



Diophantine Geometry — *A Note on Maurin’s Theorem*, by E. BOMBIERI, P. HABEGGER, D. MASSER, U. ZANNIER, presented by Umberto Zannier on 16 April 2010.

ABSTRACT. — We combine the strategy described in a paper of the first, third and fourth authors with a recent result of the second author to obtain a new proof of Maurin’s Theorem to the effect that the points satisfying two independent multiplicative relations on a fixed algebraic curve form a finite set when there is no natural obstacle.

KEY WORDS: Diophantine geometry, multiplicative dependence, Zilber conjecture.

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1. INTRODUCTION

This paper concerns conjectures on “unlikely intersections” due independently to Zilber [Z] in 2002 and Pink [P] in 2005. We treat the special but significant case of a curve defined over the field of algebraic numbers and lying in a multiplicative group \mathbb{G}_m^n . In this case the conjecture was stated, also independently, by the first, third and fourth authors [BMZ3] in 2006 after they had raised the question [BMZ1] in 1999. It was proved by Maurin [M] in 2008.

THEOREM (Maurin). *For $n \geq 2$ let C be an irreducible curve in \mathbb{G}_m^n defined over the algebraic numbers such there is no non-zero (c_1, \dots, c_n) in \mathbb{Z}^n with $x_1^{c_1} \dots x_n^{c_n} = 1$ identically on C . Then there are at most finitely many points $P = (\xi_1, \dots, \xi_n)$ on C for which there exist $(a_1, \dots, a_n), (b_1, \dots, b_n)$ in \mathbb{Z}^n , linearly independent over the rationals, with*

$$\xi_1^{a_1} \dots \xi_n^{a_n} = \xi_1^{b_1} \dots \xi_n^{b_n} = 1.$$

The object of the present note is to show that the theorem is a fairly quick consequence of a very recent result of the second author [H]. We hope that this will lead to a fully effective version. It is not so clear to us that Maurin’s proof can lead to such a version, due to the use of a Vojta-type inequality which in its original form was applied to prove the Mordell Conjecture. This latter is still not fully effective; one can bound the cardinalities of the finite sets involved but one cannot find the sets themselves.

To explain our proof we must go back into history. The first, third and fourth authors [BMZ1] proved a weaker form of the above theorem when no non-trivial $x_1^{c_1} \dots x_n^{c_n}$ is identically constant on C ; Maurin’s theorem is the statement that the result holds as long as only the constant 1 is avoided.

The problem of extending the theorem in this way proved to be an intriguing question. Partial progress was made in [BMZ3], where the authors proved the above theorem for $n = 2, 3, 4, 5$ and mentioned that the key to the full theorem is almost certainly the study of surfaces. They began this study in [BMZ4] with a Structure Theorem for anomalous subvarieties of a variety of arbitrary dimension in \mathbb{G}_m^n , and they also stated a Bounded Height Conjecture with the idea that this conjecture for surfaces would lead to the full theorem. In [BMZ5] they proved the conjecture for planes; unfortunately this had no implications back to curves. In the meantime Maurin proved his theorem by a slightly different route.

It was the second author [H] who turned the Bounded Height Conjecture into a Bounded Height Theorem. This result supplies for every variety V in \mathbb{G}_m^n defined over the algebraic numbers an upper bound $\mathcal{B}(V)$ —see Section 2 below for more details. And here we confirm that the strategy of [BMZ3] does succeed; in fact for each C as above we can effectively construct finitely many surfaces S such that all the points P above can be found effectively in terms of C and the $\mathcal{B}(S)$. It remains only to make these quantities $\mathcal{B}(S)$ effective and we see no reason why this cannot be done.

Of course the original conjectures of [Z] and [P] made sense for varieties defined over a general field of zero characteristic, and in [BMZ2] the first, third and fourth authors proved that their 1999 curves result is valid in this generality. More recently in [BMZ6] they established a specialization principle that does the same for Maurin's Theorem.

The third author heartily thanks Aurélien Galateau for discussions leading to Lemma 6(b).

2. VARIETIES

Here we describe the main Structure Theorem of [BMZ4] (p. 4) and the Bounded Height Theorem of [H] (p. 862).

