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Algebra — *Symmetric group actions on the cohomology of configurations in* $\mathbb{R}^d$, by Giacomo d'Antonio and Giovanni Gaiffi, presented by Corrado De Concini on 12 March 2010.

Abstract. — In this paper we deal with the action of the symmetric group on the cohomology of the configuration space $C_n(d)$ of n points in $\mathbb{R}^d$. This topic has been studied by several authors and it is well-known that for d even $H^*(C_n(d);\mathbb{C}) \cong 2\,\mathrm{Ind}_{S_2}^{S_n} 1$ while, for d odd, $H^*(C_n(d);\mathbb{C}) \cong \mathbb{C}S_n$.

On the cohomology algebra $H^*(C_n(d);\mathbb{C})$ there is, in addition to the natural S_n-action, an *extended action* of S_{n+1}; this was shown for the case when d is even by Mathieu, Robinson and Whitehouse and the second author using three different methods. For the case when d is odd it was shown by Mathieu (anyway we will give an elementary algebraic construction of the extended action for this case).

The purpose of this article is to present some results that can be obtained, in an elementary way, exploiting the interplay between the extended action and the standard action. Among these we will recall a quick proof for the formula cited above for the case when d is even and show how to extend this proof to the case when d is odd. We will also show how to locate among the homogeneous components of the graded algebra $H^*(C_n(d);\mathbb{C})$ the copies of the standard, sign and standard tensor sign representations and we will give explicit formulas for both the extended and the canonical actions on the low-degree cohomology modules.

Key words: Configuration spaces, symmetric group, representations.

AMS Subject Classification: 20C30, 55R80.

1. Introduction

We are concerned with the action of the symmetric group S_n on the cohomology algebra of the *configuration space of n points in* $\mathbb{R}^d$, that is the space

$$C_n(d) = \{(p_1,\dots,p_n) \in (\mathbb{R}^d)^n : p_i \neq p_j, \forall i \neq j\}.$$

S_n acts on $C_n(d)$ permuting coordinates and this action induces an action on the cohomology algebra $H^*(C_n(d);\mathbb{C})$.

In the particular case $d = 2$, $C_n(2) = M(\mathscr{B}_n)$ is the complement of the (complex) braid arrangement $\mathscr{B}_n$. In [Arn69] Arnol'd gave a presentation for the cohomology algebra $H^*(C_n(2);\mathbb{C})$; he proved that it is the skew-commutative algebra with generators $\{A_{i,j} : i < j\}$ of degree 1 and relations:

$$A_{i,j}A_{i,k} - A_{i,j}A_{j,k} + A_{i,k}A_{j,k} = 0, \quad \forall i < j < k. \tag{1}$$

This result can be generalized to hyperplane arrangements (cfr. [OS80], [OT92], [Yuz01] and [BZ92]).

The action of $\sigma \in S_n$ on $H^*(C_n(2);\mathbb{C})$ can be described as $\sigma A_{i,j} = A_{\sigma i, \sigma j}$ and was studied by Lehrer and Solomon in [LS86] and by Lehrer in [Leh87]. Among other results they proved a formula for the character of this action; precisely the following holds:

$$\chi_{H^*(C_n(2);\mathbb{C})} = 2\,\mathrm{Ind}_{S_2}^{S_n}(1) \tag{2}$$

where 1 is the character of the trivial representation of S_2. In [Gai96] the second author gave a quick proof for this formula introducing an action of S_{n+1} on $H^*(C_n(2);\mathbb{C})$ which restricts to the natural S_n action (this action is called the *extended* or *hidden* action). A similar approach was exploited by Mathieu in [Mat96] and by Robinson and Whitehouse in [RW96].

In the general case, when $d \geq 2$, the algebra $H^*(C_n(d);\mathbb{C})$ can be presented as follows (cfr. [CLM76] and [CT93]): it is the associative graded algebra with generators $\{A_{i,j} : 1 \leq i, j \leq n\}$ (with $i \neq j$; for convenience of notation we allow $i = j$ and then $A_{i,i} = 0$) of degree $d-1$ and relations

$$A_{i,j} = (-1)^d A_{j,i}, \tag{3}$$

$$A_{i,j}A_{h,k} = (-1)^{d-1} A_{h,k}A_{i,j}, \tag{4}$$

$$A_{i,j}A_{i,k} = A_{k,j}(A_{i,k} - A_{i,j}) \quad \text{for } 1 \leq i \leq n \text{ and } j \leq k. \tag{5}$$

Again these results generalize to the complement of a subspace arrangement (cfr. [FZ00], [dLS01] and [DGM00]).

The action of S_n on $H^*(C_n(d);\mathbb{C})$ for arbitrary d was studied by Cohen and Taylor in [CT93] and by Lehrer in [Leh00]. Lehrer provided formulas for the generalized Poincaré polynomials associated to the representations $H^*(C_n(d);\mathbb{C})$. It turns out that there is a qualitative difference between the case when d is even and the case when d is odd. The argument for the case $d = 2$ translates literally to the case when d is even (cfr. [Leh00]) and formula (2) still holds. Also the construction of the extended action can be translated to the case when d is even. For the case when d is odd both Cohen and Taylor in [CT93] and Lehrer in [Leh00] proved, with different arguments, that $H^*(C_n(d);\mathbb{C})$ is the regular representation $\mathbb{C}S_n$.

We will describe (section 2) in an elementary way an S_{n+1} action on $H^*(C_n(d);\mathbb{C})$ for the case when d is odd (the same action was pointed out with a different method in [Mat96]) and use it in section 3 to prove quickly some results of [CT93] and [Leh00]. In addition we will show how the extended action can be used, both in the case when d is even and in the case when d is odd, to locate the copies of the standard, sign and standard tensor sign representations of S_n on the homogeneous components $H^{k(d-1)}(C_n(d);\mathbb{C})$ (section 4) and to prove explicit formulas for the decomposition of the degrees $d-1$ and $2(d-1)$ (section 5).

2. The extended S_{n+1} action

We now discuss the definition of an extended action on $H^*(C_n(d);\mathbb{C})$. We distinguish the case when d is odd and the case when d is even. In the former case we see from relations (3)–(5) that there is an isomorphism of graded S_n-modules

$$H^*(C_n(d); \mathbb{C}) \to H^*(C_n(2); \mathbb{C}) \otimes 1 \tag{6}$$

where 1 is the graded S_n-module whose only non-zero component is the trivial representation at degree $d-1$. There are (at least) three different ways to extend the action of S_n on $H^*(C_n(2); \mathbb{C})$ to an S_{n+1}-action (see [Gai96], [Mat96] and [RW96]); the isomorphism (6) allows us to carry this extended action to $H^*(C_n(d); \mathbb{C})$.

