



**History of Mathematics** — *Peano on derivative of measures: strict derivative of distributive set functions*, by GABRIELE H. GRECO, SONIA MAZZUCCHI, ENRICO M. PAGANI, presented by Giorgio Letta on 16 April 2010.

*On the occasion of the 150<sup>th</sup> anniversary of the birth of Giuseppe Peano.*

ABSTRACT. — By retracing research on *coexistent magnitudes* (*grandeurs coexistantes*) by CAUCHY [9, (1841)], PEANO in *Applicazioni geometriche del calcolo infinitesimale* [48, (1887)] defines the “density” (strict derivative) of a “mass” (a distributive set function) with respect to a “volume” (a positive distributive set function), proves its continuity (whenever the strict derivative exists) and shows the validity of the *mass-density paradigm*: “mass” is recovered from “density” by integration with respect to “volume”. It is remarkable that PEANO’s strict derivative provides a consistent mathematical ground to the concept of “infinitesimal ratio” between two magnitudes, successfully used since KEPLER. In this way the classical (i.e., pre-Lebesgue) measure theory reaches a complete and definitive form in PEANO’s *Applicazioni geometriche*.

A primary aim of the present paper is a detailed exposition of PEANO’s work of 1887 leading to the concept of strict derivative of distributive set functions and their use. Moreover, we compare PEANO’s work and LEBESGUE’s *La mesure des grandeurs* [35, (1935)]: in this memoir LEBESGUE, motivated by coexistent magnitudes of CAUCHY, introduces a uniform-derivative of certain additive set functions, a concept that coincides with PEANO’s strict derivative. Intriguing questions are whether LEBESGUE was aware of the contributions of PEANO and which role is played by the notions of strict derivative or of uniform-derivative in today mathematical practice.

KEY WORDS: Derivative of measures, strict derivative of set functions, distributive set functions, mass-density paradigm, Peano-Jordan measure.

MATHEMATICS SUBJECT CLASSIFICATION (2010): 01A65, 28-03, 28A15, 28A25.

## 1. INTRODUCTION

By referring to CAUCHY [9, (1841)] PEANO introduces in *Applicazioni geometriche del calcolo infinitesimale* [48, (1887)] the concept of *strict derivative* of set functions. The set functions considered by him are not precisely finitely additive measures. The modern concept of finite additivity is based on partitions by disjoint sets, while PEANO’s additivity property coincides with a traditional supple concept of “decompositions of magnitudes”, which PEANO implements in his proofs as *distributive set functions*.

Contrary to PEANO’s strict derivative (*rapporto*), CAUCHY’s derivative (*rapport différentiel*) of a set function corresponds to the usual derivative of functions of one variable. In PEANO’s Theorem 7.1 on strict derivative of distributive set

functions the (physical) *mass-density paradigm* is realized: the “mass” (a distributive set function) is recovered from the “density” (the strict derivative) by integration with respect to the “volume” (a positive distributive set function of reference).

PEANO expresses CAUCHY’s ideas in a more precise and modern language and completes the program proposed by CAUCHY, who, at the end of his article [9, (1841) p. 229], writes:

Dans un autre Mémoire nous donnerons de nouveaux développements aux principes ci-dessus exposés [on coexistent magnitudes], en les appliquant d’une manière spéciale à l’évaluation des longueurs, des aires et des volumes.<sup>1</sup>

Among numerous applications of PEANO’s strict derivative of set functions which can be found in *Applicazioni geometriche*, there are formulae on oriented integrals, in which the geometric vector calculus by GRASSMANN plays an important role. For instance, PEANO proves the formula of area starting by his definition of area of a surface, that he proposed in order to solve the drawbacks of SERRET’s definition of area [62, (1879)].

The didactic value of PEANO’s strict derivative of set functions is transparent: in *La mesure des grandeurs* [35, (1935)] LEBESGUE himself uses a similar approach to differentiation of measures in order to simplify the exposition of his measure theory.

In Section 2, PEANO’s and LEBESGUE’s derivative are compared in view of the paradigm of mass-density and of the paradigm of primitives, that motivated mathematical research between 19<sup>th</sup> century and the beginning of 20<sup>th</sup> century. In the celebrated paper *L’intégration des fonctions discontinues* [29, (1910)] LEBESGUE defines a derivative of  $\sigma$ -additive measures with respect to the volume. He proves its existence and its measurability. In the case of absolute continuity of the  $\sigma$ -additive measures, LEBESGUE proves that the measure is given by the integral of his derivative with respect to the volume. As it will be seen later in details, PEANO’s *strict derivative* of distributive set functions does not necessarily exist and, moreover, whenever it exists, PEANO’s strict derivative is continuous, while LEBESGUE’s derivative in general is not.

Section 3 presents an overview of PEANO’s work on pre-Lebesgue classical measure theory which is completed in Sections 5–6.

Section 4 is devoted to an analysis of CAUCHY’s *Coexistent magnitudes* [9, (1841)]<sup>2</sup>, by emphasizing the results that will be found, in a different language, in PEANO’s *Applicazioni geometriche* or in LEBESGUE’s *La mesure des grandeurs*.

Section 5 concerns the concept of “distributive family” and of “distributive set function” as presented by PEANO in *Applicazioni geometriche* and in his paper *Le grandezze coesistenti di Cauchy* [55, (1915)].

<sup>1</sup> [In another memoir we will give new developments to the above mentioned statements [on coexistent magnitudes], and we will apply them to evaluate lengths, areas and volumes.]

<sup>2</sup> From now on we refer to CAUCHY’s paper *Mémoire sur le rapport différentiel de deux grandeurs qui varient simultanément* [9, (1841)] as to *Coexistent magnitudes*.

Section 6 presents a definition of strict derivative of set functions, main results and some applications, while in Section 7 we discuss PEANO's definition of integral of set functions and a related theorem that realizes the mentioned physical paradigm of mass-density.

Section 8 presents the approach of LEBESGUE in *La mesure des grandeurs* to CAUCHY's coexistent magnitudes, leading to introduction of a new notion of derivative: the uniform-derivative.

We observe that this paper is mainly historical. From a methodological point of view, we are focussed on primary sources, that is, on mathematical facts and not on the elaborations or interpretations of these facts by other scholars of history of mathematics. For convenience of the reader, original statements and, in some case, terminology are presented in a modern form, preserving, of course, their content.

Historical investigations on forgotten mathematical achievements are not useless (from the point of view of mathematics), because some of them carry ideas that remain innovative today. This thought was very well expressed by MASCHERONI before the beginning of the study of the geometrical problems leading to the *Geometria del compasso* (1797):

[...] mentre si trovano tante cose nuove progredendo nelle matematiche, non si potrebbe forse trovare qualche luogo ancora incognito retrocedendo?<sup>3</sup>

By respect for historical sources and for the reader's convenience, the quotations in the sequel will appear in the original tongue with a translation in double square brackets, placed in footnote.

## 2. THE PHYSICAL PARADIGM OF MASS-DENSITY VERSUS THE PARADIGM OF PRIMITIVES

In *Philosophiae Naturalis Principia Mathematica* (1687) the first definition concerns mass and density:

Quantitas materiae est mensura ejusdem orta ex ilius densitate et magnitudine conjunctim [...]. Hanc autem quantitatem sub nomine corporis vel massa in sequentibus passim intelligo.<sup>4</sup>

In this sentence NEWTON presents the *mass-density paradigm* (i.e., the mass can be computed in terms of the density and, conversely, the density can be obtained from the mass) as a fundament of Physics.

In *Coexistent magnitudes* [9, (1841)] CAUCHY, with a clear didactic aim, uses the mass-density paradigm in order to give a unitary exposition of several problems related to differential calculus.

---

<sup>3</sup> [While we can find so many new things by moving forward in mathematics, why can't we find some still unknown area by retroceding?]

<sup>4</sup> [The quantity of matter is a measure of the matter itself, arising from its density and magnitude conjunctly [...]. It is this quantity that I mean hereafter everywhere under the name of body or mass.]

From a mathematical point of view the implementation of this physical paradigm presents some difficulties and it does not assure a univocal answer. The first difficulty is in defining what is a “mass”, the second is in choosing a procedure for evaluating “density” and, finally, in determining under what condition and how it is possible “to recover” the mass from the density.

All these critical aspects that we find in CAUCHY [9, (1841)], are overcome in a precise and clear way by PEANO in *Applicazioni geometriche* [48, (1887)].

Natural properties that connect density and mass are the following:

- (2.1) *The density of a homogenous body is constant.*
- (2.2) *The greater is the density, the greater is the mass.*
- (2.3) *The mass of a body, as well as its volume, is the sum of its parts.*

The realization of the physical paradigm can be mathematically expressed by the following formula

$$(2.4) \quad \mu(A) = \int_A g \, d(\text{vol}_n)$$

where  $\mu$  is the “mass”,  $g$  is the “density” and  $\text{vol}_n$  is the  $n$ -dimensional volume.<sup>5</sup>

The properties (2.1), (2.2) and (2.3) do not allow for a direct derivation of (2.4) without further conditions depending on the meaning of integral; for instance, having in mind the Riemann integral, an obvious necessary condition is the Riemann integrability of the density  $g$ .

In PEANO’S *Applicazioni Geometriche* [48, (1887)]:

- the “masses” and the “volumes” are represented by *distributive set functions*, as it will be shown in detail in §5,
- the “densities” (strict derivatives) are computed using a limit procedure, as we shall see in the sequel (see formula (2.5)),
- the “mass” is recovered by integration using (2.4). This final step is strengthened by the fact that PEANO’S strict derivative is continuous.

The mathematical realization of mass-density paradigm is directly connected with mathematical paradigm of primitives, that is with the study of conditions assuring that integration is the inverse operation of differentiation.

At the beginning of the 20<sup>th</sup> century the problem of looking for primitives is the cornerstone of the new theory of measure, founded by LEBESGUE [28, (1904)]. The problem of primitives becomes arduous when one has to pass from functions of one variable to functions of more variables. LEBESGUE in *L’intégration des fonctions discontinues* [29, (1910)] overcomes these difficulties by substituting the integral of a generic function  $g$  with a set function  $\mu$  described by formula (2.4).

The paradigm of primitives gives more importance to the operations (of differentiation and integration) than to the set functions. On the contrary, in the mass-density paradigm the primary aim is the evaluation of the infinitesimal ratio

---

<sup>5</sup>In today terminology, the realization of (2.4) is expressed by saying that  $g$  is the *Radon-Nikodym derivative* of  $\mu$  with respect to  $\text{vol}_n$ .

between two set functions (for instance, mass and volume) in order to have the “density” and, consequently, to recover the “mass” by integrating the “density” with respect to “volume”. On the other hand in the paradigm of primitives the main problem is extending of the notion of integral in order to describe a primitive of a given function and, consequently, to preserve fundamental theorem of calculus.

In LEBESGUE’s works the two paradigms appear simultaneously for the first time in the second edition of his famous book *Leçons sur l’intégration et la recherche des fonctions primitives* [32, (1928) pp. 196–198]. In 1921 (see [37, vol. I, p. 177]) LEBESGUE has already used some physical concept in order to make the notion of set function intuitive; analogously in [30, (1926)] and [32, (1928) pp. 290–296] he uses the mass-density paradigm in order to make more natural the operations of differentiation and integration. In his lectures *Sur la mesure des grandeurs* [35, (1935)], the physical paradigm leads LEBESGUE to an alternative definition of derivative: he replaces his derivative of 1910 with the new uniform-derivative (equivalent to the strict derivative introduced by PEANO), thus allowing him to get continuity of the derivative.

Before comparing PEANO’s and LEBESGUE’s derivative of set functions, we recall the definitions of derivative given by PEANO and CAUCHY.

