Analisi Funzionale — *Mackey convergence and bornological topological modules*, by Nilson C. Bernardes Jr. and Dinamérico P. Pombo Jr., communicated on 14 May 2010.

Abstract. — The main results concerning Mackey convergent sequences are extended to the context of topological modules, including a characterization of bornological topological modules.

In his important work [5], Mackey introduced the notion of a Mackey convergent sequence in a locally convex space and characterized bornological locally convex spaces by means of that notion. The notion of bornologicalness appeared explicitly for the first time in an important work of Bourbaki [2], where it is observed that a space is bornological if and only if any linear mapping on it transforming bounded sets into bounded sets is continuous. A lucid presentation of the basic facts about the subject may be found in Grothendieck's book [3].

In this paper we define the notion of Mackey convergence in the context of topological modules over metrizable topological rings and prove that the basic facts about Mackey convergence may be extended to this general setting. The main result obtained here is a characterization of bornological topological modules showing that the approaches of Mackey and Bourbaki remain equivalent in our case.

Throughout this paper R shall denote a metrizable topological ring with a non-zero identity element such that the product of two arbitrary neighborhoods of 0 in R is a neighborhood of 0 in R, and all topological R-modules under consideration shall be unitary left topological R-modules.

Remark 1. If S is a topological ring with a non-zero identity element such that $0 \in \overline{S^{\times}}$, where $S^{\times}$ is the multiplicative group of all invertible elements of S, then the product of two arbitrary neighborhoods of 0 in S is a neighborhood of 0 in S. In particular, this property holds if $(S, \|\cdot\|)$ is a seminormed ring ([7], Definition 16.8) with a non-zero identity element such that there exists a $\lambda \in S^{\times}$ with $\|\lambda\| < 1$, and hence if $(\mathbb{K}, |\cdot|)$ is a non-trivially valued division ring. On the other hand, the product of two neighborhoods of 0 in the metrizable topological ring $\mathbb{Z}_p$ of p-adic integers is a neighborhood of 0 in $\mathbb{Z}_p$, although $0 \notin \overline{\mathbb{Z}_p^{\times}}$.

Definition 2. Let E be a topological R-module and $(x_n)_{n \in \mathbb{N}}$ a sequence in E. We shall say that $(x_n)_{n \in \mathbb{N}}$ *converges to* 0 *in the Mackey sense*, and write

$(x_n)_{n \in \mathbb{N}} \xrightarrow{M} 0$, if there exist a null sequence $(\lambda_n)_{n \in \mathbb{N}}$ in R and a null sequence $(y_n)_{n \in \mathbb{N}}$ in E such that $x_n = \lambda_n y_n$ for all $n \in \mathbb{N}$.

PROPOSITION 3. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a topological R-module E and consider the following conditions:*

(a) $(x_n)_{n \in \mathbb{N}} \xrightarrow{M} 0$;
(b) *there exists a bounded subset B of E satisfying the following property: for each neighborhood W of 0 in R there exists an $n_0 \in \mathbb{N}$ such that $x_n \in WB$ for all $n \geq n_0$;*
(c) $(x_n)_{n \in \mathbb{N}} \to 0$ *in E.*

Then (a) *and* (b) *are equivalent, and* (b) *implies* (c).

PROOF. (a) implies (b): By hypothesis, there are a null sequence $(\lambda_n)_{n \in \mathbb{N}}$ in R and a null sequence $(y_n)_{n \in \mathbb{N}}$ in E such that $x_n = \lambda_n y_n$ for all $n \in \mathbb{N}$. Put $B = \{y_n; n \in \mathbb{N}\}$, which is a bounded subset of E by Theorem 15.4 of [7], and let W be an arbitrary neighborhood of 0 in R. Then there exists an $n_0 \in \mathbb{N}$ such that $\lambda_n \in W$ for all $n \geq n_0$, and hence $x_n \in WB$ for all $n \geq n_0$. This proves (b).