In the case when $d \geq 3$ is odd we can rewrite the relations (3)–(5) as follows:

$$\begin{aligned} &A_{i,j} = -A_{j,i}, \\ &A_{i,j}A_{h,k} = A_{h,k}A_{i,j}, \\ &A_{i,j}A_{i,k} - A_{i,j}A_{j,k} + A_{i,k}A_{j,k} = 0 \end{aligned}$$

We first look at the degree $d-1$; let $V = \mathbb{C}^n$ be the permutation representation, we have an equivariant isomorphism of S_n modules

$$\begin{aligned} \textstyle\bigwedge^2 V &\to H^{d-1}(C_n(d); \mathbb{C}) \\ e_i \wedge e_j &\mapsto A_{i,j}. \end{aligned}$$

The action of S_n on V can be extended to an S_{n+1}-action; from Pieri's rule we see that any extended action must be isomorphic to the standard representation of S_{n+1}, that is $V_{(n,1)} = \ker(x_0 + \cdots + x_n) \subseteq \mathbb{C}^{n+1}$. We choose a basis for $V_{(n,1)}$ of elements $\{v_1, \ldots, v_n\}$ where $v_i = e_i - e_0$; identifying $S_n = \{\sigma \in S_{n+1} : \sigma(0) = 0\}$ we have an S_n-equivariant isomorphism

$$\begin{aligned} \operatorname{Res}_{S_n}^{S_{n+1}} V_{(n,1)} &\to V \\ v_j &\mapsto e_j \end{aligned} \tag{7}$$

and we can define the S_{n+1} action on V as the unique action that makes (7) into an equivariant isomorphism $V_{(n,1)} \to V$.

The S_{n+1} action on V induces an S_{n+1} action on $\bigwedge^2 V \cong H^{d-1}(C_n(d); \mathbb{C})$. We can describe this action as follows: if $\sigma \in S_n$ then $\sigma A_{i,j} = A_{\sigma i, \sigma j}$ and

$$(0,1)A_{i,j} = A_{i,j} - A_{1,j} + A_{1,i} \quad \text{if } 1 < i < j \tag{8}$$

$$(0,1)A_{1,j} = -A_{1,j}. \tag{9}$$

These formulas are obtained easily from the observations that $(0,1)(v_i \wedge v_j) = (v_i - v_1) \wedge (v_j - v_1)$ when $1 < i < j$ and $(0,1)(v_1 \wedge v_j) = -v_1 \wedge (v_j - v_1)$.

In particular we are able to decompose $H^{d-1}(C_n(d); \mathbb{C})$ for every n and d odd ($d \geq 3$):

Proposition 1. *There is an isomorphism of S_{n+1}-modules*

$$H^{d-1}(C_n(d); \mathbb{C}) \cong \bigwedge^2 V_{(n,1)} \cong V_{(n-1,1,1)}.$$

Remark 1. Using Pieri's rules we obtain the following isomorphism of S_n-modules

$$H^{d-1}(C_n(d);\mathbb{C}) = V_{(n-1,1)} \oplus V_{(n-2,1,1)}.$$

From relations (3)–(5) we see that there is an equivariant isomorphism

$$H^*(C_n(d);\mathbb{C}) \cong S(H^{d-1}(C_n(d);\mathbb{C}))/I_{n,d}$$

where $S(H^{d-1}(C_n(d);\mathbb{C}))$ is the symmetric algebra on $H^{d-1}(C_n(d);\mathbb{C})$ and $I_{n,d}$ is the ideal of relations:

$$I_{n,d} = \langle A_{i,j}A_{i,k} - A_{k,j}(A_{i,k} - A_{i,j})\text{: for } 1 \le i \le n \text{ and } j \le k\rangle.$$

In particular, in order to extend the S_n action on $H^*(C_n(d);\mathbb{C})$ we only need to prove that the ideal $I_{n,d}$ is invariant under the action of S_{n+1} on $S(H^{d-1}(C_n(d);\mathbb{C}))$. This is indeed the case and is proved with a short explicit computation. One has to check the equalities (in $H^*(C_n(d);\mathbb{C})$)

$$(0,1)A_{i,j}A_{i,k} = (0,1)A_{k,j}(A_{i,k} - A_{i,j}).$$

We notice that, since the expression above is symmetric in j and k, it suffices to distinguish three cases: the case when $i = 1$, the case when $j = 1$ and the case when $i, j, k \neq 1$.

3. The character of the S_n action on $H^*(C_n(d);\mathbb{C})$

In this section we will recall some results and proofs from [Gai96], [Mat96] and [RW96] and use them to show a quick proof of formula $H^*(C_n(d);\mathbb{C}) \cong \mathbb{C}S_n$ for d odd.

When d is even isomorphism (6) provides us an analogous of [Gai96, Theorem 4.1] (see also [Mat96] and [RW96]), i.e.

$$H^{k(d-1)}(C_n(d);\mathbb{C}) \cong H^{k(d-1)}(C_{n-1}(d);\mathbb{C}) \oplus (H^{(k-1)(d-1)}(C_{n-1}(d);\mathbb{C}) \otimes V_{(n-1,1)})$$

which connects the canonical S_n-action (on the left) with the extended S_n-action on $H^*(C_{n-1}(d);\mathbb{C})$ (on the right).

Consider now the case when d is odd; let $\eta : H^*(C_{n-1}(d);\mathbb{C}) \to H^*(C_n(d);\mathbb{C})$ be the map $A_{i,j} \mapsto A_{i,j}$ (i.e. the map induced by the projection on the first $n-1$ factors $C_n(d) \to C_{n-1}(d)$). If we call $s_j = (j, j+1) \in S_{n+1}$ we have from formulas (8) and (9) that the map η is $\langle s_0, \ldots, s_{n-2}\rangle$-equivariant. Recall the following well known result (see [CT93]):

Proposition 2. *The algebra $H^*(C_n(d);\mathbb{C})$ has a basis given by the elements*

$$A_{i_1,j_1}A_{i_2,j_2}\ldots A_{i_k,j_k}$$

with $i_h < j_h$ and $1 < j_1 < j_2 < \cdots < j_k \le n$.

Such elements are usually called *admissible monomials*. We will write $\chi(n,k)$ for the character of the action of S_n on $H^k(C_n(d);\mathbb{C})$ and $\tilde{\chi}(n,k)$ for the character of the extended action of S_{n+1} on $H^k(C_n(d);\mathbb{C})$.

With these ingredients we can translate almost verbatim the proof of [Gai96, Theorem 4.1] to obtain the following result, which we state for arbitrary $d \geq 2$ (see also [Mat96, Theorem 4.4]).

THEOREM 3. *For any n, k (and for any $d \geq 2$) it holds:*

$$\chi(n,k) = \tilde{\chi}(n-1,k) + p_n\tilde{\chi}(n-1,k-1),$$

where p_n is the character of the standard representation of S_n.