PEANO’s strict derivative of a set function (for instance, the “density” of a “mass”  $\mu$  with respect to the “volume”) at a point  $\bar{x}$  is computed, when it exists, as the limit of the quotient of the “mass” with respect to the “volume” of a cube  $Q$ , when the supremum of the distances of the points of the cube from  $\bar{x}$  tends to 0 (in symbols  $Q \rightarrow \bar{x}$ ). In formula, PEANO’s strict derivative  $g_P(\bar{x})$  of a mass  $\mu$  at  $\bar{x}$  is given by:

$$(2.5) \quad g_P(\bar{x}) := \lim_{Q \rightarrow \bar{x}} \frac{\mu(Q)}{\text{vol}_n(Q)}.$$

Every limit procedure of a quotient of the form  $\frac{\mu(Q)}{\text{vol}_n(Q)}$  with  $Q \rightarrow \bar{x}$  and the point  $\bar{x}$  not necessarily belonging to  $Q$ , will be referred to as *derivative à la Peano*.

On the other hand, CAUCHY’s derivative [9, (1841)] is obtained as the limit of the ratio between “mass” and “volume” of a cube  $Q$  including the point  $\bar{x}$ , when  $Q \rightarrow \bar{x}$ . In formula, CAUCHY’s derivative  $g_C(\bar{x})$  of a mass  $\mu$  at  $\bar{x}$  is given by:

$$(2.6) \quad g_C(\bar{x}) := \lim_{\substack{Q \rightarrow \bar{x} \\ \bar{x} \in Q}} \frac{\mu(Q)}{\text{vol}_n(Q)}.$$

Every limit procedure of a quotient of the form  $\frac{\mu(Q)}{\text{vol}_n(Q)}$  with  $Q \rightarrow \bar{x}$  and the point  $\bar{x}$  belonging to  $Q$ , will be referred to as *derivative à la Cauchy*.

LEBESGUE’s derivative of set functions is computed *à la Cauchy*. Notice that LEBESGUE considers finite  $\sigma$ -additive and absolutely continuous measures as “masses”, while PEANO considers distributive set functions. LEBESGUE’s derivative exists (i.e., the limit (2.6) there exists for *almost every*  $\bar{x}$ ), it is measurable and the reconstruction of a “mass” as the integral of the derivative is assured by absolute continuity of the “mass” with respect to volume. On the contrary,

PEANO's strict derivative does not necessarily exist, but when it exists, it is continuous and the mass-density paradigm holds.<sup>6</sup>

The constructive approaches to differentiation of set functions corresponding to the two limits (2.5) and (2.6) are opposed to the approach given by RADON [61, (1913)] and NIKODYM [44, (1930)], who define the derivative in a more abstract and wider context than those of LEBESGUE and PEANO. As in the case of LEBESGUE, a Radon-Nikodym derivative exists; its existence is assured by assuming absolute continuity and  $\sigma$ -additivity of the measures.

In concluding this Section, let us remark that the physical properties (2.1), (2.2) and (2.3), that stand at the basis of the mass-density paradigm, lead to the following direct characterization of the Radon-Nikodym derivative. Let  $\mu$  and  $\nu$  be finite  $\sigma$ -additive measures on a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of  $X$  and let  $\nu$  be positive and  $\mu$  be absolutely continuous with respect to  $\nu$ . A function  $g : X \rightarrow \mathbb{R}$  is a *Radon-Nikodym derivative* of  $\mu$  with respect to  $\nu$  (i.e.,  $\mu(A) = \int_A g \, d\nu$  for every  $A \in \mathcal{A}$ ) if and only if the following two properties hold for every real number  $a$ :

$$(2.7) \quad \mu(A) \geq a\nu(A) \text{ for every } A \subset \{g \geq a\} \text{ and } A \in \mathcal{A},$$

$$(2.8) \quad \mu(A) \leq a\nu(A) \text{ for every } A \subset \{g \leq a\} \text{ and } A \in \mathcal{A},$$

where  $\{g \leq a\} := \{x \in X : g(x) \leq a\}$  and, dually,  $\{g \geq a\} := \{x \in X : g(x) \geq a\}$ . These properties (2.7) and (2.8), expressed by NIKODYM [44, (1930)] in terms of Hahn decomposition of measures, are a natural translation of properties (2.1), (2.2) and (2.3).

### 3. PEANO ON (PRE-LEBESGUE) CLASSICAL MEASURE THEORY

The interest of PEANO in measure theory is rooted in his criticism of the *definition of area* (1882), of the *definition of integral* (1883) and of the *definition of derivative* (1884). This criticism leads him to an innovative measure theory, which is extensively exposed in Chapter V of *Applicazioni geometriche* [48, (1887)].

The definition of area given by SERRET in [62, (1879)] contrasted with the traditional definition of area: in 1882 PEANO, independently of SCHWARZ, observed (see [51, (1890)]) that the area of a cylindrical surface cannot be evaluated as the limit of inscribed polyhedral surfaces, as prescribed by SERRET's definition. In *Applicazioni geometriche*, PEANO provides a consistent definition of area and proves the integral formula of area.<sup>7</sup>

---

<sup>6</sup>Clearly, if PEANO's strict derivative of a finite  $\sigma$ -additive measure exists, then it coincides with LEBESGUE derivative and the "mass" is absolutely continuous.

Nowadays it is not surprising that LEBESGUE's derivative can be seen as PEANO's strict derivative by lifting both measures on a  $\sigma$ -algebra  $\mathcal{A}$  and  $\mathcal{A}$ -measurable functions to measures on the Stone space associated to  $\mathcal{A}$  and the related continuous functions, respectively.

<sup>7</sup>This topic will be extensively analyzed in a forthcoming paper by GRECO, MAZZUCCHI, PAGANI [19].

PEANO's criticism of the definition of Riemann integral of a function and of its relation with the area of the *ordinate-set* (i.e., hypograph of the function) [45, (1883)], forces him to introduce outer/inner measure as the set-theoretic counterparts of upper/lower integral: he defines the latter in terms of infimum/supremum (instead of limits, as done traditionally) of the Darboux sums.<sup>8</sup> PEANO, in introducing the inner and outer measure as well as in defining area [51, (1890)], is also influenced by ARCHIMEDES's approach on calculus of area, length and volume of convex figures.

In 1884, by analyzing the proof of mean value theorem, given by JORDAN<sup>9</sup> in the first edition of *Cours d'analyse*, PEANO stresses the difference between differentiable functions and functions with continuous derivative. The continuity of the derivative is expressed by PEANO in terms of the existence of the limit

$$(3.1) \quad \lim_{\substack{x, y \rightarrow \bar{x} \\ x \neq y}} \frac{f(x) - f(y)}{x - y}$$

for any  $\bar{x}$  in the domain of  $f$ .<sup>10</sup> Moreover, PEANO, in his correspondence with JORDAN [46, 47, (1884)], observes that uniform convergence of the difference quotient is equivalent to the continuity of the derivative.<sup>11</sup> This notion of contin-

---

<sup>8</sup>According with LETTA [38, 39], the notion of negligible set is introduced after an arduous process of investigation on "similar" notions related to cardinality and topology, between 1870 and 1882. Afterward, the definition of *Inhalt* (content) appears in the works by STOLZ [64, (1884)], CANTOR [5, (1884)], HARNACK [20, (1885)]. The notions of inner and outer measure are introduced by PEANO in [45, (1883) p. 446] and in [48, (1887) pp. 152–161], and later by JORDAN [24, (1892)]. In the following we will refer to the inner and to the outer measures as to *Peano-Jordan measures*.

In [38] LETTA appraises PEANO's contributions by the lapidary phrase:

[C]on uno sforzo di astrazione veramente notevole per il suo tempo, egli [Peano] osserva che i concetti di lunghezza, area, volume sono altrettanti casi particolari di un unico concetto, la cui definizione (tradotta in linguaggio moderno) può essere [...] [formulata come] *funzione additiva d'insieme*.

[[With an effort of abstraction, really remarkable for his epoch, he [Peano] notices that the concepts of length, area, volume are instances of the same notion corresponding, in modern terms, to finitely additive set function.]

<sup>9</sup>JORDAN, famous geometer and algebraist, publishes only a few papers on mathematical analysis. His most famous work is the *Cours d'analyse*, published in several editions. To our knowledge the relationship between PEANO and JORDAN was good and based on reciprocal appreciation, as one can deduce from two letters conserved in Archives de la Bibliothèque Centrale de l'École Polytechnique (Paris).

<sup>10</sup>Later, in a paper with didactic value [53, (1892)], PEANO re-proposes the distinction between Definition (3.1) and the usual derivative of a function, and underlines the correspondence of (3.1) with the definition of density in Physics.

Nowadays the function  $f$  is said *strictly differentiable* at the point  $\bar{x}$  if the limit (3.1) exists; consequently, the value of the limit (3.1) is called *strict derivative* of  $f$  at  $\bar{x}$ .

<sup>11</sup>Section 80 of JORDAN's *Cours d'analyse* [25, (1893) p. 68], titled "*Cas où  $\frac{f(x+h) - f(x)}{h}$  tend uniformément vers  $f'(x)$* ", contains a trace of it.

uous derivative will be the basis of PEANO's strict derivative of distributive set functions.

*Applicazioni geometriche* is a detailed exposition (more than 300 pages) of several topics of geometric applications of infinitesimal calculus.<sup>12</sup> In *Applicazioni geometriche* PEANO refounds the notion of Riemann integral by means of inner and outer measures<sup>13</sup>, and extends it to abstract measures. The development of the theory is based on solid topological and logical ground and on a deep knowledge of set theory. He introduces the notions of closure, interior and boundary of sets.

PEANO in *Applicazioni geometriche* [48, (1887)], and later JORDAN in the paper [24, (1892)] and in the second edition of *Cours d'Analyse* [25, (1893)], develops the well known concepts of classical measure theory, namely, measurability, change of variables, fundamental theorems of calculus, with some methodological differences between them.<sup>14</sup>

The mathematical tools employed by PEANO were really innovative at that time (and maybe are even nowadays), both on a geometrical and a topological level. PEANO used extensively the geometric vector calculus introduced by GRASSMANN. The geometric notions include oriented areas and volumes (called *geometric forms*).

Our main interest concerns Chapter V of PEANO's *Applicazioni geometriche*, where we find differentiation of distributive set functions.

*Applicazioni geometriche* is widely cited, but we have the feeling that the work is not sufficiently known. The revolutionary character of PEANO's book is remarked by J. TANNERY [65, (1887)]:

---

<sup>12</sup>As detailed in DOLECKI, GRECO [13], between several interesting concepts studied in *Applicazioni geometriche* that are not directly connected with measure theory, we recall the limit of sequences of sets (now called *Kuratowski limits*), the introduction of the concept of differentiability of functions (nowadays called *Fréchet differentiability*), the definition of tangent cone (nowadays called *Bouligand cone*), the necessary condition of optimality (nowadays called *Fermat conditions*) and a detailed study of problems of maximum and minimum.

<sup>13</sup>The simultaneous construction of inner and outer measure is the basis of the evolution of the theory leading to Lebesgue measure. Fortunately, CARATHÉODORY [6, (1914)] and HAUSDORFF [21, (1919)] put an end to the intoxication due to the presence of inner measure, as CARATHÉODORY writes [14, (2004) p. 72]:

Borel and Lebesgue (as well as Peano and Jordan) assigned an outer measure  $m^*(A)$  and an inner measure  $m_*(A)$  to every point set  $A$  [...]. The main advantage, however, is that the new definition [i.e., the exterior measure of Charathéodory] is independent of the concept of an *inner measure*.

<sup>14</sup>In a first paper of JORDAN [24, (1892)] and in a more extensive way in his *Cours d'analyse* [25, (1893)], we find several PEANO's results. There are, however, methodological differences between their approaches: PEANO constructs his measure by starting from polygons, while JORDAN considers (in the 2-dimensional case) squares. The definition proposed by PEANO does not have the simplicity of that of JORDAN, but it is independent of the reference frame and it is, by definition, invariant under isometries, without any need of further proof. Moreover, PEANO's definition allows for a direct computation of the proportionality factor appearing under the action of affine transformation; in previous works PEANO had developed a formalism allowing for computation of areas of polygons in a simple way (see [19] for details).