(b) implies (c): Let U be an arbitrary neighborhood of 0 in E. By the boundedness of B, there exists a neighborhood W of 0 in R such that $WB \subset U$ and, by hypothesis, there exists an $n_0 \in \mathbb{N}$ such that $x_n \in WB$ for all $n \geq n_0$. Thus $x_n \in U$ for all $n \geq n_0$. Therefore $(x_n)_{n \in \mathbb{N}} \to 0$ in E, proving (c).

(b) implies (a): Let $W_0 \supset W_1 \supset \cdots \supset W_n \supset W_{n+1} \supset \cdots$ be a countable fundamental system of neighborhoods of 0 in R. We claim that there exists a sequence $(m_i)_{i \in \mathbb{N}}$ of natural numbers such that $m_i > m_{i-1}$ ($m_{-1} = 0$) and $x_n \in (W_i)^2 B$ for all $n \geq m_i$. To justify the existence of $(m_i)_{i \in \mathbb{N}}$ we shall argue by induction, as follows. First, since $(W_0)^2$ is a neighborhood of 0 in R, there is an integer $m_0 > 0$ such that $x_n \in (W_0)^2 B$ for all $n \geq m_0$. Second, let s be an integer ≥ 0 and suppose that integers m_i ($0 \leq i \leq s$) satisfying the required properties have been constructed. Then, since $(W_{s+1})^2$ is a neighborhood of 0 in R, there is an integer $m_{s+1} > m_s$ such that $x_n \in (W_{s+1})^2 B$ for all $n \geq m_{s+1}$. Thus the existence of $(m_i)_{i \in \mathbb{N}}$ is justified. Finally, for $i \in \mathbb{N}$ and $m_i \leq n < m_{i+1}$ we can write $x_n = \lambda_n^{(1)} \lambda_n^{(2)} z_n$, where $\lambda_n^{(1)}, \lambda_n^{(2)} \in W_i$ and $z_n \in B$. It is clear that $(\lambda_n^{(1)})_{n \geq m_0}$ and $(\lambda_n^{(2)})_{n \geq m_0}$ converge to 0 in R. Then $(\lambda_n^{(2)} z_n)_{n \geq m_0}$ converges to 0 in E and, consequently, $(x_n)_{n \in \mathbb{N}} \xrightarrow{M} 0$, proving (a).

This completes the proof.

Before proceeding, let us recall a known example of a null sequence which does not converge to 0 in the Mackey sense.

EXAMPLE 4. Let X be an arbitrary infinite-dimensional Banach space and let X' be its topological dual. By the Josefson-Nissenzweig theorem [4, 6], there is a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in X' such that $\|\varphi_n\| = 1$ for all $n \in \mathbb{N}$, and $(\varphi_n)_{n \in \mathbb{N}} \to 0$ in $(X', \sigma(X', X))$. If $(\lambda_n)_{n \in \mathbb{N}}$ is an arbitrary null sequence of strictly positive real numbers, then the sequence $(\varphi_n / \lambda_n)_{n \in \mathbb{N}}$ is not bounded in $(X', \sigma(X', X))$ because

$\|\varphi_n/\lambda_n\| = 1/\lambda_n$ for all $n \in \mathbb{N}$. Therefore $(\varphi_n)_{n\in\mathbb{N}}$ does not converge to 0 in the Mackey sense in the space $(X', \sigma(X', X))$, which is not metrizable.

PROPOSITION 5. *Let E and F be two topological R-modules and $u : E \to F$ an R-linear mapping. Then the following conditions are equivalent:*

(a) *for every bounded subset B of E, $u(B)$ is a bounded subset of F;*
(b) *for every sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to 0 in the Mackey sense, $(u(x_n))_{n\in\mathbb{N}}$ converges to 0 in the Mackey sense;*
(c) *for every sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to 0 in the Mackey sense, $(u(x_n))_{n\in\mathbb{N}}$ converges to 0 in F;*
(d) *for every sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to 0 in the Mackey sense, $(u(x_n))_{n\in\mathbb{N}}$ is bounded.*

PROOF. First, (b) implies (c) by Proposition 3, (c) implies (d) because every null sequence is bounded, and (a) implies (d) by Proposition 3 and the fact that every null sequence is bounded.