PROOF. We discuss only the case when d is odd. Consider the $\langle s_0,\dots,s_{n-2}\rangle$-submodule $\Omega_{n-1,k} = \eta(H^k(C_{n-1}(d);\mathbb{C})) \subseteq H^k(C_n(d);\mathbb{C})$. We can write

$$H^k(C_n(d);\mathbb{C}) = \Omega_{n-1,k} \oplus N \cdot \Omega_{n-1,k-1} \tag{10}$$

where $N = \bigoplus_{j=1}^{n-1}\mathbb{C}A_{j,n}$ is certainly $\langle s_1,\dots,s_{n-2}\rangle$-invariant but, in general, is not an $\langle s_0,\dots,s_{n-2}\rangle$-submodule.

Now consider the case $k=1$, we have

$$H^1(C_n(d);\mathbb{C}) = \Omega_{n-1,1} \oplus N = \Omega_{n-1,1} \oplus T$$

where T is an $\langle s_0,\dots,s_{n-2}\rangle$-invariant complement of $\Omega_{n-1,1}$ (in particular its restriction to $\langle s_1,\dots,s_{n-2}\rangle$ is isomorphic to N). But $S_{n-1} = \langle s_1,\dots,s_{n-2}\rangle$ permutes the elements $A_{1,n},\dots,A_{n-1,n}$ and therefore

$$\mathrm{Res}^{S_n}_{S_{n-1}} T \cong N \cong V_{(n-1)} \oplus V_{(n-2,1)}$$

where $V_{(n-1)}$ is the trivial representation and $V_{(n-2,1)}$ is the standard representation. By Pieri's rule we have $T \cong V_{(n-1,1)}$ as S_n-module.

We can still write

$$H^k(C_n(d);\mathbb{C}) = \Omega_{n-1,k} \oplus T \cdot \Omega_{n-1,k-1},$$

indeed we have $A_{i,n} \in H^1(C_n(d);\mathbb{C}) \Rightarrow A_{i,n} = \gamma_i^{(1)} + \gamma_i^{(2)}$ with $\gamma_i^{(1)} \in \Omega_{n-1,1}$ and $\gamma_i^{(2)} \in T$. Now let $z = z_0 + \sum_{j=1}^n A_{j,n}z_j \in H^k(C_n(d);\mathbb{C})$ with $z_0 \in \Omega_{n-1,k}$ and for $j>0$, $z_j \in \Omega_{n-1,k-1}$; then we have

$$z = \underbrace{z_0 + \sum_{j=1}^n \gamma_j^{(1)} z_j}_{\in \Omega_{n-1,k}} + \underbrace{\sum_{j=1}^n \gamma_j^{(2)} z_j}_{\in T\cdot\Omega_{n-1,k-1}} .$$

Therefore $H^k(C_n(d);\mathbb{C}) = \Omega_{n-1,k} + T\cdot\Omega_{n-1,k-1}$ and the sum is direct by a dimension argument. In particular we have $\dim T\cdot\Omega_{n-1,k-1} = \dim(T\otimes\Omega_{n-1,k-1})$ and therefore there is an equivariant isomorphism $T\cdot\Omega_{n-1,k-1} \cong T\otimes\Omega_{n-1,k-1}$.

We have proved a decomposition of $\langle s_0, \dots, s_{n-2}\rangle$-modules

$$H^k(C_n(d);\mathbb{C}) \cong H^k(C_{n-1}(d);\mathbb{C}) \oplus (P_n \otimes H^{k-1}(C_{n-1}(d);\mathbb{C})).$$

Now consider the subgroups $H_1 = \langle s_0, \dots, s_{n-2}\rangle$ and $H_2 = \langle s_1, \dots, s_{n-1}\rangle$ of S_{n+1}; these are conjugate subgroups and therefore

$$\mathrm{Res}^{S_{n+1}}_{H_1} H^k(C_n(d);\mathbb{C}) \cong \mathrm{Res}^{S_{n+1}}_{H_2} H^k(C_n(d);\mathbb{C})$$

and the term on the right is the natural S_n action on $H^k(C_n(d);\mathbb{C})$. $\square$

As a consequence we immediately have the following.

Corollary 4. *For any $n > 2$ and any $d \geq 2$ the following equality of S_n-modules holds:*

$$H^*(C_n(d);\mathbb{C}) = \mathrm{Ind}^{S_n}_{S_{n-1}} H^*(C_{n-1}(d);\mathbb{C}).$$

Proof. Call $\chi_n = \sum_{k=0}^{n-1} \chi(n,k)$ the character of the action of S_n on $H^*(C_n(d);\mathbb{C})$ and $\tilde{\chi}_n = \sum_{k=0}^{n-1} \tilde{\chi}(n,k)$ the character of the extended action. Then from theorem 3 and from the fact that $\tilde{\chi}(-1, n-1) = \tilde{\chi}(n-1, n-1) = 0$ we have

$$\chi_n = \sum_{k=0}^{n-1} \chi(n,k) = \sum_{k=0}^{n-1} (\tilde{\chi}(n-1,k) + p_n \tilde{\chi}(n-1,k-1)) = (1+p_n)\tilde{\chi}_{n-1}.$$

Recall that if $H \subseteq G$ is a subgroup and M is a G-module we have $\mathrm{Ind}^G_H \mathrm{Res}^G_H M = M \otimes \mathrm{Ind}^G_H(1)$. In our case we have

$$\mathrm{Ind}^{S_n}_{S_{n-1}} \chi_{n-1} = \mathrm{Ind}^{S_n}_{S_{n-1}} \mathrm{Res}^{S_n}_{S_{n-1}} \tilde{\chi}_{n-1} = (\mathrm{Ind}^{S_n}_{S_{n-1}} 1)\tilde{\chi}_{n-1} = (1+p_n)\tilde{\chi}_{n-1} = \chi_n \qquad \square$$

As remarked in [Gai96, Theorem 4.4], corollary 4 provides a quick proof of Lehrer and Solomon result for d even: $H^*(C_n(d);\mathbb{C}) = 2\,\mathrm{Ind}^{S_n}_{S_2} 1$, since $H^*(C_2(d);\mathbb{C})$ consists of two copies of the trivial representation of S_2. Analogously, when d is odd we can now prove the following result of [CT93] and [Leh00]:

Theorem 5. *When d is odd we have:*

$$H^*(C_n(d);\mathbb{C}) \cong \mathbb{C}S_n.$$

Proof. When $d = 1$ the connected components of $C_n(1)$ are in bijection with the Weyl chambers and the action of S_n on $H^0(C_n(1);\mathbb{C})$ is naturally identified with the regular representation $\mathbb{C}S_n$. When $d \geq 3$ we proceed by induction on n; it is easy to check that $H^*(C_2(d);\mathbb{C}) \cong \mathbb{C}S_2$ (we have $H^0(C_2(d);\mathbb{C}) \cong \square\!\square$ and $H^{d-1}(C_2(d);\mathbb{C}) \cong \begin{smallmatrix}\square\\\square\end{smallmatrix}$). Now, using the inductive hypothesis and corollary 4 we have

$$H^*(C_n(d);\mathbb{C}) = \mathrm{Ind}^{S_n}_{S_{n-1}} H^*(C_{n-1}(d);\mathbb{C}) \cong \mathrm{Ind}^{S_n}_{S_{n-1}} \mathbb{C}S_{n-1} \cong \mathbb{C}S_n. \qquad \square$$