Le Chapitre V porte ce titre: *Grandeurs géométriques*. C'est peut-être le plus important et le plus intéressant, celui, du moins, par lequel le Livre de M. Peano se distingue davantage des Traités classiques: les définitions qui se rapportent aux *champs de points*, aux points extérieurs, intérieurs ou limites par rapport à un champ, aux fonctions distributives (coexistantes d'après Cauchy), à la longueur (à l'aire ou au volume) externe, interne ou propre d'un champ, la notion d'intégrale étendue à un champ sont présentées sous une forme abstraite, très précise et très claire.<sup>15</sup>

Only a few authors fully realized the innovative value of Chapter V of *Applicazioni geometriche*. As an instance, ASCOLI [1, (1955) pp. 26–27] says:

In [*Applicazioni geometriche*] vi sono profusi, in forma così semplice da parere definitiva, idee e risultati divenuti poi classici, come quelli sulla misura degli insiemi, sulla rettificazione delle curve, sulla definizione dell'area di una superficie, sull'integrazione di campo, sulle funzioni additive di insieme; ed altri che sono tutt'ora poco noti o poco studiati [...].<sup>16</sup>

Most of the modern historians are aware of the contributions to measure theory given by PEANO and JORDAN concerning inner and outer measure and measurability.<sup>17</sup>

Only a few historians mention PEANO's contributions to derivative of set functions: PESIN [59], MEDVEDEV [42] and HAWKINS [22] and others.

PESIN [59, (1970) pp. 32–33], who does “not intend to overestimate the importance of PEANO's results”, recalls some results of PEANO's work without giving details or appropriate definitions.

MEDVEDEV in [42, (1983)] recalls PEANO's contributions giving detailed information both on the integral as a set function and on the PEANO's derivative. In our opinion he gives an excessive importance to mathematical priorities without pointing out the differences between PEANO's contribution of 1887 and LEBESGUE's contribution of 1910.<sup>18</sup>

---

<sup>15</sup> [Chapter V is titled: Geometric magnitudes. This chapter is probably the most relevant and interesting, the one that marks the difference of the Book of Peano with respect to other classical Treatises: definitions concerning *sets of points*, exterior, interior and limit points of a given set, distributive functions (coexistent magnitudes in the sense of Cauchy), exterior, interior and proper length (or area or volume) of a set, the extension of the notion of integral to a set, are stated in an abstract, very precise and very clear way.]

<sup>16</sup> [In *Applicazioni geometriche* it is possible to find a clear and definitive exposition of many mathematical concepts and results, nowadays of common knowledge: results on measure of sets, on length of arcs, on the definition of area of a surface, on the integration on a set, on additive set functions; and other results that are not well known [...]]

<sup>17</sup> To our knowledge the latest example of historian who forgot to quote any PEANO's contributions, is HOCHKIRCHEN [23, (2003)]. Ironically, the symbols  $\int$  and  $\bar{\int}$  which VOLTERRA (1881) introduced for denoting lower and upper integral, were ascribed to PEANO by HOCHKIRCHEN.

<sup>18</sup> DIEUDONNÉ, reviewing in [12, (1983)] the MEDVEDEV's paper [42, (1983)], with his usual sarcasm denies any logical value of PEANO's definitions concerning limits and sets. Against any historical evidence, DIEUDONNÉ forgets several PEANO's papers on several notions of limit, and ignores

HAWKINS does not describe PEANO's results on differentiation and integration in detail, as they are too far from the main aim of his book, but he is aware of PEANO's contributions to differentiation of set functions [22, p. 88, 185], and appraises PEANO's book *Applicazioni geometriche*:

the theory is surprisingly elegant and abstract for a work of 1887 and strikingly modern in his approach [22, p. 88].

None of the historian quoted above, establishes a link between PEANO's work on differentiation of measure in *Applicazioni geometriche* with his paper *Grandezze coesistenti* [55] and with Lebesgue's comments on differentiation presented in *La mesures des grandeurs* [35, (1935)].

Beside *Applicazioni Geometriche* main primary sources on which our paper is based are [9, 55, 29, 35, 67, 68, 16, 17].

#### 4. CAUCHY'S COEXISTENT MAGNITUDES

CAUCHY's seminal paper *Coexistent magnitudes* [9, (1841)] presents some difficulties for the modern reader: the terms he introduces are rather obscure (for instance, *grandeurs*, *coexistantes*, *éléments*, ...), and the reasonings are based on vague geometric language, accordingly to the CAUCHY's taste. Actually, CAUCHY's aim was to make mathematical analysis as well rigorous as geometry [8, (1821) p. ii]:

Quant aux méthodes, j'ai cherché à leur donner toute la rigueur qu'on exige en géométrie, de manière à ne jamais recourir aux raisons tirées de la généralité de l'algèbre.<sup>19</sup>

In his *Leçons de mécanique analytique* [43, (1868) pp. 172–205] MOIGNO, a follower of CAUCHY, reprints the paper *Coexistent magnitudes*. He puts into evidence the vagueness of some terms of CAUCHY, unfortunately without adding any comment that may help the reader to a better understanding of CAUCHY's paper itself.

The meaning of the terms “*grandeurs*” and “*coexistantes*” can be made precise by analyzing the list of examples given by CAUCHY. He implicitly postulates the following properties of the “*grandeurs*”:

---

the *Formulario mathematico* where PEANO presents a large amount of mathematical results, including set axiomatization, through his logical *ideography*. Besides, DIEUDONNÉ forgets BOURBAKI's comments about PEANO in *Elements of the history of mathematics* and ignores that the logical building blocks of PEANO's ideography are the atomic propositions:  $x \in X$  and  $x = y$ .

<sup>19</sup>[About methods, I have tried to be rigorous as required in geometry, in order to avoid the general reasonings occurring in algebra.]

Not all mathematicians at that time considered geometry as a model of rigor. Indeed LOBACHEVSKY starts his famous book “Theory of parallels” [40, (1829) p. 11] with the following sentence:

In geometry I find certain imperfections which I hold to be the reason why this science, apart from transition into analytics, can as yet make no advance from that state in which it has come to us from Euclid.

- (4.1) a magnitude can be divided into finitely many infinitesimal equal elements (using the terminology of CAUCHY), where infinitesimal is related to magnitude and diameter;
- (4.2) the ratio between coexistent magnitudes (not necessarily homogeneous) is a numerical quantity.

Concerning the term “*coexistantes*”, coexistent magnitudes are defined by CAUCHY as “magnitudes which exist together, change simultaneously and the parts of one magnitude exist and change in the same way as the parts of the other magnitude”.<sup>20</sup> Despite of the vagueness of this definition, the meaning of “*coexistantes*” is partially clarified by many examples of coexistent magnitudes given by CAUCHY [9, (1841) pp. 188–189], such as the volume and the mass of a body, the time and the displacement of a moving point, the radius and the surface of a circle, the radius and the volume of a sphere, the height and the area of a triangle, the height and the volume of a prism, the base and the volume of a cylinder, and so on.

Vagueness of the CAUCHY’s definition of “*grandeurs coexistantes*” was pointed out by PEANO. In *Applicazioni geometriche* [48, (1887)] and in *Grandezze coesistenti* [55, (1915)], PEANO defines them as set functions over the same given domain, satisfying additivity properties in a suitable sense.

The primary aim of CAUCHY is pedagogic: he wants to write a paper making easier the study of infinitesimal calculus and its applications. As it is easy to understand, CAUCHY bases himself on the mass-density paradigm and introduces the limit of the average of two coexistent magnitudes, calling it *differential ratio*. In a modern language we could say that the coexistent magnitudes are set functions, while the differential ratio is a point function. CAUCHY points out that the differential ratio is termed in different ways depending on the context, namely, on the nature of the magnitudes themselves (for instance, mass density of a body at a given point, velocity of a moving point at a given time, hydrostatic pressure at a point of a given surface, ...).

Now we list the most significant theorems that are present in the paper of CAUCHY, preserving, as much as possible, his terminology.

**THEOREM 4.1** [9, Theorem 1, p. 190]. *The average between two coexistent magnitudes is bounded between the supremum and the infimum of the values of the differential ratio.*

**THEOREM 4.2** [9, Theorem 4, p. 192]. *A magnitude vanishes whenever its differential ratio, with respect to another coexistent magnitude, is a null function.*

---

<sup>20</sup>CAUCHY says in [9, (1841) p. 188]:

Nous appellons *grandeurs* ou quantités *coexistantes* deux grandeurs ou quantités qui existent ensemble et varient simultanément, de telle sorte que les éléments de l’une existent et varient, ou s’évanouissent, en même temps que les éléments de l’autre.

**THEOREM 4.3** [9, Theorem 5, p. 198]. *If the differential ratio between two co-existent magnitudes is a continuous function, then the “mean value property” holds.*<sup>21</sup>

**THEOREM 4.4** [9, Theorem 13, p. 202]. *If two magnitudes have the same differential ratio with respect to another magnitude, then they are equal.*

Even if CAUCHY presents proofs that are rather “vanishing”, his statements (see theorems listed above) and his use of the differential ratio allow PEANO to rebuild his arguments on solid grounds. PEANO translates the coexistent magnitudes into the concept of distributive set functions, restating the theorems presented by CAUCHY and proving them rigorously.

In PEANO, the property of continuity of the differential ratio (whenever it exists) is a consequence of its definition. On the contrary, CAUCHY’s definition of differential ratio does not guarantee its continuity. CAUCHY is aware of the fact that the differential ratio can be discontinuous, nevertheless he thinks that, in the most common “real” cases, it may be assumed to be continuous; see [9, (1841), p. 196]:

Le plus souvent, ce rapport différentiel sera une fonction continue de la variable dont il dépend, c’est-à-dire qu’il changera de valeur avec elle par degrés insensibles.<sup>22</sup>

and [9, (1841) p. 197]:

Dans un grand nombre de cas, le rapport différentiel  $\rho$  est une fonction continue [...].<sup>23</sup>

In evaluating the differential ratio as a “limit of average values  $\frac{\mu(A)}{v(A)}$  at a point  $P$ ”, for PEANO the set  $A$  does not necessarily include the point  $P$ , while for CAUCHY  $A$  includes  $P$  (as CAUCHY says: *A renferme le point P*).

This difference is fundamental also in case of linearly distributed masses. Indeed a linear mass distribution, described in terms of a function of a real variable, admits a differential ratio in the sense of PEANO if the ordinary derivative exists and is continuous, whilst it admits a differential ratio in the sense of CAUCHY<sup>24</sup> only if the function is differentiable [9, (1841) p. 208]:

<sup>21</sup> Let  $\mu, v : \mathcal{A} \rightarrow \mathbb{R}$  be two magnitudes and let  $g$  be the differential ratio of  $\mu$  with respect to  $v$ . We say that the *mean value property* holds if, for any set  $A \in \mathcal{A}$ , with  $v(A) \neq 0$ , there exists a point  $P \in A$  such that  $g(P) = \frac{\mu(A)}{v(A)}$ .

<sup>22</sup> [Almost always, this differential ratio is a continuous function of the independent variable, i.e., its values change in a smooth way.]

<sup>23</sup> [Almost always, the differential ratio  $\rho$  is a continuous function [...].]

<sup>24</sup> Using the identity

$$\frac{f(x+h) - f(x-k)}{h+k} = \frac{h \frac{f(x+h) - f(x)}{h} + k \frac{f(x) - f(x-k)}{k}}{h+k} \quad \text{for every } k, h > 0$$

the reader can easily verify that the differential ratio in the sense of CAUCHY exists (i.e., the limit of  $\frac{f(x+h) - f(x-k)}{h+k}$  exists for  $k \rightarrow 0^+$  and  $h \rightarrow 0^+$ , with  $h+k > 0$ ) whenever  $f'(x)$  exists.

