\mathrm{d}\xi \\
&= 2\int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(\xi,t)\hat{f}(p\xi,t)\hat{f}(q\xi,t)\,\mathrm{d}\xi - 2\|f(t)\|_{H^\rho}^2 \\
&\le -2\|f(t)\|_{H^\rho}^2 + \int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(\xi,t)^2\,\mathrm{d}\xi + \int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(p\xi,t)^2\hat{f}(q\xi,t)^2\,\mathrm{d}\xi \\
&= -\|f(t)\|_{H^\rho}^2 + \int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(p\xi,t)^2\hat{f}(q\xi,t)^2\,\mathrm{d}\xi.
\end{aligned}$$

Now, it would be enough to obtain the following inequality

$$\int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(p\xi,t)^2\hat{f}(q\xi,t)^2\,\mathrm{d}\xi \le \frac{1}{2}\|f(t)\|_{H^\rho}^2 + K, \quad t \ge t_0 \tag{25}$$

where $K > 0$ is independent of t. We split the integral in (25) into two parts

$$\int_{\mathbb{R}} |\xi|^{2\rho}\hat{f}(p\xi,t)^2\hat{f}(q\xi,t)^2\,\mathrm{d}\xi = \int_{|\xi|\le R} + \int_{|\xi|>R} = A + B$$

where R will be chosen later. Let us estimate first the term in A. Since $|\hat{f}(\xi,t)| \le 1$ for $\xi \in \mathbb{R}$ and $t \ge 0$ we simply get

$$\begin{aligned}
&\int_{|\xi|\le R} |\xi|^{2\eta}\hat{f}(p\xi,t)^2\hat{f}(q\xi,t)^2\,\mathrm{d}\xi \\
&\qquad \le \int_{|\xi|\le R} |\xi|^{2\eta}\,\mathrm{d}\xi = \frac{2}{2\eta+1}R^{2\eta+1}, \quad t \ge 0.
\end{aligned}$$

Let us come to the term in B, where we are going to exploit the H functional. It is crucial that not only $H(f)(t)$ is a decreasing function, but more precisely that

$$\lim_{t\to\infty} H(f)(t) = H(M) \tag{26}$$

where M is the Maxwellian $M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$. We will come back to this point afterward. As a consequence, thanks to the Csiszár-Kullback-Pinsker inequality

$$\|f(t) - M\|_{L^1}^2 \leq 2(H(f)(t) - H(M)), \quad t > 0$$

we get

$$|\hat{f}(\xi, t) - e^{-\xi^2/2}|^2 \leq 2(H(f)(t) - H(M)), \quad \xi \in \mathbb{R},\ t > 0$$

so that, for all $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$\hat{f}(\xi, t) \leq \varepsilon + e^{-\xi^2/2}, \quad \xi \in \mathbb{R},\ t \geq t_0.$$

We remark therefore that

$$|\hat{f}(p\xi, t)| \leq \varepsilon + e^{-(p\xi)^2/2} \leq \varepsilon + e^{-(pR)^2/2}, \quad |\xi| > R,\ t \geq t_0$$

so, if $R^2 = \frac{2}{p^2} \log \frac{1}{\varepsilon}$ we get

$$|\hat{f}(p\xi, t)| \leq 2\varepsilon, \quad |\xi| > R,\ t \geq t_0.$$

We can deduce for $t \geq t_0$:

$$\begin{aligned} \int_{|\xi|>R} |\xi|^{2\rho} \hat{f}(p\xi, t)^2 \hat{f}(q\xi, t)^2 \,\mathrm{d}\xi &\leq (2\varepsilon)^2 \int_{|\xi|>R} |\xi|^{2\rho} \hat{f}(q\xi, t)^2 \,\mathrm{d}\xi \\ &\leq \frac{\varepsilon}{q^{2\rho+1}} \int_{\mathbb{R}} |\xi|^{2\rho} \hat{f}(\xi, t)^2 \,\mathrm{d}\xi = \varepsilon \|f(t)\|_{H^\rho}^2. \end{aligned}$$

We have obtained

$$\int_{\mathbb{R}} |\xi|^{2\rho} \hat{f}(p\xi, t)^2 \hat{f}(q\xi, t)^2 \,\mathrm{d}\xi \leq \varepsilon \|f(t)\|_{H^\rho}^2 + \frac{2}{2\eta + 1} R^{2\eta+1}, \quad t \geq t_0.$$

Letting ε be fixed such that $\varepsilon \leq \frac{1}{2}$, we get the desired estimate.

At this point, let us recall how to prove that

$$\lim_{t \to \infty} H(f)(t) = H(M) \tag{27}$$

where M is the Maxwellian $M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$.