Table 1. Decomposition of $H^*(M(d\mathscr{A}_{n-1}); \mathbb{C})$.

degrees		0	1	2	3
$n=2$	can.	$\ydiagram{2}$			
	ext.	$\ydiagram{3}$			
$n=3$	can.	$\ydiagram{3}$	$\ydiagram{2,1}$		
	ext.	$\ydiagram{4}$	$\ydiagram{2,2}$		
$n=4$	can.	$\ydiagram{4}$	$\ydiagram{2,2} \oplus \ydiagram{3,1}$	$\ydiagram{3,1} \oplus \ydiagram{2,1,1}$	
	ext.	$\ydiagram{5}$	$\ydiagram{3,2}$	$\ydiagram{3,1,1}$	
$n=5$	can.	$\ydiagram{5}$	$\ydiagram{3,2} \oplus \ydiagram{4,1}$	$2\ydiagram{3,1,1} \oplus \ydiagram{3,2} \oplus \ydiagram{4,1} \oplus \ydiagram{2,2,1}$	$\ydiagram{4,1} \oplus \ydiagram{2,1,1,1} \oplus \ydiagram{3,1,1} \oplus \ydiagram{3,2} \oplus \ydiagram{2,2,1}$

For low n, the recursive relation of Theorem 3 allows us to compute the graded character of the S_n action, as is shown in tables 2 and 3.

4. Locating some irreducible representations

Using the recursive formula of theorem 3 it is possible to locate some irreducible representations of S_n in the homogeneous components $H^{k(d-1)}(C_n(d); \mathbb{C})$; namely we will locate the copies of the *standard*, the *sign* and the *standard tensor sign* representations. As before we need to distinguish the case when d is even and the case when d is odd.

4.1. The case d even

Using isomorphism (6) we reduce ourselves to study the action of S_n on $H^*(M(\mathscr{B}_n); \mathbb{C})$; more precisely we study the action of S_n on the cohomology of the complement of the *essential braid arrangement* $\mathscr{A}_{n-1}$ (i.e. the arrangement in $\mathbb{C}^n/\langle(1, 1, \ldots, 1)\rangle$ induced by $\mathscr{B}_n$ or equivalently the *Coxeter arrangement* of type A_{n-1}).

Recall the *deconing construction* from the theory of arrangements; i.e. the deconing of the essential braid arrangement is the arrangement $d\mathscr{A}_{n-1}$ on the vector space $\mathbb{C}^{n-2}$ such that $M(d\mathscr{A}_{n-1}) \cong M(\mathscr{A}_{n-1})/\mathbb{C}^*$. There is an S_{n+1}-equivariant isomorphism of graded algebras ([Gai96, Proposition 2.2])

$$H^*(M(\mathscr{A}_{n-1}); \mathbb{C}) \cong H^*(M(d\mathscr{A}_{n-1}); \mathbb{C}) \otimes \mathbb{C}[\varepsilon]/\varepsilon^2 \tag{11}$$

where ε has degree 1 and S_{n+1} acts trivially on $\mathbb{C}[\varepsilon]/\varepsilon^2$. Futhermore theorem 3 and corollary 4 still hold for the S_n-module $H^*(M(d\mathscr{A}_{n-1}); \mathbb{C})$. There is an analogous of (2) for $H^*(M(d\mathscr{A}_{n-1}); \mathbb{C})$, namely:

$$H^*(M(d\mathscr{A}_{n-1}); \mathbb{C}) = \mathrm{Ind}_{S_2}^{S_n} 1. \tag{12}$$

Moreover isomorphism (11) allows us to know the location of an irreducible representation in $H^*(M(\mathscr{A}_{n-1}); \mathbb{C})$ once we know its location in $H^*(M(d\mathscr{A}_{n-1}); \mathbb{C})$.

Table 2. Decomposition of $H^*(C_n(d);\mathbb{C})$ when d is even.

degrees		0	$d-1$	$2(d-1)$	$3(d-1)$	$4(d-1)$
$n=2$	can.	$[2]$	$[2]$			
	ext.	$[3]$	$[3]$			
$n=3$	can.	$[3]$	$[3]\oplus[2,1]$	$[2,1]$		
	ext.	$[4]$	$[4]\oplus[2,2]$	$[2,2]$		
$n=4$	can.	$[4]$	$[4]\oplus[2,2]\oplus[3,1]$	$[2,2]\oplus 2[3,1]\oplus[2,1,1]$	$[3,1]\oplus[2,1,1]$	
	ext.	$[5]$	$[4]\oplus[3,2]$	$[3,2]\oplus[3,1,1]$	$[3,1,1]$	
$n=5$	can.	$[5]$	$[4]\oplus[3,2]\oplus[4,1]$	$2[3,1,1]\oplus 2[3,2]\oplus 2[4,1]\oplus[2,2,1]$	$2[4,1]\oplus[2,1,1,1]\oplus 3[3,1,1]\oplus 2[3,2]\oplus 2[2,2,1]$	$[4,1]\oplus[2,1,1,1]\oplus[3,1,1]\oplus[3,2]\oplus[2,2,1]$

Table 3. Decomposition of $H^*(C_n(d);\mathbb{C})$ when d is odd.

degrees		0	$d-1$	$2(d-1)$	$3(d-1)$	$4(d-1)$
$n=2$	can.	$[2]$	$[1,1]$			
	ext.	$[3]$	$[1,1,1]$			
$n=3$	can.	$[3]$	$[2,1]\oplus[1,1,1]$	$[2,1]$		
	ext.	$[4]$	$[2,1,1]$	$[2,2]$		
$n=4$	can.	$[4]$	$[2,1,1]\oplus[3,1]$	$[1,1,1,1]\oplus[3,1]\oplus[2,1,1]\oplus 2[2,2]$	$[3,1]\oplus[2,1,1]$	
	ext.	$[5]$	$[3,1,1]$	$[1,1,1,1,1]\oplus[3,2]\oplus[2,2,1]$	$[3,1,1]$	
$n=5$	can.	$[5]$	$[3,1,1]\oplus[4,1]$	$[1,1,1,1,1]\oplus[2,1,1,1]\oplus[4,1]\oplus 2[3,2]\oplus 2[2,2,1]\oplus[3,1,1]$	$3[3,1,1]\oplus 2[2,1,1,1]\oplus[4,1]\oplus 2[3,2]\oplus 2[2,2,1]$	$[4,1]\oplus[2,1,1,1]\oplus[3,1,1]\oplus[3,2]\oplus[2,2,1]$

REMARK 2. We recall that a formula for the generalized Poincarè series associated to the S_{n+1} action on $H^*(M(d\mathscr{A}_{n-1});\mathbb{C})$ has been shown in [Get95], given that $M(d\mathscr{A}_{n-1})$ is homeomorphic to the moduli space $\mathcal{M}_{0,n+1}$ of genus zero $n+1$-pointed curves (and its minimal De Concini-Procesi wonderful model—see [DCP95]—is isomorphic to the Deligne-Mumford compatification of $\mathcal{M}_{0,n+1}$).