Lorsque deux grandeurs ou quantités coexistantes se réduisent à une variable  $x$  et à une fonction  $y$  de cette variable, le rapport différentiel de la fonction à la variable est précisément ce qu'on nomme la *dérivée* de la fonction ou le *coefficient différentiel*.<sup>25</sup>

Concerning the existence of the differential ratio, CAUCHY is rather obscure; indeed whenever he defines the differential ratio, he specifies that “it will converge in general to a certain limit different from 0”. As CAUCHY does not clarify the meaning of the expression “in general”, the conditions assuring the existence of the differential ratio are not given explicitly. On the other hand, CAUCHY himself is aware of this lack, as in several theorems he explicitly assumes that the differential ratio is “completely determined at every point”.

Concerning the mass-density paradigm, in CAUCHY's *Coexistent magnitudes* an explicit formula allowing for constructing the mass of a body in terms of its density is also lacking. In spite of this, CAUCHY provides a large amount of theorems and corollaries giving an approximate calculation of the mass under the assumption of continuity of the density. We can envisage this approach as a first step toward the modern notion of integral with respect to a general abstract measure.

We can summarize further CAUCHY's results into the following theorem:

**THEOREM 4.5** [9, (1841) pp. 208–215]. *Let us assume that the differential ratio  $g$  between two coexistent magnitudes  $\mu$  and  $\nu$  exists and is continuous. Then  $\mu$  can be computed in terms of the integral of  $g$  with respect to  $\nu$ .*

CAUCHY concludes his memoir [9, (1841) pp. 215–229] with a second section in which he states the following theorem in order to evaluate lengths, areas and volumes of homothetic elementary figures.

**THEOREM 4.6** [9, Theorem 1, p. 216]. *Two coexistent magnitudes are proportional, whenever to equal parts of one magnitude there correspond equal parts of the other.*<sup>26</sup>

Even if the CAUCHY's paper contains several innovative procedures, to our knowledge only a few authors (MOIGNO, PEANO, VITALI, PICONE and LEBESGUE) quote it, and only PEANO and LEBESGUE analyze it in details.

## 5. DISTRIBUTIVE FAMILIES, DECOMPOSITIONS AND PEANO ADDITIVITY

In his paper *Le grandezze coesistenti* [55, (1915)], PEANO introduces a general concept of *distributive function*, namely a function  $f : A \rightarrow B$ , where  $(A, +)$ ,

---

<sup>25</sup> [When two coexistent magnitudes are a variable  $x$  and a function  $y$  of  $x$ , the differential ratio of the function with respect to the variable  $x$  coincides with the *derivative* of the function.]

<sup>26</sup> One can observe that this theorem holds true by imposing condition (4.1).

$(B, +)$  are two sets endowed with binary operations, denoted by the same symbol  $+$ , satisfying the equality

$$(5.1) \quad f(x + y) = f(x) + f(y)$$

for all  $x, y$  belonging to  $A$  and, if necessary, verifying suitable assumptions.<sup>27</sup> PEANO presents several examples of distributive functions. As a special instance,  $A$  stands for the family  $\mathcal{P}(X)$  of all subsets of a finite dimensional Euclidean space  $X$ , “+” in the left hand side of (5.1) is the set-union, and “+” in the right hand side of (5.1) is the logical OR (denoted in PEANO’s ideography by the same symbol of set-union); therefore, equation (5.1) becomes:

$$(5.2) \quad f(x \cup y) = f(x) \cup f(y).$$

To make (5.2) significant, PEANO chooses a family  $\mathcal{U} \subset \mathcal{P}(X)$  and defines “ $f(x)$ ” as “ $x \in \mathcal{U}$ ”. Consequently (5.2) becomes:

$$(5.3) \quad x \cup y \in \mathcal{U} \Leftrightarrow x \in \mathcal{U} \text{ or } y \in \mathcal{U}$$

for all  $x, y \in \mathcal{P}(X)$ . A family  $\mathcal{U}$  satisfying (5.3) is called by PEANO a *distributive family*.<sup>28</sup>

Moreover, PEANO considers *semi-distributive families*  $\mathcal{F} \subset \mathcal{P}(X)$ , i.e., families of non empty sets such that

$$(5.4) \quad x \cup y \in \mathcal{F} \Rightarrow x \in \mathcal{F} \text{ or } y \in \mathcal{F}$$

for all  $x, y \in \mathcal{P}(X)$ .

A distributive family of subsets of  $X$  is obtained by a semi-distributive family  $\mathcal{F}$  by adding to  $\mathcal{F}$  any supersets of its elements. PEANO states the following theorem, and attributes to CANTOR [5, (1884) p. 454] both its statement and its proof.

**THEOREM 5.1 (Cantor compactness property).** *Let  $\mathcal{F}$  be a semi-distributive family of subsets of a finite-dimensional Euclidean space, and let  $S$  be a bounded non-empty set belonging to  $\mathcal{F}$ . Then there exists a point  $\bar{x}$ , belonging to the closure of  $S$ , such that every neighborhood of  $\bar{x}$  contains some set belonging to  $\mathcal{F}$ .*

The notion of distributive family is essential in the study of the derivation of distributive set functions by PEANO. Distributive families have been introduced by PEANO in *Applicazioni geometriche* in 1887. Moreover, he uses them in his famous paper on the existence of solutions of differential equations [52, (1890) pp. 201–202] and, later, in his textbook *Lezioni di analisi infinitesimale* [54, (1893)

---

<sup>27</sup> Among distributive functions considered by PEANO, there are the usual linear functions and particular set functions. The reader has to pay attention in order to avoid the interpretation of distributive set functions as finitely additive set functions.

<sup>28</sup> This notion of distributive family will be rediscovered later by CHOQUET [10, (1947)], who called it *grill* and recognized it as the dual notion of CARTAN’s *filter* [7, (1937)].

vol. 2, pp. 46–53]. The role played by this notion is nowadays recovered by “compactness by coverings” or by “existence of accumulation points”.<sup>29</sup>

In proving Theorem 5.1, PEANO decomposes a subset of the Euclidean space  $\mathbb{R}^n$  following a grid of  $n$ -intervals implemented by cutting sets along hyperplanes parallel to coordinate axes. We may formalize this procedure in the following way.

Let us denote by  $H$  a hyperplane of the form  $H := \{x \in \mathbb{R}^n : \langle x, e_i \rangle = a\}$  where  $e_i$  is a vector of the canonical basis of  $\mathbb{R}^n$  and  $a \in \mathbb{R}$ . Let us denote by  $H^+$  and  $H^-$  the two closed half-spaces delimited by  $H$ .

A family  $\mathcal{F}$  of non empty subsets of  $\mathbb{R}^n$  is called *semi-distributive by cutting along hyperplanes* if

$$A \cap H^+ \in \mathcal{F} \quad \text{or} \quad A \cap H^- \in \mathcal{F}$$

for every  $A \in \mathcal{F}$  and for every hyperplane  $H$  of  $\mathbb{R}^n$  of the form indicated above. Under this restrictions a new version of Theorem 5.1 still holds:

**THEOREM 5.2** (Cantor compactness property by interval-decompositions). *Let  $\mathcal{F}$  be semi-distributive by cutting along hyperplanes and let  $S$  be a bounded non-empty set belonging to  $\mathcal{F}$ . Then there exists a point  $\bar{x}$  belonging to the closure of  $S$  such that every neighborhood of  $\bar{x}$  contains some set belonging to  $\mathcal{F}$ .*

To express additivity properties of set functions, PEANO, as it was common at his time<sup>30</sup>, uses the term *decomposition*. PEANO writes in *Applicazioni geometriche* [48, (1887) p. 164, 167]:

Se un campo  $A$  è decomposto in parti  $A_1, A_2, \dots, A_n$  esso si dirà *somma* delle sue parti, e si scriverà

$$A = A_1 + A_2 + \dots + A_n.$$

[...] Una grandezza dicesi *funzione distributiva* d’un campo, se il valore di quella grandezza corrispondente ad un campo è la somma dei valori di essa corrispondenti alle parti in cui si può decomporre il campo dato.<sup>31</sup>

In order to formalize in modern language both the operation of “decomposing” and his use in PEANO’s works, we can pursuit a “minimal” way, leading

<sup>29</sup> Two examples of distributive families considered by PEANO are  $\mathcal{U} := \{A \subset \mathbb{R}^n : \text{card}(A) = \infty\}$ , and  $\mathcal{U}_h := \{A \subset \mathbb{R}^n : \sup_A h = \sup_{\mathbb{R}^n} h\}$ , where  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is a given real function.

<sup>30</sup> A similar expression is used also by JORDAN [24]:

[C]haque champ  $E$  a une étendue déterminée; [...] si on le décompose en plusieurs parties  $E_1, E_2, \dots$ , la somme des étendues de ces parties est égale à l’étendue totale de  $E$ .

[Every set  $E$  has a defined extension; [...] if  $E$  is decomposed into parts  $E_1, E_2, \dots$ , the sum of the extensions of these parts is equal to the extension of  $E$ .]

<sup>31</sup> [If a set  $A$  is decomposed into the parts  $A_1, A_2, \dots, A_n$ , it will be called *sum* of its parts, and it will be denoted by  $A = A_1 + A_2 + \dots + A_n$ . [...] A magnitude is said to be a *distributive set function* if its value on a given set is the sum of the corresponding values of the function on the parts decomposing the set itself.]

to “families of interval-decompositions”, and a “proof-driven” way, leading to “families of finite decompositions”.

First, the minimal way consists in implementing the procedure of decomposing by cutting along hyperplanes used by PEANO in proving Theorem 5.1. More precisely, let  $\mathcal{A}$  be a family of subsets of the Euclidean space  $\mathbb{R}^n$ ; a finite family  $\{A_i\}_{i=1}^m$  of elements of  $\mathcal{A}$  is called an *interval-decomposition* of  $A \in \mathcal{A}$  if it is obtained by iterating the procedure of cutting by hyperplanes parallel to coordinate axes. In other words, an interval-decomposition  $\{A_i\}_{i=1}^m$  of a set  $A \in \mathcal{A}$  is a finite sub-family of  $\mathcal{A}$  defined recursively as follows:

- for  $m = 1$ ,  $A_1 = A$ ;
- for  $m = 2$ , there exists a hyperplane  $H$  such that  $A_1 = A \cap H^-$ ,  $A_2 = A \cap H^+$  and  $A_1, A_2 \in \mathcal{A}$ ;
- for  $m > 2$ , there exist two distinct indices  $i_0, i_1 \leq m$  such that  $\tilde{A} := A_{i_0} \cup A_{i_1} \in \mathcal{A}$  and the families  $\{A_i : 1 \leq i \leq m, i \neq i_0, i \neq i_1\} \cup \{\tilde{A}\}$  and  $\{A_{i_0}, A_{i_1}\}$  are interval-decompositions of  $A$  and  $\tilde{A}$ , respectively.

The totality of these interval-decompositions will be denoted by  $\mathbb{D}_{\text{int}}(\mathcal{A})$ . In the case where  $\mathcal{A}$  is the family of all the closed bounded subintervals of a given closed interval  $[a, b]$  of the real line, an arbitrary interval-decomposition of an interval  $[a', b'] \subset [a, b]$  is a family  $\{[a_{i-1}, a_i]\}_{i=1}^m$  where  $a' = a_0 \leq a_1 \leq \dots \leq a_{m-1} \leq a_m = b'$ . The totality of these interval-decompositions are denoted by  $\mathbb{D}_{\text{int}}(a, b)$ .

The second way consists in summarizing explicitly the properties of the decompositions themselves, as used by PEANO in defining the integral and in proving related theorems<sup>32</sup>, as it will be seen in Section 7. This leads to the following definitions of *family of finite decompositions* and of the related *semi-distributive family*, *Cantor compactness property* and *distributive set function*.