Of course the condition on C in Maurin's Theorem means that C lies in no algebraic subgroup of dimension $n - 1$ in the algebraic group \mathbb{G}_m^n . In general an algebraic subgroup H is either \mathbb{G}_m^n or has dimension $s < n$ and is defined by $n - s$ equations of the form $x^{\mathbf{c}} = x_1^{c_1} \dots x_n^{c_n} = 1$, where the $n - s$ exponent vectors $\mathbf{c} = (c_1, \dots, c_n)$ in \mathbb{Z}^n are linearly independent over the rationals \mathbb{Q} . See for example [BG] (pp. 82–88). Denote by \mathcal{H}_s the union of all algebraic subgroups of dimension s , with of course $\mathcal{H}_n = \mathbb{G}_m^n$. We define a coset QH to be a translate of an algebraic subgroup H by a point Q . It will be called proper if $H \neq \mathbb{G}_m^n$. Later on it will be called torsion if Q has finite order.

Let V be an irreducible variety in \mathbb{G}_m^n . As in [BMZ4] (p. 3) we define the subset V^{oa} as what remains after removing from V all irreducible subvarieties W lying in some coset of some dimension $n - h$ and satisfying

$$\dim W > \max\{0, \dim V - h\}.$$

This simply means that the dimension of W is larger than the lower bound given by the intersection theory of varieties in projective space.

Such W are called anomalous in V , and such a W is called maximal if it is not contained in another strictly larger anomalous subvariety of V .

We can now state the Structure Theorem for anomalous subvarieties.

THEOREM (Bombieri–Masser–Zannier). *Let V be an irreducible variety in \mathbb{G}_m^n of positive dimension defined over \mathbb{C} .*

(a) *For any torus H with*

$$(1) \quad 1 \leq h = n - \dim H \leq \dim V$$

the union \mathcal{L}_H of all subvarieties W of V contained in any coset of H with

$$(2) \quad \dim W = \dim V - h + 1$$

is a closed subset of V , and the product $H\mathcal{L}_H$ is not dense in \mathbb{G}_m^n .

(b) *There is a finite collection $\Phi = \Phi_V$ of such tori H such that every maximal anomalous subvariety W of V is a component of $V \cap gH$ for some H in Φ satisfying (1) and (2) and some g in \mathcal{L}_H ; and V^{oa} is obtained from V by removing the \mathcal{L}_H for all H in Φ . In particular V^{oa} is open in V .*

In the sequel we shall refer to this as ST.

Denote by $h(\xi)$ the absolute logarithmic height of ξ in the field $\overline{\mathbb{Q}}$ of all algebraic numbers, and for $P = (\xi_1, \dots, \xi_n)$ in $V(\overline{\mathbb{Q}})$ define

$$h(P) = h(\xi_1) + \dots + h(\xi_n).$$

We can now state the Bounded Height Theorem.

THEOREM (Habegger). *For $1 \leq m \leq n$ let V be an irreducible variety in \mathbb{G}_m^n of dimension m defined over $\overline{\mathbb{Q}}$. Then there is $\mathcal{B}(V)$ such that $h(P) \leq \mathcal{B}(V)$ for all P in $V^{oa}(\overline{\mathbb{Q}}) \cap \mathcal{H}_{n-m}$.*

In the sequel we shall refer to this as BHT.

3. SURFACES

In order to construct a surface from a curve we use as in [BMZ3] the quotient map

$$\varphi : \mathbb{G}_m^{2n} \rightarrow \mathbb{G}_m^n$$

given by (use the natural isomorphism $\mathbb{G}_m^{2n} \simeq \mathbb{G}_m^n \times \mathbb{G}_m^n$):

$$\varphi(x_1, \dots, x_n, y_1, \dots, y_n) = \left(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n} \right).$$

LEMMA 1. *For $n \geq 2$ let C be an irreducible curve in \mathbb{G}_m^n defined over $\overline{\mathbb{Q}}$ and not lying in any proper coset. Then*

- (a) the closure S in \mathbb{G}_m^n of $\varphi(C \times C)$ is an irreducible surface,
- (b) there is an effectively computable set $Y = Y(C)$ of at most finitely many anomalous curves such that $S = S^{oa} \cup Y$,
- (c) there is an effectively computable finite set $Z = Z(C)$ in $S(\overline{\mathbb{Q}})$ and an effective positive constant κ_1 such that $h(P) + h(Q) \leq \kappa h(\varphi(P, Q)) + \kappa_1$ for any P, Q in $C(\overline{\mathbb{Q}})$ with $\varphi(P, Q)$ not in Z .