As before theorem 3 suffices to compute the graded character of the S_n action on $H^*(M(d\mathscr{A}_{n-1}),\mathbb{C})$ for low n, as is shown in table 1. As a first observation we see that formula (12) and Fröbenius reciprocity allow us to know the number of copies of each irreducible representation in the whole $H^*(M(d\mathscr{A}_{n-1});\mathbb{C})$; in particular

(i) there is only one copy of the trivial representation in $H^*(M(d\mathscr{A}_{n-1});\mathbb{C})$ (and must be at the degree 0),
(ii) there are $n-2$ copies of the standard representation in $H^*(M(d\mathscr{A}_{n-1});\mathbb{C})$,
(iii) there are no copies of the sign representation in $H^*(M(d\mathscr{A}_{n-1});\mathbb{C})$,
(iv) there is one copy of the standard tensor sign representation in $H^*(M(d\mathscr{A}_{n-1});\mathbb{C})$.

We will use the notation $\chi^*(n,k)$ for the character of the action of S_n on $H^k(M(d\mathscr{A}_{n-1});\mathbb{C})$ and $\tilde{\chi}^*(n,k)$ for the character of the extended action of S_{n+1} on $H^k(M(d\mathscr{A}_{n-1});\mathbb{C})$.

PROPOSITION 6. *For $n\geq 3$ there is exactly one copy of the standard representation $V_{(n-1,1)}$ in $H^k(M(d\mathscr{A}_{n-1});\mathbb{C})$ for each $0<k<n-1$.*

PROOF. By induction on n, for $n=3,4,5$ it follows from an explicit computation (see table 1). Let $n>5$; we have

$$\langle\chi^*(n,k),p_n\rangle = \langle\tilde{\chi}^*(n-1,k),p_n\rangle + \langle p_n\tilde{\chi}^*(n-1,k-1),p_n\rangle.$$

If $k=1$ we know from theorem 3 that $H^1(M(d\mathscr{A}_{n-1});\mathbb{C})\cong H^1(M(d\mathscr{A}_{n-2});\mathbb{C})\oplus V_{(n-1,1)}$ and there is (at least) one copy of the standard representation at the degree 1. Consider the case $k>1$. By inductive hypothesis $\mathrm{Res}^{S_n}_{S_{n-1}}\tilde{\chi}^*(n-1,k-1)=\chi^*(n-1,k-1)$ contains exactly one copy of the standard representation therefore $\tilde{\chi}^*(n-1,k-1)$ must contain an irreducible representation which restricts to the standard representation of S_{n-1}; $V_{(n-1,1)}$ is not suitable because there is no copy of the trivial representation in $\chi^*(n-1,k-1)$, so $\tilde{\chi}^*(n-1,k-1)$ must contain exactly one of the following

$$V_{(n-2,2)},\quad V_{(n-2,1,1)}.$$

Using Pieri's rule we see that both $V_{(n-2,1,1)}\otimes V_{(n-1,1)}$ and $V_{(n-2,2)}\otimes V_{(n-1,1)}$ contain exactly one copy of the standard representation.

In particular $H^k(M(d\mathscr{A}_{n-1});\mathbb{C})$ contains exactly one copy of the standard representation for every $1<k<n-1$ and since there are $n-2$ copies of the standard representation in $H^*(M(d\mathscr{A}_{n-1});\mathbb{C})$ also $H^1(M(d\mathscr{A}_{n-1});\mathbb{C})$ contains exactly one copy of the standard representation. $\square$

Remark 3. Proposition 6 can be used for instance to compute the cohomology of the quotient space $M(\mathscr{A}_{n-1})/S_{n-1}$. Indeed, using the theorem on transfer we know that there is an isomorphism of graded algebras $H^*(M(\mathscr{A}_{n-1})/S_{n-1};\mathbb{C}) \cong H^*(M(\mathscr{A}_{n-1});\mathbb{C})^{S_{n-1}}$. So, in order to compute the $\mathbb{C}$-vector space structure of $H^*(M(\mathscr{A}_{n-1})/S_{n-1};\mathbb{C})$ we need to look at those representations of S_n whose restriction to S_{n-1} contain a copy of the trivial representation, i.e. the trivial representation and the standard representation. Therefore, when $k=0$ or $k=n-1$ $H^k(M(\mathscr{A}_{n-1})/S_{n-1};\mathbb{C})$ is one dimensional, while $H^k(M(\mathscr{A}_{n-1})/S_{n-1};\mathbb{C})$ is two dimensional when $0<k<n-1$.

Proposition 7. *For $n \geq 3$ the copy of the standard tensor sign representation $V_{(2,1,\dots,1)}$ appears in the top cohomology $H^{n-2}(M(d\mathscr{A}_{n-1});\mathbb{C})$.*

Proof. By induction on n; as in proposition 6 for $n=3,4,5$ it follows from an explicit computation. Let $n>5$, from theorem 3 we have

$$H^{n-2}(M(d\mathscr{A}_{n-1});\mathbb{C}) \cong V_{(n-1,1)} \otimes H^{n-3}(M(d\mathscr{A}_{n-2});\mathbb{C}).$$

Again there must be exactly one irreducible representation of S_n in $H^{n-3}(M(d\mathscr{A}_{n-2});\mathbb{C})$ whose restriction to S_{n-1} contains a copy of $V_{(2,1,\dots,1)}$. This can't be $V_{(2,1,\dots,1)}$ because there is no copy of the alternating representation of S_{n-1} in $H^{n-3}(M(d\mathscr{A}_{n-2});\mathbb{C})$. Therefore $H^{n-3}(M(d\mathscr{A}_{n-2});\mathbb{C})$ must contain one of the following representations of S_n:

$$V_{(2,2,1,\dots,1)}, \quad V_{(3,1,\dots,1)}.$$

But $V_{(2,2,1,\dots,1)} \otimes V_{(n-1,1)}$ and $V_{(3,1,\dots,1)} \otimes V_{(n-1,1)}$ contain exactly one copy of $V_{(2,1,\dots,1)}$; therefore $H^{n-2}(M(d\mathscr{A}_{n-1});\mathbb{C})$ contains exactly one copy of $V_{(2,1,\dots,1)}$. □

4.2. The case d odd

From theorem 5 we know that $H^*(C_n(d);\mathbb{C})$ is the regular representation, in particular it contains $\dim V_{(n-1,1)} = n-1$ copies of the standard representation, $\dim V_{(2,1,\dots,1)} = n-1$ copies of the standard tensor sign representation, one copy of the trivial and one copy of the sign representations.