Let  $\mathcal{A}$  be again a family of subsets of the Euclidean space  $\mathbb{R}^n$  and let us denote by  $\mathcal{P}_f(\mathcal{A})$  the set of all non-empty finite subfamily of  $\mathcal{A}$ . Define  $\mathbb{U}(\mathcal{A})$  by

$$\mathbb{U}(\mathcal{A}) := \left\{ \mathcal{H} \in \mathcal{P}_f(\mathcal{A}) : \bigcup \mathcal{H} \in \mathcal{A} \right\}.$$

Let  $\mathbb{D}$  be a subset of  $\mathbb{U}(\mathcal{A})$ ; we will say that  $\mathcal{H}$  is a  $\mathbb{D}$ -decomposition of  $A$  if  $\mathcal{H} \in \mathbb{D}$  and  $A = \bigcup \mathcal{H}$ .

**DEFINITION 5.3.**  $\mathbb{D} \subset \mathbb{U}(\mathcal{A})$  is called a family of finite decompositions relative to  $\mathcal{A}$  if the following properties are satisfied:

$$(5.5) \quad \{A\} \in \mathbb{D} \text{ for every } A \in \mathcal{A};$$

$$(5.6) \quad \text{if } \mathcal{H} \text{ and } \mathcal{G} \text{ are } \mathbb{D}\text{-decompositions of a set } A, \text{ then}$$

$$\{H \cap G : H \in \mathcal{H}, G \in \mathcal{G}\}$$

is a  $\mathbb{D}$ -decomposition of  $A$ ;

---

<sup>32</sup>See pages 165 and 186–188 of *Applicazioni geometriche* [48, (1887)].



(5.7) if  $\mathcal{H}$  and  $\mathcal{G}$  are  $\mathbb{D}$ -decompositions of  $A$ , then for every  $G \in \mathcal{G}$  the family

$$\mathcal{H}_G := \{H \cap G : H \in \mathcal{H}\}$$

is a  $\mathbb{D}$ -decomposition of  $G$ ;

(5.8) if  $\mathcal{H}$  is a  $\mathbb{D}$ -decomposition of  $A$  and, moreover, for every  $H \in \mathcal{H}$  the family  $\mathcal{G}_H$  is a  $\mathbb{D}$ -decomposition of  $H$ , then

$$\bigcup \{\mathcal{G}_H : H \in \mathcal{H}\}$$

is a  $\mathbb{D}$ -decomposition of  $A$ .

DEFINITION 5.4. A family  $\mathbb{D}$  of finite decompositions relative to  $\mathcal{A}$  is called infinitesimal if for  $\mathcal{B} \subset \mathcal{A}$ , every bounded set  $A \in \mathcal{B}$  and for every real number  $\varepsilon > 0$ , there is a  $\mathbb{D}$ -decomposition  $\mathcal{H}$  of  $A$  such that  $\mathcal{H} \subset \mathcal{B}$  and the diameter of every  $H \in \mathcal{H}$  is less than  $\varepsilon$ .

DEFINITION 5.5. Let  $\mathbb{D}$  be a family of finite decompositions relative to  $\mathcal{A}$ . Then a set function  $\mu : \mathcal{A} \rightarrow \mathbb{R}$  is said to be distributive with respect to  $\mathbb{D}$ , if  $\emptyset \in \mathcal{A}$ ,  $\mu(\emptyset) = 0$  and

$$(5.9) \quad \mu\left(\bigcup \mathcal{H}\right) = \sum_{H \in \mathcal{H}} \mu(H) \quad \text{for every } \mathcal{H} \in \mathbb{D}.$$

Consequently,

DEFINITION 5.6. Let  $\mathbb{D}$  be a family of finite decompositions relative to  $\mathcal{A}$ . A family  $\mathcal{F}$  of non empty subsets of the Euclidean space  $\mathbb{R}^n$  is said to be semi-distributive with respect to  $\mathbb{D}$ , if

$$(5.10) \quad \mathcal{H} \in \mathbb{D} \quad \text{and} \quad \bigcup \mathcal{H} \in \mathcal{F} \Rightarrow \exists H \in \mathcal{H} \quad \text{such that } H \in \mathcal{F}.$$

THEOREM 5.7 (Cantor compactness property by an arbitrary family of decompositions). Let  $\mathbb{D}$  be an infinitesimal family of finite decompositions relative to  $\mathcal{A}$  and let  $\mathcal{F}$  be a semi-distributive family with respect to  $\mathbb{D}$ . If  $S$  is a bounded non-empty set belonging to  $\mathcal{F}$ , then there exists a point  $\bar{x}$  belonging to the closure of  $S$  such that every neighborhood of  $\bar{x}$  contains some set belonging to  $\mathcal{F}$ .

In the following, an expression of type “ $\mu : (\mathcal{A}, \mathbb{D}) \rightarrow \mathbb{R}$  is a distributive set function” stands for “ $\mathbb{D}$  is a family of finite decompositions relative to  $\mathcal{A}$  and  $\mu : \mathcal{A} \rightarrow \mathbb{R}$  is a distributive set function with respect to  $\mathbb{D}$ .”

---

<sup>33</sup>In (5.11) we assume that  $A \cap H^+, A \cap H^- \in \mathcal{A}$  for every  $A \in \mathcal{A}$  and hyperplane  $H$  parallel to a coordinate axis. Hence, a set function  $\mu : \mathcal{A} \rightarrow \mathbb{R}$  is distributive with respect to  $\mathbb{D}_{\text{int}}(\mathcal{A})$ , if  $\mu(A) = \mu(A \cap H^+) + \mu(A \cap H^-)$  for every  $A \in \mathcal{A}$  and hyperplane  $H$  parallel to a coordinate axis. Inner and upper Peano-Jordan measures are both distributive in this sense, but they are not finitely additive.

If  $\mathcal{A}$  is *stable by finite intersections*, examples of families of decompositions are  $\mathbb{U}(\mathcal{A})$ , and

- (5.11) the family  $\mathbb{D}_{\text{int}}(\mathcal{A})$  of all interval-decompositions introduced above;<sup>33</sup>
- (5.12) the family of all  $\mathcal{H} \in \mathbb{U}(\mathcal{A})$  such that the interiors of two arbitrary distinct elements of  $\mathcal{H}$  have empty intersection and every  $H \in \mathcal{H}$  is Peano-Jordan measurable;
- (5.13) the family of all  $\mathcal{H} \in \mathbb{U}(\mathcal{A})$  such that the intersection of the closure of two arbitrary distinct elements of  $\mathcal{H}$  have null Peano-Jordan measure and every  $H \in \mathcal{H}$  is bounded;
- (5.14) the family of all  $\mathcal{H} \in \mathbb{U}(\mathcal{A})$  such that two arbitrary distinct elements of  $\mathcal{H}$  have empty intersection.

The interval-decompositions (in particular  $\mathbb{D}_{\text{int}}(a, b)$ ) occurs frequently in PEANO'S works. Distributive set functions related to the last example (5.14) are well known as *finitely additive set functions*; this type of additivity, expressed in terms of partitions of sets, was introduced for the first time in BOREL [3, (1898), pp. 46–50], and, more clearly, in LEBESGUE [27, (1902), p. 6].

As far as we know, all historians interpreted PEANO'S distributive set functions as “finitely additive” set functions.<sup>34</sup> For instance, in the proof of the integrability of functions [48, (1887) p. 188], PEANO uses distributivity properties of the upper and lower integral with respect to the domain of integration; clearly neither the upper nor the lower integral are finitely additive.

## 6. PEANO'S STRICT DERIVATIVE OF DISTRIBUTIVE FUNCTIONS AND ITS APPLICATIONS

In *Applicazioni geometriche* [48, (1887)] PEANO translates in terms of “distributive functions” the “magnitudes” of CAUCHY, so that two CAUCHY'S magnitudes are “coexistent” if they are distributive functions with the same domain.

PEANO'S distributive set functions are called *positive* if their values are positive. PEANO'S *strict derivative* is defined by<sup>35</sup>

**DEFINITION 6.1.** *Let  $\mu, \nu : (\mathcal{A}, \mathbb{D}) \rightarrow \mathbb{R}$  be distributive set functions, and let  $\nu$  be positive. A real function  $g$  over a set  $S$  is called a “strict derivative of  $\mu$  with respect*

<sup>34</sup>Observe that inner and outer Peano-Jordan measures on Euclidean spaces are not finitely additive, but they are distributive set functions with respect to the families of decomposition of type (5.11) or (5.12). Moreover, notice that outer Peano-Jordan measure is a distributive set function with respect to a family of decompositions of type (5.13).

<sup>35</sup>In PEANO'S words [48, (1887) p. 169]:

Diremo che, in un punto  $P$ , il *rapporto* delle due funzioni distributive  $y$  ed  $x$  d'un campo vale  $\rho$ , se  $\rho$  è il limite verso cui tende il rapporto dei valori di queste funzioni, corrispondenti ad un campo di cui tutti i punti si avvicinano indefinitamente a  $P$ .

[[Given two distributive functions  $y$  and  $x$  defined over a given set, we say that their *ratio*, at a given point  $P$ , is  $\rho$ , if  $\rho$  is the limit of the ratio between the values of the two functions, taken along sets for which all its points approach the point  $P$ .]]

to  $v''$  on  $S$  (denoted by  $\frac{d\mu}{dv}$  and termed *rapporto* in *Applicazioni geometriche*) if, for every point  $x \in S$  and for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that<sup>36</sup>

$$(6.1) \quad \left| \frac{\mu(A)}{v(A)} - g(x) \right| < \epsilon \quad \text{for every } A \in \mathcal{A}, \text{ with } v(A) \neq 0, A \subset B_\delta(x).$$

It is worth noticing that the concept of strict derivative given by PEANO provides a consistent mathematical ground to the concept of “infinitesimal ratio” between two magnitudes, successfully used since KEPLER. A remarkable example given by PEANO is the evaluation of a rectifiable arc length by integrating the “infinitesimal arc length”  $ds$ . Notice that, whenever  $ds$  exists in the sense of PEANO, the corresponding integral provides the length of the arc. On the contrary, the integration of the infinitesimal arc length  $ds$ , evaluated in the sense of LEBESGUE (1910), provides the length of the arc only in case of absolute continuity of the arc parametrization (see TONELLI [66, (1908)]).

The existence of PEANO’s strict derivative is not assured in general; its characterizing properties are clearly presented in *Applicazioni geometriche* and can be summarized in the following theorems.

First, PEANO gives a precise form to CAUCHY’s Theorem 4.1, stating the following:

**THEOREM 6.2** (see PEANO [48, Theorem 13, p. 170] for  $\mathbb{D} = \mathbb{D}_{\text{int}}$ ). *Let  $\mu, v : (\mathcal{A}, \mathbb{D}) \rightarrow \mathbb{R}$  be distributive set functions with  $\mathbb{D}$  infinitesimal for  $\mathcal{B} := \{A \in \mathcal{A} : v(A) \neq 0\}$  and  $v$  positive. If  $S \in \mathcal{A}$  is a closed and bounded non-empty set and  $g$  is the strict derivative of  $\mu$  with respect to  $v$  on  $S$ , then*

$$(6.2) \quad \inf_B g \leq \frac{\mu(A)}{v(A)} \leq \sup_B g$$

for all  $A, B \in \mathcal{A}$  with  $A \subset B \subset S$  and  $v(A) > 0$ .

In the case  $\mathbb{D} = \mathbb{D}_{\text{int}}$ , PEANO proves this fundamental theorem by applying Theorem 5.2 to the semi-distributive families  $\mathcal{F}_a := \{A \in \mathcal{A} : \mu(A) > av(A)\}$  and  $\mathcal{G}_a := \{A \in \mathcal{A} : \mu(A) < av(A)\}$ , for real numbers  $a$ . Observe that (6.2) amounts to (2.7)–(2.8) and also, indirectly, to (2.1)–(2.3).