PROOF. It was shown in Lemma 2(a) (p. 2250) of [BMZ3] (actually the necessary condition $n \geq 2$ was there omitted) that S is a surface, clearly irreducible, as required in (a) above. It is itself not anomalous, for otherwise some x^a/y^a ($\mathbf{a} \neq 0$) would be constant on $C \times C$, and then specializing (y_1, \dots, y_n) we would find that x^a is constant on C .

Thus every anomalous subvariety is a curve, automatically maximal. Now the only obstacle to their finiteness is in fact the algebraic dependence on S of some pair x^a, x^b for independent \mathbf{a}, \mathbf{b} . We can see this from ST with $V = S$ and $h = 2$ as follows. From ST(b) we see that the anomalous curve lies in a translate of one of finitely many algebraic subgroups H of dimension $n - 2$. Each of these can be effectively computed; see equation (3.4) of [BMZ4] (p. 19). Consider \mathcal{L}_H in ST(a), a subvariety of S . If $\mathcal{L}_H \neq S$ then \mathcal{L}_H would be a finite union of anomalous curves which is also effectively computable; see for example the remarks in [BMZ4] (pp. 5, 20) on an effective Fibre Dimension Theorem. If this were to happen for each of the finitely many H , then the number of anomalous curves would be finite and (b) above would follow.

So we can assume that $\mathcal{L}_H = S$ for some H . After an automorphism we can suppose as in the subsequent discussion in [BMZ4] that H is defined by $x_1 = x_2 = 1$. Now the sets $H\mathcal{L}_H$ and S have the same image U_H under the projection π from \mathbb{G}_m^n to $\mathbb{G}_m^2 = \mathbb{G}_m^2 \times \{1\}^{n-2}$ in \mathbb{G}_m^n . As $H\mathcal{L}_H = U_H \times \mathbb{G}_m^{n-2}$ is not dense in \mathbb{G}_m^n by ST(a), it follows that U_H is not dense in \mathbb{G}_m^2 . So x_1, x_2 must be algebraically dependent on S .

We deduce that $\pi(S) = U_H$ is a curve, and so $\pi(\varphi(C \times C))$ too. However this latter is $\varphi(\pi(C) \times \pi(C))$, and so it would follow again from Lemma 2(a) of [BMZ3] that the curve $\pi(C)$ is contained in a proper coset. But then C would be too. This contradiction establishes (b) above.

Finally (c) above is Lemma 2(b) of [BMZ3], and because the proof there is fully effective this completes the proof of the present lemma. □

REMARK 2. In fact it can be shown with a bit more effort that $Z(C)$ is the stabilizer of C , as suspected by Sinnou David, and therefore consists only of torsion points. Using this fact would slightly simplify the proof in Section 5.

4. CURVES

We prove here two results of the same type as Maurin’s Theorem on the set $C \cap \mathcal{H}_{n-2}$, but each with an extra restriction on the points P . We note that even $C \cap \mathcal{H}_{n-1}$ lies in $C(\overline{\mathbb{Q}})$, so these P are certainly defined over $\overline{\mathbb{Q}}$. In the first result the height $h(P)$ is assumed to be bounded above.

LEMMA 3. For $n \geq 2$ let C be an irreducible curve in \mathbb{G}_m^n defined over $\overline{\mathbb{Q}}$ and not lying in \mathcal{H}_{n-1} . Then for any $\mathcal{B} > 0$ there are at most finitely many points P in $C \cap \mathcal{H}_{n-2}$ with $h(P) \leq \mathcal{B}$ and these can be effectively found.

PROOF. If $n \geq 3$ then apart from the effectivity this is Lemma 1 (p. 2249) of [BMZ3]. But the results in [AD] and [AZ] referred to are effective in nature, and the version of Liardet's Theorem referred to is that just for torsion points, which is well known to be fully effective; see also the remarks at the end of the proof of Lemma 8.1 (p. 73) in [BMZ5]. This also covers the case $n = 2$. □

In the second result, some fixed power of P is assumed to be defined over a fixed number field.

LEMMA 4. For $n \geq 2$ let C be an irreducible curve in \mathbb{G}_m^n defined over $\overline{\mathbb{Q}}$ and not lying in \mathcal{H}_{n-1} . Then for any positive integer e and any number field K there are at most finitely many points P in $C \cap \mathcal{H}_{n-2}$ with P^e in $\mathbb{G}_m^n(K)$, and these can be effectively found.