With the same argument as in proposition 6 we can prove the following:

Proposition 8. *For $n \geq 3$ and d odd $(d \geq 3)$ there is exactly one copy of the standard representation in the degree $k(d-1)$ for each $1 \leq k \leq n-1$.*

Remark 4. As in remark 3, proposition 8 can be used to compute the cohomology algebra of the quotient space $H^*(C_n(d)/S_{n-1};\mathbb{C})$. In particular $H^{k(d-1)}(C_n(d)/S_{n-1};\mathbb{C})$ is one dimensional for every $0 \leq k \leq n-1$.

Next we look at the sign representation; this was located by Lehrer in [Leh00] using a formula for the generalized Poincarè polynomial. Our proof is different: we show an explicit generator.

PROPOSITION 9. *Let $n = 2k$ or $n = 2k + 1$ and d odd ($d \geq 3$), then the copy of the sign representation appears in the component $H^{k(d-1)}(C_n(d); \mathbb{C})$.*

PROOF. Consider the case $n = 2k$ and the following antisymmetrizer

$$x = \sum_{\sigma \in S_n} (-1)^\sigma A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \ldots A_{\sigma(n-1),\sigma(n)} \in H^{k(d-1)}(C_n(d); \mathbb{C}).$$

Of course S_n acts on $\mathbb{C}x$ as $\tau x = (-1)^\tau x$, the non trivial part of the argument consists in proving that $x \neq 0$. Consider the action of S_n on the set of 2-*partitions of* $\{1, \ldots, n\}$ (that is partitions in which every block has cardinality 2); let Λ be a 2-partition and consider the following ordering on Λ

$$\Lambda = \{\Lambda_1, \ldots, \Lambda_k\}, \quad \Lambda_h = \{i_h, j_h\} \quad \text{with } i_h < j_h \text{ and } j_1 < \cdots < j_k.$$

In particular we can associate to every Λ a permutation $\sigma_\Lambda \in S_n$ such that $\sigma_\Lambda\{\{1, 2\}, \{3, 4\}, \ldots, \{n - 1, n\}\} = \Lambda$ as follows

$$\sigma_\Lambda(2s) = j_s, \qquad \sigma_\Lambda(2s + 1) = i_{s+1}.$$

Note that from this definition we have that $\sigma_\Lambda(A_{1,2}A_{3,4} \ldots A_{n-1,n})$ is an element of the basis of admissible monomials (proposition 2).

Using the fact that $H^*(C_n(d); \mathbb{C})$ is commutative and relation $A_{i,j} = -A_{j,i}$ it can be easily seen that if $\tau \in S_n$ and $\tau\{\{1, 2\}, \{3, 4\}, \ldots, \{n - 1, n\}\} = \Lambda$ then

$$(-1)^\tau A_{\tau(1),\tau(2)} \ldots A_{\tau(n-1),\tau(n)} = (-1)^{\sigma_\Lambda} \sigma_\Lambda(A_{1,2}A_{3,4} \ldots A_{n-1,n}).$$

In particular the expression of x with respect to the basis of admissible monomials appears as follows

$$x = m \sum_{\Lambda} (-1)^{\sigma_\Lambda} \sigma_\Lambda(A_{1,2}A_{3,4} \ldots A_{n-1,n}) \tag{13}$$

where Λ runs over the 2-partitions of $\{1, \ldots, n\}$ and $m = k!2^k$ is the number of permutations of S_n that fix the partition $\{\{1, 2\}, \ldots, \{n - 1, n\}\}$, from which we conclude $x \neq 0$.

Now consider the case $n = 2k + 1$ and the element

$$x = \sum_{\sigma \in S_n} (-1)^\sigma A_{\sigma 1,\sigma 2} A_{\sigma 3,\sigma 4} \ldots A_{\sigma n-2,\sigma n-1} \in H^{k(d-1)}(C_n(d); \mathbb{C}).$$

With a similar argument as before we see that an analogous of (13) applies and therefore $x \neq 0$. $\square$

Next we look at the standard tensor sign representation $V_{(2,1,\ldots,1)}$.

PROPOSITION 10. *Consider $d \geq 3$ odd and $k \geq 2$; if $n = 2k$ there is one copy of $V_{(2,1,\dots,1)}$ in $H^{(k-1)(d-1)}(C_n(d);\mathbb{C})$, one copy in $H^{k(d-1)}(C_n(d);\mathbb{C})$, one copy in $H^{(n-1)(d-1)}(C_n(d);\mathbb{C})$ and 2 copies in each $H^{j(d-1)}(C_n(d);\mathbb{C})$ for each $k < j < n-1$. If $n = 2k+1$ there is one copy of $V_{(2,1,\dots,1)}$ in $H^{k(d-1)}(C_n(d);\mathbb{C})$, one copy in $H^{(n-1)(d-1)}(C_n(d);\mathbb{C})$ and 2 copies in each $H^{j(d-1)}(C_n(d);\mathbb{C})$ for each $k < j < n-1$.*

PROOF. By induction on k; the case $k = 2$ is trivial (see table 3). When $k > 2$, we use the recursive formula of theorem 3:

$$H^{j(d-1)}(C_n(d);\mathbb{C}) \cong H^{j(d-1)}(C_{n-1}(d);\mathbb{C}) \oplus (H^{(j-1)(d-1)}(C_{n-1}(d);\mathbb{C}) \otimes V_{(n-1,n)}).$$

Consider the case $n = 2k$.

(i) If $j = (k-1)$ then, by proposition 9 we know that the extended action on $H^{j(d-1)}(C_{n-1}(d);\mathbb{C})$ must contain a copy of $V_{(2,1,\dots,1)}$.
(ii) If $j = k$ then by inductive hypothesis the extended action of S_n on $H^{(k-1)(d-1)}(C_{n-1}(d);\mathbb{C})$ must contain an S_n-irreducible representation that restricts to $V_{(2,1,\dots,1)}$ and as in proposition 6 we know that $H^{k(d-1)}(C_n(d);\mathbb{C})$ must contain a copy of $V_{(2,1,\dots,1)}$.
(iii) If $j = (n-1)$ then $H^{(n-1)(d-1)}(C_n(d);\mathbb{C}) \cong H^{(n-2)(d-1)}(C_n(d);\mathbb{C}) \otimes V_{(n-1,1)}$ and as before $H^{(n-1)(d-1)}(C_n(d);\mathbb{C})$ must contain a copy of $V_{(2,1,\dots,1)}$.
(iv) if $k < j < n-1$ then $k-1 < j-1 < n-2$ and by inductive hypothesis the extended action on $H^{j(d-1)}(C_{n-1}(d);\mathbb{C})$ must contain two irreducible representations whose restrictions contain a copy of $V_{(2,1,\dots,1)}$; as before we conclude that $H^{j(d-1)}(C_n(d);\mathbb{C})$ contains at least two copies of $V_{(2,1,\dots,1)}$.