In *Applicazioni geometriche*, Theorem 6.2 is followed by three corollaries, which we summarize into the following:

**COROLLARY 6.3** [48, (1987) p. 171]. *Under the same hypothesis as in the previous theorem:*

$$(6.3) \quad \text{if the strict derivative } \frac{d\mu}{dv} \text{ is a constant } b \text{ on } S, \text{ then } \mu(A) = bv(A), \text{ for all } A \in \mathcal{A} \text{ with } A \subset S;$$

---

<sup>36</sup>One can note that for the definition of strict derivative at a point  $x$ , the point  $x$  itself must be an accumulation point with respect to the family  $\mathcal{A}$  and the measure  $v$ , that is, for all  $\delta > 0$ , there exists a  $A \in \mathcal{A}$  such that  $v(A) \neq 0$  and  $A \subset B_\delta(x)$ , where  $B_\delta(x)$  denotes the Euclidean ball of center  $x$  and radius  $\delta$ .

- (6.4) if the strict derivative  $\frac{d\mu}{dv}$  vanishes at every points of  $S$ , then  $\mu(A) = 0$ , for all  $A \in \mathcal{A}$  with  $A \subset S$ ;
- (6.5) if two distributive set functions have equal strict derivatives with respect to  $v$  on  $S$ , then they are equal on subsets of  $S$  belonging to  $\mathcal{A}$ .<sup>37</sup>

The following fundamental PEANO's result point out the difference of PEANO's approach with respect to both approaches of CAUCHY and of LEBESGUE (1910).

**THEOREM 6.4.** *Under the same hypothesis as in the previous theorem, if the strict derivative of  $\mu$  with respect to  $v$  exists on  $S$ , then it is continuous on  $S$ .*

The importance of these results is emphasized in *Applicazioni geometriche* by a large amount of evaluations of derivatives of distributive set functions. As a consequence of the existence of the strict derivative, PEANO gives, for the first time, several examples of measurable sets. The most significant examples, observations and results are listed below.

- (6.6) *Measurability of the hypograph of a continuous function* [48, (1887) pp. 172–174]. Let  $f$  be a continuous positive real function defined on an interval  $[a, b]$ , let  $\mathcal{A}$  be the family of all sub-intervals of  $[a, b]$  and let  $v$  be the Euclidean measure on 1-dimensional intervals. Define  $\mu_f : \mathcal{A} \rightarrow \mathbb{R}$  on every  $A$  belonging to  $\mathcal{A}$ , by the inner (respectively, the outer) 2-dimensional measure (in the sense of Peano-Jordan) of the *positive-hypograph* of  $f$ , restricted to  $A$ .<sup>38</sup> In any case, independently of the choice of inner or outer measure, we have that  $\mu_f$  and  $v$  are distributive set functions with respect to  $\mathbb{D}_{\text{int}}(a, b)$ , and that  $\frac{d\mu_f}{dv}(x) = f(x)$  for every  $x \in [a, b]$ . From (6.5) of Corollary 6.3 it follows that the inner measure of the positive-hypograph of the continuous function  $f$  coincides with its outer measure; therefore it is measurable in the sense of Peano-Jordan.
- (6.7) Analogously, PEANO considers continuous functions of two variables and the volume of their *positive-hypographs* [48, (1887) p. 175].
- (6.8) *Area of a plane star-shaped subset delimited by a continuous closed curve* [48, (1887) pp. 175–176]. Consider a continuous closed curve that can be described in polar coordinates in terms of a continuous function  $\rho : [0, 2\pi] \rightarrow \mathbb{R}_+$ , with  $\rho(0) = \rho(2\pi)$ . Let  $\mathcal{A}$  be the family of all subintervals of  $[0, 2\pi]$ ; and for every  $A \in \mathcal{A}$ , let  $v(A)$  denote the Euclidean measure of the area of the circular sector  $\{(\rho \cos(\theta), \rho \sin(\theta)) : \theta \in A, \rho \in [0, 1]\}$ . Moreover, let  $\mu(A)$  denote inner (or outer, indifferently) Peano-Jordan 2-dimensional measure of the set  $\{(\rho \cos(\theta), \rho \sin(\theta)) : \theta \in A, \rho \in [0, \rho(\theta)]\}$ . Then the strict derivative  $\frac{d\mu}{dv}(\theta)$  is equal to  $\rho^2(\theta)$ . From the fact that this derivative does not depend on the choice of inner or outer measure, it

<sup>37</sup> It is evident that properties (6.3)–(6.5) are equivalent. To prove (6.5), PEANO shows that the strict derivative of a sum of two distributive set functions is the sum of their derivatives.

<sup>38</sup> By *positive-hypograph* of  $f$  restricted to  $A$  we mean the set  $\{(x, y) \in [a, b] \times \mathbb{R}_+ : x \in A \text{ and } y \leq f(x)\}$ , where  $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ .

follows Peano-Jordan measurability of plane star-shaped sets delimited by continuous closed curves.

- (6.9) Analogously, PEANO considers the volume of a star-shaped set bounded by simple continuous closed surface [48, (1887) p. 177].
- (6.10) *Cavalieri's principle between a prism and a spatial figure* [48, (1887) pp. 177–179]. Consider a straight line  $r$  in the tri-dimensional space, an unbounded cylinder  $P$  parallel to  $r$  with polygonal section of non null area, and a spatial figure  $F$ . Let  $\pi_x$  denote the plane perpendicular to  $r$  at the point  $x \in r$ . Assume non null area of all sections of the boundary of  $F$  perpendicular to  $r$ , namely

$$(*) \quad \mu_e(\partial F \cap \pi_x) = 0 \quad \text{for all } x \in r$$

where  $\mu_e$  denotes 2-dimensional Peano-Jordan outer measure and  $\partial F$  denotes the boundary of  $F$ . Let  $\mathcal{A}$  be the family of all segments of  $r$ . Given a set  $A \in \mathcal{A}$ , let  $\mu : \mathcal{A} \rightarrow \mathbb{R}$  denote the outer (or inner, indifferently) 3-dimensional measure of the set  $\bigcup_{x \in A} (F \cap \pi_x)$ , and  $\nu(A)$  denote Peano-Jordan 3-dimensional measure of the set  $\bigcup_{x \in A} (P \cap \pi_x)$ . The set functions  $\mu$  and  $\nu$  are distributive with respect to the family  $\mathbb{D}_{\text{int}}(r)$  of interval-decompositions of  $r$  and

$$\frac{d\mu}{d\nu}(x) = \frac{\mu_e(F \cap \pi_x)}{\mu_e(P \cap \pi_x)} \quad \text{for every } x \in r.$$

From the fact that this derivative does not depend on the choice of the inner or outer measure involved in defining  $\mu$ , it follows Peano-Jordan measurability of the spatial figure  $F$ .

- (6.11) *Cavalieri's principle between two spatial figures* [48, (1887) p. 180]. Consider two spatial figures  $F$  and  $G$  such that all sections of their boundaries with planes perpendicular to a given straight line  $r$  have null area. Let  $\mathcal{A}$  be the family of all segments of  $r$ . Given a set  $A \in \mathcal{A}$ , let  $\mu(A)$  and  $\nu(A)$  denote outer (or inner, indifferently) Peano-Jordan 3-dimensional measures of the sets  $\bigcup_{x \in A} (F \cap \pi_x)$  and  $\bigcup_{x \in A} (G \cap \pi_x)$ , respectively. The set functions  $\mu$  and  $\nu$  are distributive with respect to the family  $\mathbb{D}_{\text{int}}(r)$  of interval-decompositions of  $r$  and

$$\frac{d\mu}{d\nu}(x) = \frac{\mu_e(F \cap \pi_x)}{\mu_e(G \cap \pi_x)} \quad \text{for every } x \in r.$$

Hence, from (6.3) it follows the classical Cavalieri's principle: two figures whose corresponding sections have equal areas, have the same volume.

- (6.12) *Cavalieri's principle for 3 dimensional figures with respect to one dimensional sections* [48, (1887) p. 180]. Consider a plane  $\pi$ . Let  $\mathcal{A}$  be the family of all rectangles of  $\pi$  and let  $r_x$  be the straight line perpendicular to  $\pi$  at  $x \in \pi$ . Moreover, consider a spatial figure  $F$  such that for any  $x \in \pi$

$$(**) \quad \mu_e(\partial F \cap r_x) = 0 \quad \text{for every } x \in \pi$$

where  $\mu_e$  denotes the Peano-Jordan 1-dimensional outer measure and  $\partial F$  denotes the boundary of  $F$ . Given a set  $Q \in \mathcal{A}$ , let  $\mu(Q)$  denote the outer (or inner, indifferently) measure of the set  $\bigcup_{x \in Q} (F \cap r_x)$ , and  $\nu(Q)$  denote the elementary usual measure of  $Q$ . Then  $\mu$  and  $\nu$  are distributive with respect interval-decompositions of rectangles of  $\pi$  and

$$\frac{d\mu}{d\nu}(x) = \mu_e(F \cap r_x) \quad \text{for every } x \in \pi.$$

(6.13) *Cavalieri's principle for 2 dimensional figures* [48, (1887) p. 180]. Analogously to (6.10), PEANO considers Cavalieri's principle for planar figures.

(6.14) *Derivative of the length of an arc* [48, (1887) p. 181]. In order to compare the length of an arc with the length of its orthogonal projection on a straight line  $r$ , PEANO assumes that the orthogonal projection is bijective on a segment  $\rho$  of  $r$ , and that the arc can be parametrized through a function with continuous non null derivative.<sup>39</sup> Let  $\mathcal{A}$  be the family of all closed bounded segments of  $\rho$ . For every segment  $A \in \mathcal{A}$ , let  $\mu(A)$  denote the length of the arc whose orthogonal projection over  $r$  is  $A$  and let  $\nu(A)$  denote the length of  $A$ . Then

$$(***) \quad \frac{d\mu}{d\nu}(x) = \frac{1}{\cos \theta_x}$$

where  $\theta_x$  is the angle between  $r$  and the straight line that is tangent to the arc at the point (of the arc) corresponding to  $x \in \rho$ .<sup>40</sup>

(6.15) *Derivative of the area of a surface* [48, (1887) pp. 182–184]. By adapting the previous argument, PEANO shows that the strict derivative between the area of a surface and its projection on a plane is given by (\*\*\*), where  $\cos \theta$  is the cosinus of the angle between the tangent plane and the projection plane.

## 7. DISTRIBUTIVE SET FUNCTIONS: INTEGRAL AND STRICT DERIVATIVE

PEANO introduces also the notion of integral with respect to a positive distributive set function. The *proper integral* of a bounded function  $\rho$  on a set  $A \in \mathcal{A}$  with respect to a positive distributive set function  $\nu : (\mathcal{A}, \mathbb{D}) \rightarrow \mathbb{R}$ , is denoted by

<sup>39</sup>The requirement that the derivative of the arc with respect to a parameter be continuous and non null is expressed by PEANO in geometrical terms, namely by requiring that “the tangent straight line exists at every point  $P$  of the arc, and it is the limit of the straight lines passing through two points of the arc, when they tend to  $P$ ”. PEANO was aware that these geometrical conditions are implied by the existence of a parametrization with a continuous non-null derivative [48, (1887) p. 59, 184].

<sup>40</sup>Of course, to avoid  $\cos \theta_x = 0$  along the arc, PEANO assumes that the tangent straight line at every point of the arc is not orthogonal to  $r$ .