Let $W_0 \supset W_1 \supset \cdots \supset W_n \supset W_{n+1} \supset \cdots$ be a countable fundamental system of neighborhoods of 0 in R.

Now, we claim that (d) implies (a). Indeed, assume there exists a bounded subset B of E such that $u(B)$ is not bounded. Since each $(W_n)^3$ is a neighborhood of 0 in R, there exists a neighborhood V of 0 in F such that $W_n u((W_n)^2 B) = (W_n)^3 u(B) \not\subset V$ for all $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there are $\lambda_n^{(1)}, \lambda_n^{(2)} \in W_n$ and $z_n \in B$ so that $W_n u(\lambda_n^{(1)} \lambda_n^{(2)} z_n) \not\subset V$ for all $n \in \mathbb{N}$. As we have already observed, $(\lambda_n^{(1)} \lambda_n^{(2)} z_n)_{n\in\mathbb{N}} \overset{\text{M}}{\to} 0$; but $(u(\lambda_n^{(1)} \lambda_n^{(2)} z_n))_{n\in\mathbb{N}}$ is not bounded.

Finally, let us prove that (d) implies (b). Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E such that $(x_n)_{n\in\mathbb{N}} \overset{\text{M}}{\to} 0$ and let B be a bounded subset of E as in condition (b) of Proposition 3. By arguing as in the proof of Proposition 3, with $(W_i)^4$ in place of $(W_i)^2$, we obtain a sequence $(m_i)_{i\in\mathbb{N}}$ of natural numbers such that $m_i > m_{i-1}$ $(m_{-1} = 0)$ and $x_n \in (W_i)^4 B$ for $n \geq m_i$. Therefore we can write $x_n = \lambda_n^{(1)} \lambda_n^{(2)} \lambda_n^{(3)} \lambda_n^{(4)} z_n$ for $n \geq m_0$, where $(\lambda_n^{(1)})_{n\geq m_0}, (\lambda_n^{(2)})_{n\geq m_0}, (\lambda_n^{(3)})_{n\geq m_0}$ and $(\lambda_n^{(4)})_{n\geq m_0}$ are null sequences in R and $(z_n)_{n\geq m_0}$ is a sequence in B. Since $(\lambda_n^{(4)} z_n)_{n\geq m_0} \to 0$ in E, $(\lambda_n^{(3)} \lambda_n^{(4)} z_n)_{n\geq m_0} \overset{\text{M}}{\to} 0$, and hence $(u(\lambda_n^{(3)} \lambda_n^{(4)} z_n))_{n\geq m_0}$ is bounded by hypothesis. Consequently, $(u(x_n))_{n\in\mathbb{N}} \overset{\text{M}}{\to} 0$ since $u(x_n) = \lambda_n^{(1)} (\lambda_n^{(2)} u(\lambda_n^{(3)} \lambda_n^{(4)} z_n))$ for $n \geq m_0$ and $(\lambda_n^{(2)} u(\lambda_n^{(3)} \lambda_n^{(4)} z_n))_{n\geq m_0} \to 0$ in F.

This completes the proof.

PROPOSITION 6. *Let E be a metrizable topological R-module such that the product of any neighborhood of 0 in R by any neighborhood of 0 in E is a neighborhood of 0 in E. If $(x_n)_{n\in\mathbb{N}} \to 0$ in E, then $(x_n)_{n\in\mathbb{N}} \overset{\text{M}}{\to} 0$.*

PROOF. Let $W_0 \supset W_1 \supset \cdots \supset W_n \supset W_{n+1} \supset \cdots$ (resp. $U_0 \supset U_1 \supset \cdots \supset U_n \supset U_{n+1} \supset \cdots$) be a countable fundamental system of neighborhoods of 0 in R (resp. of 0 in E). Since $W_i U_i$ is a neighborhood of 0 in E for all $i \in \mathbb{N}$, we can argue as in the proof of Proposition 3 to construct a sequence $(m_i)_{i\in\mathbb{N}}$ of natural

numbers so that $m_i > m_{i-1}$ ($m_{-1} = 0$) and $x_n \in W_i U_i$ for $n \geq m_i$. Hence we can write $x_n = \lambda_n y_n$ for $n \geq m_0$, where $(\lambda_n)_{n \geq m_0} \to 0$ in R and $(y_n)_{n \geq m_0} \to 0$ in E. Therefore $(x_n)_{n \in \mathbb{N}} \overset{M}{\to} 0$, as was to be shown.