Observing that $H^*(C_n(d);\mathbb{C}) \cong \mathbb{C}S_n$ contains $n-1$ copies of $V_{(2,1,\dots,1)}$ we obtain the thesis. Now consider the case $n = 2k+1$.

(i) If $j = k$ then by inductive hypothesis we know that the S_{n-1}-action on $H^{(k-1)(d-1)}(C_{n-1}(d);\mathbb{C})$ contains a copy of $V_{(2,1,\dots,1)}$ and as before $H^{k(d-1)}(C_{n-1}(d);\mathbb{C})$ contains a copy of $V_{(2,1,\dots,1)}$.
(ii) If $j = k+1$, we know that the S_{n-1} action on $H^{k(d-1)}(C_{n-1}(d);\mathbb{C})$ contains a copy of the alternating representation and a copy of $V_{(2,1,\dots,1)}$. Anyway the extended action of S_n on $H^{k(d-1)}(C_{n-1}(d);\mathbb{C})$ cannot contain a copy of $V_{(2,1,\dots,1)}$ because $V_{(2,1,\dots,1)} \otimes V_{(n-1,1)}$ contains a copy of the alternating representation (contradicting proposition 9). Therefore the extended action on $H^{k(d-1)}(C_{n-1}(d);\mathbb{C})$ must contain a copy of the alternating representation of S_n and an irreducible representation of S_n whose restriction contains a copy of $V_{(2,1,\dots,1)}$. The copy of the alternating gives, after tensoring with $V_{(n-1,n)}$, one copy of $V_{(2,1,\dots,1)}$ and the other irreducible representation gives another one.
(iii) If $k+1 < j < n-1$ then $k < j-1 < n$ and as before we have that $H^{j(d-1)}(C_n(d);\mathbb{C})$ contains 2 copies of $V_{(2,1,\dots,1)}$.

(iv) If $j = n - 1$ we have $H^{(n-1)(d-1)}(C_n(d); \mathbb{C}) \cong H^{(n-2)(d-1)}(C_{n-2}(d); \mathbb{C}) \otimes V_{(n-1,1)}$, which contains a copy of $V_{(2,1,\dots,1)}$.

Again we conclude using the fact that $H^*(C_n(d); \mathbb{C}) \cong \mathbb{C}S_n$ contains $n-1$ copies of $V_{(2,1,\dots,1)}$. □

5. The degrees $d-1$ and $2(d-1)$

It is interesting to notice that the recurrence formula of Theorem 3 suffices to determine, for every $n \geq 3$ and $d \geq 2$, an explicit decomposition of $H^{d-1}(C_n(d); \mathbb{C})$ and $H^{2(d-1)}(C_n(d); \mathbb{C})$, both as S_n and as S_{n+1}-modules.

5.1. *The case d even*

As in section 4.1 it suffices to study the cohomology algebra of the *deconed* braid arrangement $H^*(M(d\mathscr{A}_{n-1}); \mathbb{C})$; the isomorphism (11) allows to infere formulas for the decomposition of $H^1(M(\mathscr{A}_{n-1}); \mathbb{C})$ and $H^2(M(\mathscr{A}_{n-1}); \mathbb{C})$ from the analogous formulas for $M(d\mathscr{A}_{n-1})$.

Proposition 11. *For every $n \geq 3$ the following equality of S_{n+1} modules holds:*

$$H^1(M(d\mathscr{A}_{n-1}); \mathbb{C}) \cong V_{(n-1,2)}.$$

In particular we have the following decomposition of S_n-modules:

$$H^1(M(d\mathscr{A}_{n-1}); \mathbb{C}) \cong V_{(n-1,1)} \oplus V_{(n-2,2)}.$$

Proof. By induction on n; we have already discussed the case $n = 3$ (see table 1). Let $n > 3$, from theorem 3 and the inductive hypothesis we have

$$H^1(M(d\mathscr{A}_{n-1}); \mathbb{C}) \cong H^1(M(d\mathscr{A}_{n-1}); \mathbb{C}) \oplus V_{(n-1,1)} \cong V_{(n-2,2)} \oplus V_{(n-1,1)}$$

and it is easily seen, using Pieri's rule, that $V_{(n-1,2)}$ is the only representation of S_{n+1} that restricts to $V_{(n-2,2)} \oplus V_{(n-1,1)}$. □

Next we look at $H^2(M(d\mathscr{A}_{n-1}); \mathbb{C})$; its decomposition can be recursively computed for $n \leq 6$ using theorem 3 and observing that for every $m < 6$ there exists a unique action of S_{m+1} that restricts to $H^2(M(d\mathscr{A}_{m-1}); \mathbb{C})$ (see table 1). This way we obtain the following decomposition of S_6 modules:

$$H^2(M(d\mathscr{A}_5); \mathbb{C}) \cong 2\,\ydiagram{4,1,1} \oplus 2\,\ydiagram{3,2,1} \oplus \ydiagram{3,3} \oplus \ydiagram{5,1} \oplus \ydiagram{4,2}.$$

Again there is only one S_7-action that restricts to $H^2(M(d\mathscr{A}_5); \mathbb{C})$, namely

$$H^2(M(d\mathscr{A}_5); \mathbb{C}) \cong \ydiagram{3,3,1} \oplus \ydiagram{4,2,1} \oplus \ydiagram{5,1,1}.$$

Theorem 12. *For $n \geq 6$ the following equality of S_{n+1}-modules holds:*

$$H^2(M(d\mathscr{A}_{n-1}); \mathbb{C}) \cong V_{(n-1,1,1)} \oplus V_{(n-3,3,1)} \oplus V_{(n-2,2,1)}.$$

PROOF. By induction on n; for $n = 6$ the result follows from our previous discussion. Let $n > 6$, from theorem 3 and the inductive hypothesis we have

$$H^2(M(d\mathscr{A}_{n-1}); \mathbb{C}) \cong V_{(n-2,1,1)} \oplus V_{(n-4,3,1)} \oplus V_{(n-3,2,1)} \oplus \left(V_{(n-2,2)} \otimes V_{(n-1,1)}\right)$$

Next we notice that

$$V_{(n-2,2)} \otimes V_{(n-1,1)} \cong V_{(n-3,2,1)} \oplus V_{(n-3,3)} \oplus V_{(n-2,1,1)} \oplus V_{(n-2,2)} \oplus V_{(n-1,1)}.$$

and therefore

$$\begin{aligned} H^2(M(d\mathscr{A}_{n-1}); \mathbb{C}) \cong\ & 2V_{(n-3,2,1)} \oplus V_{(n-3,3)} \oplus 2V_{(n-2,1,1)} \\ & \oplus V_{(n-2,2)} \oplus V_{(n-1,1)} \oplus V_{(n-4,3,1)}. \end{aligned}$$