$\int_A \rho \, dv$  and is defined as the real number such that for any  $\mathbb{D}$ -decomposition  $\{A_i\}_{i=1}^m$  of the set  $A$ , one has

$$\int_A \rho \, dv \geq \rho'_1 v(A_1) + \rho'_2 v(A_2) + \cdots + \rho'_n v(A_m)$$

$$\int_A \rho \, dv \leq \rho''_1 v(A_1) + \rho''_2 v(A_2) + \cdots + \rho''_n v(A_m)$$

where  $\rho'_1, \rho'_2, \dots, \rho'_m$  (respectively  $\rho''_1, \rho''_2, \dots, \rho''_m$ ), are numbers defined by

$$(7.1) \quad \rho'_i := \inf_{x \in A_i} \rho(x) \quad \text{and} \quad \rho''_i := \sup_{x \in A_i} \rho(x),$$

for all  $i = 1, \dots, m$ .<sup>41</sup>

PEANO defines also the *lower*  $\int_A \rho \, dv$  and the *upper* integral  $\int_A \rho \, dv$  of a bounded function  $\rho$  on a set  $A \in \mathcal{A}$  by

$$\int_A \rho \, dv := \sup s' \quad \text{and} \quad \int_A \rho \, dv := \inf s''$$

where  $s' = \rho'_1 v(A_1) + \rho'_2 v(A_2) + \cdots + \rho'_m v(A_m)$  and  $s'' = \rho''_1 v(A_1) + \rho''_2 v(A_2) + \cdots + \rho''_m v(A_m)$ , where  $\rho'_i$  and  $\rho''_i$  are defined as in (7.1) and  $\{A_i\}_{i=1}^m$  runs over  $\mathbb{D}$ -decompositions of  $A$ .

In PEANO’s terminology, the integrals defined above are called *geometric integrals*. PEANO stresses the analogy among these integrals and the usual *elementary integral*  $\int_a^b f(x) \, dx$  of functions  $f$  defined over intervals of  $\mathbb{R}$ .

Using property (5.6) of  $\mathbb{D}$ -decompositions, PEANO shows that the lower integral is always less or equal than the upper integral. When these values coincide, their common value is called a proper integral and is denoted by  $\int_A \rho \, dv$ .

Moreover, using properties (5.7) and (5.8) of  $\mathbb{D}$ -decompositions, PEANO shows that the lower integral  $A \mapsto \int_A \rho \, dv$  and the upper integral  $A \mapsto \int_A \rho \, dv$  are distributive set functions on  $\mathcal{A}$  with respect to the same family  $\mathbb{D}$  of decompositions [48, (1887) Theorem I, p. 187].

In case of  $\rho$  continuous, using the property of “infinitesimality” of  $\mathbb{D}$  (see Definition 5.4), PEANO shows that the derivative of both lower and upper integrals with respect to  $v$  is  $\rho$  [48, (1887) Theorem II, p. 189]; consequently the

---

<sup>41</sup> This clear, simple and general definition of integral with respect to an abstract positive distributive set function is ignored until the year 1915, when FRÉCHET re-discovers it in the setting of “finitely additive” measures [15, (1915)].

proper integral  $\int_A \rho \, dv$  of a continuous  $\rho$  exists whenever  $A$  is closed and bounded [48, (1887) Cor. of Theorem II, p. 189].

The definitions introduced above allow PEANO to realize the mass-density paradigm, i.e., to prove that it is possible to recover a distributive function  $\mu$  as the integral of the strict derivative  $\frac{d\mu}{dv}$  with respect to a positive distributive function  $v$ . PEANO's results can be formulated into the following

**THEOREM 7.1** (PEANO's Theorem on strict derivative of distributive set functions, see [48, (1887) Theorem 14, p. 171, Theorems II, III, p. 189]). *Let  $\mu, v : (\mathcal{A}, \mathbb{D}) \rightarrow \mathbb{R}$  be distributive set functions, with  $v$  positive and  $\mathbb{D}$  infinitesimal. Let  $S \in \mathcal{A}$  be a closed bounded non empty set and  $\rho : S \rightarrow \mathbb{R}$  a function. The following properties are equivalent:*

(7.2)  $\rho$  is the strict derivative  $\frac{d\mu}{dv}$  of  $\mu$  with respect to  $v$  on  $S$ ;

(7.3)  $\rho$  is continuous and  $\mu(A) = \int_A \rho \, dv$  for any  $A \subset S$ ,  $A \in \mathcal{A}$ .

PEANO applies Theorem 7.1 to the list of examples of strict derivatives of distributive set functions of §6 and obtains the following results.

(7.4) *Fundamental theorem of integral calculus for continuous functions* [48, (1887) pp. 191–193]. Consider a continuous function  $f$  on  $\mathbb{R}$  and let  $F$  be a primitive of  $f$ . Define  $\mu$  and  $v$  over the family  $\mathcal{A}$  of closed bounded intervals  $[a, b]$  of  $\mathbb{R}$  by  $\mu([a, b]) := F(b) - F(a)$  and  $v([a, b]) := b - a$ . Observe that both  $\mu$  and  $v$  are distributive set functions with respect to  $\mathbb{D}_{\text{int}}(\mathbb{R})$  and

$$\frac{d\mu}{dv}(x) = \lim_{\substack{a, b \rightarrow x \\ a \neq b}} \frac{F(b) - F(a)}{b - a} = f(x)$$

since  $F$  has continuous derivative.<sup>42</sup> Therefore, by Theorem 7.1, PEANO obtains

$$F(b) - F(a) = \mu([a, b]) = \int_{[a, b]} f \, dv = \int_a^b f(x) \, dx.$$

(7.5) *Calculus of an integral as a planar area* [48, (1887) pp. 193–195]. The elementary integral of a continuous positive function is Peano-Jordan measure of the positive hypograph of the function. This is an immediate application of Theorem 7.1 to the setting (6.6).

(7.6) *Cavalieri's formula for planar figures* [48, (1887) p. 195]. Let us suppose that  $C \subset \mathbb{R}^2$ ,  $C_x := \{y \in \mathbb{R} : (x, y) \in C\}$  and  $(\partial C)_x := \{y \in \mathbb{R} : (x, y) \in \partial C\}$  for every  $x \in \mathbb{R}$ . Assume that for any  $x$  the set  $(\partial C)_x$  has vanishing outer mea-

---

<sup>42</sup>PEANO observes that continuity of derivative of  $F$  is a necessary and sufficient condition to have the existence of  $\frac{d\mu}{dv}$ .



sure. As a consequence of Theorem 7.1 and the two-dimensional version of *Cavalieri's principle* (6.13) (see [48, (1887) p. 180]), it follows that the measure of the part of the figure  $C$ , bounded by the abscissas  $a$  and  $b$ , is equal to

$$\int_a^b \mu_e(C_x) dx$$

where  $\mu_e$  denotes outer Peano-Jordan one-dimensional measure.

- (7.7) *Area of a plane star-shaped subset delimited by a continuous closed curve* [48, (1887) p. 199]. In the setting of example (6.8), PEANO shows that the area of the sector between the angles  $\theta_0$  and  $\theta_1$ , delimited by a curve described in polar coordinates by  $\rho$ , is equal to

$$\frac{1}{2} \int_{\theta_0}^{\theta_1} \rho(\theta)^2 d\theta.$$

- (7.8) *Cavalieri's formula for volumes* [48, (1887) p. 221]. In the setting (6.12), let's define  $F_x := \{(y, z) \in \mathbb{R}^2 : (x, y, z) \in F\}$  and  $(\partial F)_x := \{(y, z) \in \mathbb{R}^2 : (x, y, z) \in \partial F\}$ . Assume that for any  $x$ , the set  $(\partial F)_x$  has vanishing outer measure. From Theorem 7.1, PEANO shows that the volume of the part of the figure  $F$ , delimited by the planes  $x = a$  and  $x = b$ , is equal to

$$\int_a^b \mu_e(F_x) dx$$

where  $\mu_e$  denotes outer Peano-Jordan two-dimensional measure.

## 8. COEXISTENT MAGNITUDES IN LEBESGUE AND PEANO'S DERIVATIVE

LEBESGUE gives a final pedagogical<sup>43</sup> exposition of his measure theory in *La mesure des grandeurs* [35, (1935) p. 176], by referring directly to CAUCHY's *Coexistent magnitudes*:<sup>44</sup>

La théorie des grandeurs qui constitue le précédent chapitre avait été préparée par des recherches de Cauchy, sur ce qu'il appelait des grandeurs

---

<sup>43</sup> LEBESGUE says in [33, (1931) p. 174]:

[...] depuis trente ans [d'enseignement] [...] on ne s'étonnera pas que l'idée me soit venue d'écrire des articles de nature pédagogique; si j'ose employer ce qualificatif que suffit ordinairement pour faire fuir les mathématiciens. [[...] in the thirty years [of teaching] [...] it is not at all surprising that the idea should occur to me of writing articles on a pedagogical vein; if I may use an expression which usually puts mathematicians to flight. (transl. MAY [36, (1966) p. 10]]]

<sup>44</sup> The five parts of the essay *La mesure des grandeurs* have been published in *L'Enseignement mathématique* during the years 1931–1935. An english translation *Measure and the Integral of La mesure des grandeurs* is due to Kenneth O. May [36, (1966)].

concomitantes [*sic*], par les travaux destinés à éclaircir les notions d'aire, de volume, de mesure [...].<sup>45</sup>

LEBESGUE is aware of the obscurity of the concepts that are present in CAUCHY's *Coexistent magnitudes*, starting by the meaning of the term *magnitude* itself. In this respect, in order to put on a solid ground the ideas of CAUCHY, LEBESGUE was compelled to pursue an approach similar to that PEANO: in fact he defines a "magnitude" as a set function on a family of sets  $\mathcal{A}$ , requires infinitesimality of  $\mathcal{A}$  (in the sense that every element of  $\mathcal{A}$  can be *réduit à un point par diminutions successives*), and additivity properties that he expresses in *La mesure des grandeurs* [34, (1934) p. 275] in these words:

Si l'on divise un corps  $C$  en un certain nombre de corps partiels  $C_1, C_2, \dots, C_p$ , et si la grandeur  $G$  est, pour ces corps,  $g$  d'une part,  $g_1, g_2, \dots, g_p$  d'autre part, on doit avoir:  $g = g_1 + g_2 + \dots + g_p$ .<sup>46</sup>

In *La mesure des grandeurs* LEBESGUE considers the operations of integration and differentiation by presenting these topics in a new form with respect to his fundamental and celebrated paper *L'intégration des fonctions discontinues* [29, (1910)].

LEBESGUE theory of differentiation of 1910 concerns absolutely continuous  $\sigma$ -additive measures on Lebesgue measurable sets. On the contrary, twenty-five years later in *La mesure des grandeurs* of 1935

- $\sigma$ -additive set functions are replaced by continuous<sup>47</sup> additive<sup>48</sup> measures;

---

<sup>45</sup>[[The theory of magnitudes forming the subject of the preceding chapter was prepared by researches of Cauchy on what he called concomitant magnitudes, by studies destined to clarify the concepts of area, volume, and measure [...]] (transl. MAY [36, (1966) p. 138]]]

<sup>46</sup>[[If a body  $C$  is partitioned into a certain number of sub-bodies  $C_1, C_2, \dots, C_p$  and if for these bodies the magnitude  $G$  is  $g$  on the one hand and  $g_1, g_2, \dots, g_p$  on the other, we must have  $g = g_1 + g_2 + \dots + g_p$ . (transl. MAY [36, (1966) p. 129]]]

LEBESGUE observes that in order to make this condition rigorous, it would be necessary to give a precise meaning to the words *corp* and *partage de la figure totale en parties* [34, (1934) p. 275–276]. Moreover he observes that *diviser un corps* may be interpreted in different ways [34, (1934) p. 279].

<sup>47</sup>It is not easy to give in a few words a definition of the concept of continuity according to LEBESGUE: such a continuity is based on a convergence of sequences of sets that in the relevant cases coincides with the convergence in the sense of Hausdorff. We recall that a sequence of sets  $\Delta_n$  converges to  $\Delta$  in the sense of Hausdorff if for all  $\epsilon > 0$  there exists  $n_0$  such that  $\Delta_n \subset B_\epsilon(\Delta)$  and  $\Delta \subset B_\epsilon(\Delta_n)$  for all  $n > n_0$ , where  $B_\epsilon(A) := \{x \in \mathbb{R}^n : \text{there exists } a \in A \text{ such that } \|x - a\| < \epsilon\}$ . Therefore, a set function  $f$  is said to be *continuous* if for any  $\Delta_n$  and  $\Delta$  Peano-Jordan measurable sets, we have that  $\lim_{n \rightarrow \infty} f(\Delta_n) = f(\Delta)$ , whenever  $\Delta_n$  converges to  $\Delta$  in Hausdorff sense.