The condition concerning product of neighborhoods of 0 is essential for the validity of Proposition 6, as the following example shows.

EXAMPLE 7. Let S be a ring with an identity element $1 \neq 0$ endowed with the discrete topology. Let I be a non-empty countable set and consider the product topological ring $S' = S^I$. Note that S' is metrizable, discrete if I is finite and non-discrete if I is infinite. Fix an element $j \in I$ and let $\pi_j : (\lambda_i)_{i \in I} \in S' \mapsto \lambda_j \in S$. Let E be the product topological group $S^{\mathbb{N}}$ endowed with the following law:

$$((\lambda_i)_{i \in I}, (x_k)_{k \in \mathbb{N}}) \in S' \times E \mapsto (\lambda_j x_k)_{k \in \mathbb{N}} \in E.$$

Then E is a metrizable topological S'-module. Note that $W = \pi_j^{-1}(\{0\})$ is a neighborhood of 0 in S', $V = E$ is a neighborhood of 0 in E, but $WV = \{0\}$ is not a neighborhood of 0 in E. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in E such that $(x_n)_{n \in \mathbb{N}} \overset{M}{\to} 0$, then there are a null sequence $(\lambda_n)_{n \in \mathbb{N}}$ in S' and a null sequence $(y_n)_{n \in \mathbb{N}}$ in E such that $x_n = \lambda_n y_n$ for all $n \in \mathbb{N}$. But, since S is equipped with the discrete topology, there is an $m \in \mathbb{N}$ so that $\pi_j(\lambda_n) = 0$ for $n \geq m$, and hence $x_n = 0$ for $n \geq m$. For each $n = 1, 2, \ldots$, let $z_n = (\underbrace{0, 0, \ldots, 0}_{n \text{ times}}, 1, 1, 1, \ldots) \in E$. Obviously, $(z_n)_{n \geq 1} \to 0$ in E and, by what we have just observed, $(z_n)_{n \geq 1}$ does not converge to 0 in the Mackey sense.

Our final result is a characterization of bornological topological R-modules [1] which incorporates the approaches of Mackey and Bourbaki.

THEOREM 8. *For a topological R-module E, consider the following conditions:*

(a) *E is bornological;*
(b) *for each topological R-module F, each R-linear mapping from E into F which transforms bounded sets into bounded sets is continuous;*
(c) *for each semimetrizable topological R-module F, each R-linear mapping from E into F which transforms bounded sets into bounded sets is continuous;*
(d) *for each topological R-module F, each R-linear mapping from E into F which transforms sequences converging to 0 in the Mackey sense into sequences converging to 0 in the Mackey sense is continuous;*
(e) *for each topological R-module F, each R-linear mapping from E into F which transforms sequences converging to 0 in the Mackey sense into null sequences is continuous;*
(f) *for each topological R-module F, each R-linear mapping from E into F which transforms sequences converging to 0 in the Mackey sense into bounded sequences is continuous;*
(g) *for each topological R-module F, each set of R-linear mappings from E into F which transforms bounded sets into bounded sets is equicontinuous.*

Then conditions (a), (b), (d), (e), (f) *and* (g) *are equivalent. In addition, if* $0 \in \overline{R^{\times}}$*, then all conditions are equivalent.*

PROOF. The equivalence among (b), (d), (e) and (f) follows from Proposition 5. Obviously, (g) implies (b). The equivalence between (a) and (b) and the fact that (a) implies (g) follow from the theorem established in [1].