Using Pieri's rule we see that the only irreducible representations of S_{n+1} whose restriction contains $V_{(n-3,2,1)}$ that can appear in the decomposition of the extended action on $H^2(M(d\mathscr{A}_{n-1}); \mathbb{C})$ are $V_{(n-3,3,1)}$ and $V_{(n-2,2,1)}$ and they must both appear with multiplicity one. This forces the extended action of S_{n+1} on $H^2(M(d\mathscr{A}_{n-1}); \mathbb{C})$ to be

$$V_{(n-1,1,1)} \oplus V_{(n-3,3,1)} \oplus V_{(n-2,2,1)}. \qquad \square$$

REMARK 5. In particular for $n \geq 7$ we have the following decomposition of S_n-modules:

$$\begin{aligned} H^2(M(d\mathscr{A}_{n-1}); \mathbb{C}) \cong\ & V_{(n-1,1)} \oplus 2V_{(n-2,1,1)} \oplus V_{(n-3,3)} \\ & \oplus 2V_{(n-3,2,1)} \oplus V_{(n-4,3,1)} \oplus V_{(n-2,2)} \end{aligned}$$

5.2. *The case d odd*

We have already discussed the decomposition of $H^{d-1}(C_n(d); \mathbb{C})$ (proposition 1), so we only have to treat the degree $2(d-1)$.

As before with an explicit computation it can be seen that

$$H^{2(d-1)}(C_5(d); \mathbb{C}) \cong \ydiagram{1,1,1,1,1} \oplus \ydiagram{2,1,1,1} \oplus \ydiagram{4,1} \oplus 2\,\ydiagram{3,2} \oplus 2\,\ydiagram{2,2,1} \oplus \ydiagram{3,1,1}.$$

So, at first sight, there are two possible actions of S_6 that restrict to $H^{2(d-1)}(C_5(d); \mathbb{C})$, namely:

$$\ydiagram{2,1,1,1,1} \oplus 2\,\ydiagram{2,2,2} \oplus \ydiagram{4,1,1} \oplus 2\,\ydiagram{3,3} \quad \text{and} \quad \ydiagram{2,1,1,1,1} \oplus \ydiagram{3,2,1} \oplus \ydiagram{2,2,2} \oplus \ydiagram{4,2}.$$

Anyway if the first case holds we would have

$$H^{2(d-1)}(C_6(d); \mathbb{C}) \cong \ydiagram{2,1,1,1,1} \oplus 2\,\ydiagram{2,2,2} \oplus 2\,\ydiagram{4,1,1} \oplus 2\,\ydiagram{3,3} \oplus \ydiagram{4,2} \oplus \ydiagram{5,1} \oplus \ydiagram{3,1,1,1} \oplus \ydiagram{3,2,1}$$

which is not the restriction of an S_7 action; therefore the second case must hold.

THEOREM 13. *For $n \geq 5$ and d odd ($d \geq 3$) there is an isomorphism of S_{n+1}-modules*

$$H^{2(d-1)}(C_n(d);\mathbb{C}) \cong V_{(n-3,1,1,1,1)} \oplus V_{(n-2,2,1)} \oplus V_{(n-3,2,2)} \oplus V_{(n-1,2)}.$$

PROOF. First we observe that for every n it holds:

$$V_{(n-2,1,1)} \otimes V_{(n-1,1)} \cong V_{(n-2,1,1)} \oplus V_{(n-3,2,1)} \oplus V_{(n-3,1,1,1)} \oplus V_{(n-1,1)} \oplus V_{(n-2,2)}.$$

We prove the thesis by induction on n; we have already discussed the case $n = 5$. Let $n > 5$, from theorem 3 and the inductive hypothesis we have

$$\begin{aligned} H^{2(d-1)}(C_n(d);\mathbb{C}) &\cong H^{2(d-1)}(C_{n-1}(d);\mathbb{C}) \oplus (V_{(n-1,1)} \otimes H^{d-1}(C_{n-1}(d);\mathbb{C})) \\ &\cong V_{(n-4,1,1,1,1)} \oplus 2V_{(n-3,2,1)} \oplus V_{(n-4,2,2)} \oplus 2V_{(n-2,2)} \\ &\quad \oplus V_{(n-2,1,1)} \oplus V_{(n-3,1,1,1)} \oplus V_{(n-1,1)}. \end{aligned}$$

The copy of $V_{(n-4,2,2)}$ can not appear as a component of the restriction of $V_{(n-4,2,2,1)}$ or $V_{(n-4,3,2)}$ (the latter makes sense only for $n \geq 7$) because there are no copies of $V_{(n-4,2,1,1)}$ and $V_{(n-4,3,1)}$ in $H^{2(d-1)}(C_n(d);\mathbb{C})$. Therefore the extended action must contain a copy of $V_{(n-3,2,2)}$ and his restriction gives a copy of $V_{(n-4,2,2)}$ and a copy of $V_{(n-3,2,1)}$. The other copy of $V_{(n-3,2,1)}$ must appear as a component of the restriction of $V_{(n-2,2,1)}$ because there is only one copy of $V_{(n-4,2,2)}$ and there are no copies of $V_{(n-4,2,1,1)}$ and $V_{(n-4,3,1)}$ in $H^{2(d-1)}(C_n(d);\mathbb{C})$. The restriction of $V_{(n-2,2,1)}$ contains a copy of $V_{(n-2,2)}$, a copy of $V_{(n-2,1,1)}$ and a copy of $V_{(n-3,2,1)}$. Analogously the other copy of $V_{(n-2,2)}$ must appear as a component of the restriction of $V_{(n-1,2)}$; this gives a copy of $V_{(n-2,2)}$ and a copy of $V_{(n-1,1)}$. At this point the copies of $V_{(n-4,1,1,1,1)}$ and $V_{(n-3,1,1,1)}$ must come from the restriction of $V_{(n-3,1,1,1,1)}$.

Summarizing, there is only an action of S_{n+1} that restricts to the action $S_n \curvearrowright H^{2(d-1)}(C_n(d);\mathbb{C})$, namely:

$$V_{(n-3,1,1,1,1)} \oplus V_{(n-2,2,1)} \oplus V_{(n-3,2,2)} \oplus V_{(n-1,2)}. \qquad \square$$

REMARK 6. In particular, for $n \geq 6$ and $d \geq 3$ odd the following decomposition of S_n-modules holds:

$$\begin{aligned} H^{2(d-1)}(C_n(d);\mathbb{C}) &\cong V_{(n-4,1,1,1,1)} \oplus V_{(n-3,1,1,1)} \oplus 2V_{(n-3,2,1)} \oplus V_{(n-2,1,1)} \\ &\quad \oplus 2V_{(n-2,2)} \oplus V_{(n-4,2,2)} \oplus V_{(n-1,1)}. \end{aligned}$$

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Giacomo d'Antonio
FB 3—Mathematik und Informatik, Universität Bremen
Bibliothekstraße 1, 28359 Bremen—Germany
dantonio@math.uni-bremen.de

Giovanni Gaiffi
Dipartimento di Matematica, Università di Pisa
Largo B. Pontecorvo 1, 56127 Pisa—Italy
gaiffi@dm.unipi.it