PROOF. Replacing C by its e th power C^e in \mathbb{G}_m^n , we may assume that $e = 1$. Now we use induction on n . The case $n = 2$ follows at once from Lemma 3, since then $h(P) = 0$.

For $n \geq 3$ we can suppose, after applying an automorphism of \mathbb{G}_m^n , that C is parametrized by $(x_1, \dots, x_k, g_{k+1}, \dots, g_n)$, where x_1, \dots, x_k are multiplicatively independent modulo constants and the constants g_{k+1}, \dots, g_n are multiplicatively independent. We can also suppose that $k \geq 2$ else $C \cap \mathcal{H}_{n-2}$ is empty.

For the induction step suppose that $P = (\xi_1, \dots, \xi_k, g_{k+1}, \dots, g_n)$ is a point of $C(K)$ in \mathbb{G}_m^n ; we can assume that our curve C is defined over K . As in [BMZ3] (p. 2253) select non-zero polynomials f_{ij} ($1 \leq i < j \leq k$) over K vanishing on the projections of C to the various pairs of coordinates from x_1, \dots, x_k . Let \mathcal{V} be the set of non-archimedean valuations on K which are trivial on the non-zero coefficients of the f_{ij} and on the group generated by g_{k+1}, \dots, g_n . The complementary set \mathcal{W} of all other valuations on K is finite.

Let v be in \mathcal{V} and consider the equation $f_{12}(\xi_1, \xi_2) = 0$. Since v is non-archimedean there must appear two monomials with the same value, and since v is trivial on the coefficients of f_{12} we get an additive relation $b_1v(\xi_1) + b_2v(\xi_2) = 0$, where (b_1, b_2) in \mathbb{Z}^2 is non-zero taken from a finite set independent of P or v . The same argument applies to any pair $v(\xi_i), v(\xi_j)$ ($1 \leq i < j \leq k$).

Therefore the point $v(P) = (v(\xi_1), \dots, v(\xi_k))$ lies in a finite set of \mathbb{Q} -vector spaces of dimension at most 1, also independent of P or v .

Suppose first that $v(P) = 0$ for every v in \mathcal{V} . Then each ξ_i belongs to the finitely generated group of \mathcal{W} -units of K . So we can apply the more general form of Liardet's Theorem to the projection C' on \mathbb{G}_m^k (which lies in no proper coset) to finish the proof of the present lemma in this case. Note that this form of Liardet's Theorem was originally proved using Siegel's Theorem on integral points, which remains ineffective today, but see Theorem 5.4.5 of [BG] (p. 147) for an effective proof, also using valuations.

Suppose then instead that there is v in \mathcal{V} with $v(P) \neq 0$, so that $v(P)$ lies in our finite set of \mathbb{Q} -vector spaces of dimension 1. We have equations

$$\zeta_1^{a_1} \cdots \zeta_k^{a_k} g_{k+1}^{a_{k+1}} \cdots g_n^{a_n} = 1$$

for all $\mathbf{a} = (a_1, \dots, a_n)$ in some subgroup A of \mathbb{Z}^n of rank 2. Applying v leads to $a_1 v(\zeta_1) + \cdots + a_k v(\zeta_k) = 0$, which says that the (a_1, \dots, a_k) lie in a finite set of proper subgroups of \mathbb{Z}^k . After an automorphism of \mathbb{G}_m^k we can suppose $a_1 = 0$, and then we just project to $(x_2, \dots, x_k, g_{k+1}, \dots, g_n)$. By induction we get at most finitely many $(\zeta_2, \dots, \zeta_k)$, so also $(\zeta_1, \dots, \zeta_k)$; for example f_{12} above must involve x_1 and we can take it irreducible. \square

REMARK 5. In fact Bérczes, Evertse, Györy and Pontreau have recently given an effective version of Liardet’s Theorem for division groups (and more) as Theorem 2.2 (p. 73) of [BEGP]. Using this might also slightly simplify the proof in Section 5.

Actually, the arguments below suffice to deduce (without BHT) such an effective version from the version for finitely generated groups Γ . (Start with a curve in \mathbb{G}_m^2 and in (4) take $k = 2$ and g_3, \dots, g_n as generators of Γ .)

Finally we record a remark about Galois conjugates.