A classical procedure shows that if there exists $s > 2$ such that $\| |v|^s f(t)\|_{L^1} \leq C$ for all $t > 0$, then $f(t) \rightharpoonup M$ for $t \to \infty$ in weak L^1 ([Mor55]) and so $H(M) \leq \liminf_{t\to\infty} H(f)(t) = \lim_{t\to\infty} H(f)(t)$ ([Elm84]).

In order to conclude that $\lim_{t\to\infty} H(f)(t) = H(M)$ one can pass through an approximation procedure by convolution with a Gaussian kernel, as is done for example in [Mor55]. We recall hereafter the steps and the suitable references for the reader's convenience.

- First of all we remark that among all the positive, even functions with prescribed values of mass, momentum and energy, inequality (16) is an equality if and only if f is a Gaussian function. This is due to the fact that in (17) we have $\frac{f(v'^*)f(w'^*)}{f(v)f(w)} = 1$ if and only if f is a Gaussian function (see [Vil02], chapter 2.C, section 4.3).
- Then, for $\delta > 0$ fixed, we consider the convolution $f_\delta(t) = f(t) * \varphi_\delta$ where $\varphi_\delta = \frac{1}{\sqrt{2\pi\delta}} e^{-v^2/2\delta}$ and we remark that $f_\delta(t)$ is the solution of Equation (2) with $f_0 * \varphi_\delta$ as initial data. In this case, the energy becomes

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} v^2 f(v-w)\varphi_\delta(w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} (v-w+w)^2 f(v-w)\varphi_\delta(w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} (v-w)^2 f(v-w)\varphi_\delta(w)\,\mathrm{d}w\,\mathrm{d}v + \int_{\mathbb{R}}\int_{\mathbb{R}} f(v-w) w^2 \varphi_\delta(w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad = \int_{\mathbb{R}} \varphi_\delta(w)\Big(\int_{R} (v-w)^2 f(v-w)\,\mathrm{d}v\Big)\,\mathrm{d}w + \int_{\mathbb{R}} w^2\varphi_\delta(w)\Big(\int_{\mathbb{R}} f(v-w)\,\mathrm{d}v\Big)\,\mathrm{d}w \\
&\quad = 1 + \delta \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} v^2 e^{-v^2/2}\,\mathrm{d}v = 1 + \delta.
\end{aligned}$$

- We remark that since $f(t) \rightharpoonup M$ for $t \to \infty$ in weak L^1, then $f_\delta \to M * \varphi_\delta$ uniformly on compact sets and therefore $\lim_{t\to\infty} H(f_\delta) = H(M * \varphi_\delta)$, for fixed δ. It is crucial here to stress that $M * \varphi_\delta$ is a Gaussian function itself (see [Mor55]).
- Since $\lim_{\delta\to 0} M * \varphi_\delta = M$ in L^1, it is possibile to prove that $\lim_{\delta\to 0} H(M * \varphi_\delta) = H(M)$ (see [Tos91]).
- Following McKean [McK66], we remark that $H(f_\delta) \le H(f)$ for all $\delta > 0$.
- For $t > 0$ fixed, thanks to Proposition 6 we write

$$0 \le H(f)(t) - H(f_\delta)(t) = \frac{1}{2}\int_{r=0}^{\delta} I((f_r)(t))\,\mathrm{d}r$$

and remembering that $I((f_r)(t)) \le I(f)(t)$ ([McK66]) and that $I(f)(t)$ is decreasing in t, as we showed in Proposition 5, we can get

$$\begin{aligned}
\frac{1}{2}\int_{r=0}^{\delta} I((f_r)(t))\,\mathrm{d}r &\le \frac{1}{2}\int_{r=0}^{\delta} I((f(t))\,\mathrm{d}r \\
&\le \frac{1}{2}\int_{r=0}^{\delta} I(f_0)\,\mathrm{d}r = \frac{1}{2}\delta I(f_0)) \to 0, \quad \delta \to 0
\end{aligned}$$

and so $\lim_{\delta\to 0}(H(f)(t) - H(f_\delta)(t)) = 0$ uniformly in $t > 0$. We stress that it is essential here to assume $\|\sqrt{f_0}\|_{H^1} < \infty$ for the Fisher functional to be finite along the solution.

- The last step is as follows. Let $\varepsilon > 0$; we get

$$|H(f)(t) - H(M)| \leq |H(f)(t) - H(f_\delta)(t)| + |H(f_\delta)(t) - H(M * \varphi_\delta)| + |H(M * \varphi_\delta) - H(M)| \leq 3\varepsilon$$

for $\delta \leq \delta_0(\varepsilon)$ fixed and $t \geq t_0(\varepsilon)$ and this achieves the goal.

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Received 6 October 2009,
and in revised form 14 October 2009.

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