Clearly, (b) implies (c). So it remains to prove that (c) implies (b) under the assumption that $0 \in \overline{R^{\times}}$. Indeed, let $W_0 \supset W_1 \supset \cdots \supset W_n \supset W_{n+1} \supset \cdots$ be a countable fundamental system of neighborhoods of 0 in R. Let F be an arbitrary topological R-module and $u : E \to F$ an R-linear mapping which transforms bounded sets into bounded sets. Let V be an arbitrary symmetric neighborhood of 0 in F and let V_1 be a symmetric neighborhood of 0 in F such that $V_1 + V_1 \subset V$. Choose a neighborhood T_2 of 0 in R, $T_2 \subset W_2$, and a symmetric neighborhood V_2 of 0 in F, $V_2 \subset V_1$, so that $T_2 V_2 \subset V_1$, and choose a symmetric neighborhood V_3 of 0 in F so that $V_3 + V_3 \subset T_2 V_2 \cap V_2$. Now, let T_4 be a neighborhood of 0 in R, $T_4 \subset W_4$, and V_4 a symmetric neighborhood of 0 in F, $V_4 \subset V_3$, so that $T_4 V_4 \subset V_3$. Let V_5 be a symmetric neighborhood of 0 in F so that $V_5 + V_5 \subset T_4 V_4 \cap V_4$. By induction, we construct a sequence $(T_{2n})_{n \geq 1}$ of neighborhoods of 0 in R and a sequence $(V_n)_{n \in \mathbb{N}}$ of symmetric neighborhoods of 0 in F, with $V_0 = V$, such that $T_{2n} \subset W_{2n}$, $V_{2n} \subset V_{2n-1}$, $T_{2n} V_{2n} \subset V_{2n-1}$ and $V_{2n+1} + V_{2n+1} \subset T_{2n} V_{2n} \cap V_{2n}$ for all $n \geq 1$. We claim that the filter base $\mathscr{B} = \{V_1, V_3, V_5, \ldots\}$ on F satisfies conditions (ATG 1), (ATG 2), (TMN 1), (TMN 2) and (TMN 3) of Theorem 12.3 of [7]. In fact, (ATG 1) holds because $V_{2n+1} + V_{2n+1} \subset V_{2n} \subset V_{2n-1}$ for all $n \geq 1$, (ATG 2) holds because every element of $\mathscr{B}$ is symmetric, and (TMN 1) holds because $T_{2n} V_{2n+1} \subset T_{2n} V_{2n} \subset V_{2n-1}$ for all $n \geq 1$. Now, let $y_0 \in F$ and $n \in \mathbb{N}$ be arbitrary. Since V_{2n+1} is a neighborhood of 0 in F, there is a neighborhood W of 0 in R such that $W y_0 \subset V_{2n+1}$, and (TMN 2) holds. Finally, let $\lambda_0 \in R$ and $n \geq 1$ be arbitrary. Since $\{W_2, W_4, W_6, \ldots\}$ is a fundamental system of neighborhoods of 0 in R, there is an integer $m > n$ such that $\lambda_0 T_{2m} \subset T_{2n}$. Thus $\lambda_0 V_{2m+1} \subset \lambda_0 T_{2m} V_{2m} \subset T_{2n} V_{2n} \subset V_{2n-1}$, and (TMN 3) holds. Therefore, by the theorem just mentioned, there exists a unique R-module topology τ on F for which $\mathscr{B}$ is a fundamental system of neighborhoods of 0. Let G be the topological R-module (F, τ), which is semimetrizable because $\mathscr{B}$ is countable. Since $u : E \to G$ transforms bounded sets into bounded sets, then it is continuous by hypothesis. Consequently, $u^{-1}(V_1)$ is a neighborhood of 0 in E, and hence $u^{-1}(V)$ is a neighborhood of 0 in E because $V \supset V_1$. Therefore $u : E \to F$ is continuous, proving (b).

This completes the proof of the theorem.

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Nilson C. Bernardes Jr.
Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530
21945-970, RIO DE JANEIRO, RJ (Brasil)
bernardes@im.ufrj.br

Dinamérico P. Pombo Jr.
Instituto de Matemática
Universidade Federal Fluminense
Rua Mário Santos Braga, s/nº
24020-140, NITERÓI, RJ (Brasil)
dpombo@terra.com.br