LEMMA 6. *For a number field K there is an effectively computable positive integer d with the following properties. For $n \geq 1$ let P be in $\mathbb{G}_m^n(\bar{K})$. Then*

(a) *Suppose for some σ in $\text{Gal}(\bar{K}/K)$ that $\varphi(P^\sigma, P)$ lies in a coset QH with Q in $\mathbb{G}_m^n(K)$. Then there is \tilde{Q} with $\tilde{Q}^d = (1, \dots, 1)$ and $QH = \tilde{Q}H$; so QH is a torsion coset.*

(b) *Suppose for all σ in $\text{Gal}(\bar{K}/K)$ that $\varphi(P^\sigma, P)$ lies in $\mathbb{G}_m^n(K)$. Then P^d lies in $\mathbb{G}_m^n(K)$.*

PROOF. We take d as a positive integer such that $\zeta^d = 1$ for all roots of unity ζ in K . If $\dim H = n$ in (a) then there is nothing to do. Otherwise after an automorphism we can suppose that H is defined by $x_j = 1$ ($j = k + 1, \dots, n$) for some $k < n$. Thus QH is defined by $x_j = \alpha_j$ for α_j in K , and we get $\zeta_j^\sigma \zeta_j^{-1} = \alpha_j$ for $P = (\zeta_1, \dots, \zeta_n)$. Now the monic minimal polynomial of ζ_j over K , of degree say d_j , has a zero also at $\zeta_j^\sigma = \alpha_j \zeta_j$. So it gets multiplied by $\alpha_j^{d_j}$ when we multiply the variable by α_j ; and inspection of the non-zero constant term shows that this $\alpha_j^{d_j} = 1$. Thus $\alpha_j^d = 1$ ($j = k + 1, \dots, n$), and we can take $\tilde{Q} = (1, \dots, 1, \alpha_{k+1}, \dots, \alpha_n)$.

To prove (b) we apply this for each σ to $H = \{1\}^n$ with $Q = \varphi(P^\sigma, P)$. We find that $Q = \tilde{Q}$ so $P^{d\sigma} = P^d$, and because this holds for all σ the result follows. \square

5. PROOF OF THEOREM

The general idea is to take the bounded cosets implicit in Lemma 1(b), and intersect them with the unbounded torsion coset implicit in \mathcal{H}_{n-2} . A similar idea was

used in [BMZ4] (p. 23) and [BMZ6] (p. 317), but this time the bounded cosets are also torsion cosets, thanks to Lemma 6(a). We argue again by induction on n ; the effectivity will be clear from the preceding discussions.

We can suppose that C is parametrized by $(x_1, \dots, x_k, g_{k+1}, \dots, g_n)$, where x_1, \dots, x_k are multiplicatively independent modulo constants and the coordinates g_{k+1}, \dots, g_n are multiplicatively independent; and further $k \geq 2$. Write $C = C' \times P_0$ with C' in \mathbb{G}_m^k and $P_0 = (g_{k+1}, \dots, g_n)$.

During the proof various fixed curves C'' will turn up. These C'' , which include C' , will come from applying finitely many automorphisms of \mathbb{G}_m^k to C' and then projecting down to various \mathbb{G}_m^l . None of these C'' lies in a proper coset. The corresponding closures S'' of $\varphi(C'' \times C'')$ satisfy Lemma 1, so that S'' is the union of S''^{oa} with finitely many curves contained in cosets Q_0H_0 of codimension 2, with various $Q_0 = Q_0(C'')$. We choose once and for all a number field K containing fields of definition for the C'' , the points of $Z(C'')$, and the $Q_0(C'')$.

Next, during the proof a finite collection of positive integers b related to the Q_0H_0 will turn up. We choose once and for all a positive integer c divisible by all these.

To begin the proof proper take now an arbitrary $P = (P', P_0)$ in $C \cap \mathcal{H}_{n-2}$ with $P' = (\xi_1, \dots, \xi_k)$ in C' . We claim that we can assume that there exists $\sigma = \sigma(P)$ in $\text{Gal}(\bar{K}/K)$ such that

$$(3) \quad \varphi(P^{c\sigma}, P^c) \notin \mathbb{G}_m^n(K).$$

Namely, if this fails, then Lemma 6(b) gives a positive integer d , depending only on K , with P^{cd} in $\mathbb{G}_m^n(K)$. But then Lemma 4 leads to the finiteness of the P .

There is a subgroup A of \mathbb{Z}^n of rank 2 with

$$(4) \quad \xi_1^{a_1} \dots \xi_k^{a_k} g_{k+1}^{a_{k+1}} \dots g_n^{a_n} = 1$$

for all $\mathbf{a} = (a_1, \dots, a_k, a_{k+1}, \dots, a_n)$ in A . Define $Q = \varphi(P'^\sigma, P') = (\eta_1, \dots, \eta_k)$. This Q lies on $\varphi(C' \times C')$ in \mathbb{G}_m^k with closure say S . In fact Q lies in $S \cap \mathcal{H}_{k-2}$, because two independent \mathbf{a} in (4) project down to two independent (a_1, \dots, a_k) , else there would be a non-trivial relation (4) with $a_1 = \dots = a_k = 0$, which is ruled out by the independence of g_{k+1}, \dots, g_n . There are now two main cases (I) and (II).

CASE I: THE POINT Q LIES IN S^{oa} . Then by BHT we know that $h(Q) \leq \mathcal{B}(S)$. Now by (3) the point $(Q, 1, \dots, 1) = \varphi(P^\sigma, P)$ does not lie in $\mathbb{G}_m^n(K)$ and so Q does not lie in $\mathbb{G}_m^k(K)$. Therefore Q avoids the exceptional set $Z(C')$ by our choice of K . Hence $h(P')$ is also bounded above by Lemma 1(c). Thus $h(P)$ is bounded too. Now the required finiteness follows from Lemma 3.

CASE II: THE POINT Q DOES NOT LIE IN S^{oa} . Now Lemma 1(b) shows that Q lies in one of a finite number of anomalous curves of S , each one lying in a fixed $(k - 2)$ -dimensional coset Q_0H_0 . Recall that the number field K contains a field of definition for all the Q_0 . As $Q = \varphi(P'^\sigma, P')$ we see from Lemma 6(a) that each

Q_0H_0 is a torsion coset. Corresponding to this there is a fixed subgroup B_0 of \mathbb{Z}^k of rank at least 2 such that

$$(5) \quad \eta_1^{b_1} \dots \eta_k^{b_k} = 1$$

for all (b_1, \dots, b_k) in B_0 .

We will now attempt to prove that (5) holds even on a fixed subgroup of rank k . If $k = 2$ then there is nothing to do; so we suppose $k \geq 3$, and we proceed inductively from a fixed subgroup, also for convenience denoted by B_0 , of rank $k - l + 1$ to a fixed subgroup of rank $k - l + 2$ ($l = k - 1, \dots, 2$). We do this by considering the sum $B_0 + \pi(A)$, where now π is the projection from \mathbb{Z}^n to $\mathbb{Z}^k = \mathbb{Z}^k \times \{0\}^{n-k}$ in \mathbb{Z}^n . If this sum is not much larger than B_0 , we get information on $\pi(A)$. Otherwise we get additional relations on Q for free.

We start by noting from (5) that Q lies in a fixed torsion coset $\tilde{Q}_0\tilde{H}_0$, with \tilde{H}_0 now of dimension $l - 1$ in \mathbb{G}_m^k ; in particular Q lies in $S \cap \mathcal{H}_{l-1}$, not just $S \cap \mathcal{H}_{k-2}$. After an automorphism of \mathbb{G}_m^k we can suppose that $\tilde{H}_0 = \mathbb{G}_m^{l-1} = \mathbb{G}_m^{l-1} \times \{1\}^{k-l+1}$ in \mathbb{G}_m^k . This has the effect of allowing $B_0 = b\mathbb{Z}^{k-l+1} = \{0\}^{l-1} \times b\mathbb{Z}^{k-l+1}$ in \mathbb{Z}^k for some bounded positive integer b .

Next we apply σ to (4) to see that (5) already holds for all (b_1, \dots, b_k) in $\pi(A)$. Now there are two cases depending on the rank r of $B_0 + \pi(A)$; of course $r \geq k - l + 1$.

CASE II₁: WE HAVE $r = k - l + 1$. This means that $\pi(A)$ lies in \mathbb{Z}^{k-l+1} . Then (4) becomes

$$(6) \quad \zeta_l^{a_l} \dots \zeta_k^{a_k} g_{k+1}^{a_{k+1}} \dots g_n^{a_n} = 1$$

for any \mathbf{a} in A . We have two independent \mathbf{a} and correspondingly two independent $\pi(\mathbf{a})$; so also two independent (a_l, \dots, a_k) in (6). The resulting pair of equations represents a point on $C''' \cap \mathcal{H}_{n-l-1}$, where the curve C''' in \mathbb{G}_m^{n-l+1} is parametrized by $(x_l, \dots, x_k, g_{k+1}, \dots, g_n)$. This C''' is not in a proper torsion coset, and so we can appeal to induction as in the proof of Lemma 4 because $n - l + 1 < n$.

CASE II₂: WE HAVE $r > k - l + 1$. Now Q lies in $S \cap \mathcal{H}_{l-2}$, not just $S \cap \mathcal{H}_{l-1}$. We use the projection λ from \mathbb{G}_m^k to $\mathbb{G}_m^l = \mathbb{G}_m^l \times \{1\}^{k-l}$ in \mathbb{G}_m^k . Thus $\lambda(Q)$ lies in $\lambda(S)$ in \mathbb{G}_m^l , and even in $\lambda(S) \cap \mathcal{H}_{l-2}$, because we had $k - l + 2$ relations on Q and we lost $k - l$ coordinates in the projection, so there remain at least two relations on $\lambda(Q)$. We now imitate Cases I and II above, writing S'' for the closure of $\lambda(S)$ in \mathbb{G}_m^l .

SUBCASE (λ, I) : THE POINT $\lambda(Q)$ LIES IN S''^{oa} . We argue as in Case I above. By BHT we get $h(\lambda(Q)) \leq \mathcal{B}(S'')$. Now we show that $\lambda(Q)$ avoids the exceptional set $Z(C'')$ in \mathbb{G}_m^l , where C'' is the closure of $\lambda(C')$ parametrized by (x_1, \dots, x_l) . Namely suppose $\lambda(Q) = \varphi(\lambda(P')^\sigma, \lambda(P'))$ lies in $Z(C'')$. We have chosen K so large that $Z(C'')$ lies in $\mathbb{G}_m^l(K)$. So the η_i ($i = 1, \dots, l$) lie in K . But (5) on $B_0 = b\mathbb{Z}^{k-l+1} = \{0\}^{l-1} \times b\mathbb{Z}^{k-l+1}$ shows that the other $\eta_i^b = 1$ ($i = l + 1, \dots, k$). Thus $\varphi(P^{b\sigma}, P^b)$ lies in $\mathbb{G}_m^n(K)$. Choosing the integer c of (3) to be divisible by b ,

we get a contradiction with (3). Thus indeed $\lambda(Q)$ avoids the exceptional set $Z(C'')$.

So by Lemma 1(c) we see that $h(\lambda(P'))$ is also bounded above. Thus $h(P')$ as well, for example by using the f_{li} ($i = l + 1, \dots, k$) in the proof of Lemma 4. Thus $h(P)$ too, and now the required finiteness follows from Lemma 3.

SUBCASE (λ, II) : THE POINT $\lambda(Q)$ DOES NOT LIE IN S''^{oa} . We argue as in Case II above. Now Lemma 1(b) shows that $\lambda(Q)$ lies in one of a finite number of anomalous curves of S'' , each one lying in a fixed $(l - 2)$ -dimensional coset Q_0H_0 . As above, we have chosen the number field K to include the field of definition of all the Q_0 . As $\lambda(Q) = \varphi(\lambda(P')^\sigma, \lambda(P'))$ we see from Lemma 6(a) that each Q_0H_0 is a torsion coset. Corresponding to this there is a fixed subgroup C_0 of $\mathbb{Z}^l = \mathbb{Z}^l \times \{0\}^{k-l}$ in \mathbb{Z}^k of rank 2 such that $\eta_1^{c_1} \dots \eta_l^{c_l} = 1$ for all (c_1, \dots, c_l) in C_0 .

Now the subgroup $b\mathbb{Z}^{k-l} = \{0\} \times b\mathbb{Z}^{k-l}$ of $b\mathbb{Z}^{k-l+1} = B_0$ in (5), which is also $\{0\}^l \times b\mathbb{Z}^{k-l} \subset \mathbb{Z}^k$, generates together with C_0 a fixed group of rank $k - l + 2$ on which (5) holds. This is exactly the required inductive step.

We therefore deduce (5) for a fixed group of rank k in \mathbb{Z}^k . So $Q^b = (1, \dots, 1)$ for some bounded positive b ; and the same for Q^c . But then $\varphi(P^{c\sigma}, P^c) = (1, \dots, 1)$, contradicting (3) quite badly. \square

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