<sup>48</sup>LEBESGUE writes in [35, (1935) p. 185]:

[...] nous supposons cette fonction [ $f$ ] *additive*, c'est-à-dire telle que, si l'on divise  $\Delta$  en deux domaines quarrables  $\Delta_1$  et  $\Delta_2$  on ait  $f(\Delta) = f(\Delta_1) + f(\Delta_2)$ .

[[...] let us assume that this function is *additive*; that is, it is such that, if we partition  $\Delta$  into two quadrable domains  $\Delta_1$  and  $\Delta_2$ , we have  $f(\Delta) = f(\Delta_1) + f(\Delta_2)$ . (transl. MAY [36, (1966) p. 146]]]

- absolutely continuous measures become set functions with bounded-derivative<sup>49</sup> (*à nombres dérivés bornés*);
- Lebesgue measurable sets are replaced by Jordan-Peano measurable subsets of a given bounded set.

Let  $K$  be a bounded closed subset of the Euclidean space  $\mathbb{R}^n$ , let  $\mathcal{A}_K$  be the family of Jordan-Peano measurable (*quarrables*) subsets of  $K$  and let  $V$  be a positive, continuous, additive set function on  $\mathcal{A}_K$  with bounded-derivative. Then LEBESGUE introduces a definition of derivative. The *uniform-derivative* (*dérivée à convergence uniforme*)  $\varphi$  of a set function  $f$  with respect to  $V$ , is defined as the function  $\varphi : K \rightarrow \mathbb{R}$  such that, for every  $\epsilon > 0$ , there exists  $\eta > 0$  such that

$$(8.1) \quad \left| \frac{f(\Delta)}{V(\Delta)} - \varphi(x) \right| < \epsilon$$

for all  $x \in K$  and  $\Delta \in \mathcal{A}_K$  with  $x \in \Delta \subset B_\eta(x)$ . It is clear that LEBESGUE's new notion of uniform-derivative is strictly related to PEANO's one. In fact, LEBESGUE observes that the uniform-derivative is continuous whenever it exists; moreover, he defines the integral

$$(8.2) \quad \int_K \varphi \, dV$$

of a *continuous function*  $\varphi$  with respect to  $V$ . His definition of integral [35, (1935) pp. 188–191] is rather intricate with respect to that of PEANO.

It is worthwhile noticing that LEBESGUE recognizes the relevance of the notion of an integral with respect to set functions. LEBESGUE, not acquainted with previous PEANO's contributions, assigns the priority of this notion to RADON [61, (1913)]. On the other hand, LEBESGUE notices that the integral with respect to set functions was already present in Physics<sup>50</sup> and express his great surprise in recovering in STIELTJES's integral [63, (1894)] an instance of integral with respect to set functions; Lebesgue writes [30, (1926) p. 69–70]:

Mais son premier inventeur, Stieltjès, y avait été conduit par des recherches d'analyse et d'arithmétique et il l'avait présentée sous une forme purement analytique qui masquait sa signification physique; si bien qu'il a fallu beaucoup d'efforts pour comprendre et connaître ce qui est maintenant évident. L'historique de ces efforts citerait les nom de F. Riesz, H. Lebesgue, W. H.

---

<sup>49</sup> A set function  $f$  has a *bounded-derivative* with respect to Peano-Jordan  $n$ -dimensional measure  $\text{vol}_n$  if there exists a constant  $M$  such that  $|f(\Delta)| \leq M \text{vol}_n(\Delta)$  for any Peano-Jordan measurable set  $\Delta$ . A set function with bounded-derivatives is called *uniformly Lipschitzian* by PICONE [60, (1923) vol. 2, p. 467].

<sup>50</sup> LEBESGUE gives several examples of this. For instance, the evaluation of the heath quantity, necessary to increase the temperature of a body, as integral of the specific heath with respect to the mass.

Young, M. Fréchet, C. de la Vallé-Poussin; il montrerait que nous avons rivalisé en ingéniosité, en perspicacité, mais aussi en aveuglement.<sup>51</sup>

The first important theorem presented by LEBESGUE is the following

**THEOREM 8.1.** *Let  $K$  be a bounded closed subset of  $\mathbb{R}^n$ ,  $\varphi : K \rightarrow \mathbb{R}$  a continuous function and  $V$  a positive additive continuous set function with bounded-derivative.*

*Then the integral  $\Delta \mapsto \int_{\Delta} \varphi \, dV$  with  $\Delta \in \mathcal{A}$  is the unique additive set function with bounded-derivative which has  $\varphi$  as uniform-derivative with respect to  $V$ .<sup>52</sup>*

The main applications of this theorem, given by LEBESGUE in *La mesure des grandeurs* [35, (1935) p. 176], concern:

- (8.3) the proof that multiple integrals can be given in terms of simple integrals;
- (8.4) the formula of change of variables;<sup>53</sup>
- (8.5) several formulae for oriented integrals (Green's formula, length of curves and area of surfaces).

The uniform-derivative defined by LEBESGUE is, as observed above, a continuous function, and coincides exactly with PEANO's strict derivative. Through a different and more difficult path<sup>54</sup> than PEANO's one, LEBESGUE rediscovers the importance of the continuity of the derivative. In LEBESGUE's works there are no references to the contributions of PEANO concerning differentiation of set functions.

Several years before *La mesure des grandeurs* of 1935, LEBESGUE in [30, (1926)] outlines his contribution to the notion of integral. In the same paper he mentions CAUCHY's *Coexistent magnitudes* in the setting of derivative of measures. Moreover he cites FUBINI's and VITALI's works of 1915 and 1916 (published by Academies of Turin and of Lincei) in the context of the general problem of primitive functions.

More precisely, in 1915, the year of publication of PEANO's paper *Le grandezze coesistenti* [55], FUBINI [16, 17, (1915)] and VITALI [67, 68, (1915, 1916)]

<sup>51</sup> [But its original inventor, Stieltjes, was led to it by researches in analysis and theory of number and he presented it in a purely analytical form which masked its physical significance, so much so that it required a much effort to understand and recognizes what is nowadays obvious. The history of these efforts includes the works of F. Riesz, H. Lebesgue, W. H. Young, M. Fréchet, C. de la Vallé-Poussin. It shows that we were rivals in ingenuity, in insight, but also in blindness. (transl. MAY [36, (1966) p. 190)]]

<sup>52</sup> The proof is rather lengthy, as LEBESGUE included in it the definition of integral as well as the theorem of average value.

<sup>53</sup> Lebesgue uses the implicit function theorem.

<sup>54</sup> The exposition of 1935 is elementary, but more lengthy and difficult than those presented by LEBESGUE in 1910. Surprisingly, the terms *domain*, *decomposition*, *limit*, *additive*, *continuous* are used by LEBESGUE in a supple way.

introduce a definition of derivative of “finitely additive measures”<sup>55</sup>, oscillating themselves between definitions *à la Cauchy* and *à la Peano*.

VITALI, in his second paper [68], refers to the *Coexistent magnitudes* of CAUCHY, and presents a comparison among the notions of derivative given by FUBINI, himself, PEANO and the one of LEBESGUE of 1910, emphasizing the continuity of the PEANO’s strict derivative. VITALI writes in [68, (1916)]:

Il Prof. G. Peano nella Nota citata [*Le grandezze coesistenti*] e in un’altra sua pubblicazione anteriore [*Applicazioni geometriche*], si occupa dei teoremi di Rolle e della media e ne indica la semplice dimostrazione nel caso in cui la derivata [della funzione di insieme  $f$ ] in  $P$  sia intesa come il limite del rapporto di  $\frac{f(\tau)}{\tau}$ , dove  $\tau$  è un campo qualunque che può anche non contenere il punto  $P$ .

L’esistenza di tale simile derivata finita in ogni punto porta difatti la continuità [della derivata medesima].<sup>56</sup>

This proves that since 1926 LEBESGUE should have been aware of PEANO’s derivative and of its continuity.<sup>57</sup>

Undoubtedly, the contributions of PEANO and LEBESGUE have a pedagogical and mathematical relevance in formulating a definition of derivative having the

<sup>55</sup>FUBINI’s first paper [17] is presented by C. SEGRE at the Academy of Sciences of Turin on January 10, 1915. In the same session, PEANO, Member of the Academy, presents a multilingual dictionary and a paper written by one of his students, VACCA. SEGRE, on April 11, 1915, presents, as a Member, a second paper of FUBINI [16] to *Accademia dei Lincei*. In the session of the Academy of Turin of June 13, 1915, PEANO presents his paper *Le grandezze coesistenti*. Moreover SEGRE presents two papers by VITALI [67, (1915)] and [68, (1916)] to Academy of Turin on November 28, 1915 and to Academy of Lincei on May 21, 1916, respectively.

There is a rich correspondence between VITALI and FUBINI. In the period March–May 1916 FUBINI sends three letters to VITALI (transcribed in *Selected papers* of VITALI [69, pp. 519–520]), concerning differentiation of finitely additive measures and related theorems. In particular FUBINI suggests VITALI to quote PEANO’s paper [55, (1915)] and to compare alternative definitions of derivative. In *Selected papers* of VITALI it is also possible to find six letters by PEANO to VITALI. Among them, there is letter of March 21, 1916 concerning CAUCHY’s coexistent magnitudes; PEANO writes:

Grazie della sua nota [67, (1915)]. Mi pare che la dimostrazione che Ella dà, sia proprio quella di Cauchy, come fu rimodernata da G. Cantor, e poi da me, e di cui trattasi nel mio articolo, *Le grandezze coesistenti di Cauchy*, giugno 1915, e di cui debbo avere inviato copia.

[[Thanks for your paper [67, (1915)]. In my opinion your proof coincides with the one given by Cauchy, as formulated by Cantor and by myself in my paper “Coexistent magnitudes of Cauchy” (June 1915), that I sent you.]

To our knowledge, FUBINI [16, 17, (1915)] and VITALI [67, 68, (1915, 1916)] are not cited by other authors, with the exception of BANACH [2, (1924) p. 186], who refers to FUBINI [16, (1915)].

<sup>56</sup>[Prof. Peano, in the cited Paper [*Le grandezze coesistenti*] and in a previous publication [*Applicazioni geometriche*] deals with Rolle’s and mean value theorems, pointing out a simple proof, valid in the case in which the derivative [of the set function  $f$ ], in a given point  $P$ , is the limit of the ratio  $\frac{f(\tau)}{\tau}$ , where  $\tau$  is a set that might not contain the point  $P$ .]

<sup>57</sup>We can ask how much LEBESGUE was aware of the contributions of PEANO. In many historical papers the comment of KENNEDY [26, (1980) p. 174], a well known biographer of PEANO, occurs:

Lebesgue acknowledged Peano’s influence on his own development.

property of continuity whenever it exists. Surprisingly these contributions are not known.

Rarely the notion of derivative of set functions is presented and used in educational texts.

An example is provided by *Lezioni di analisi matematica* of FUBINI. There are several editions of these *Lezioni*: starting by the second edition [18, (1915)], FUBINI introduces a derivative *à la Peano* of additive set functions in order to build a basis for integral calculus in one or several variables. Nevertheless, in his *Lezioni*, FUBINI assumes continuity of its derivative as an additional property. Ironically, FUBINI is aware of continuity of PEANO's derivative, whenever it exists; this is clear from two letters of 1916 that he sent to VITALI [69, p. 518–520]; in particular, in the second letter, about the PEANO's paper *Grandezze coesistenti* [55, (1915)], he writes:

Sarebbe bene citare [l'articolo di] Peano e dire che, se la derivata esiste e per calcolarla in [un punto]  $A$  si adottano anche domini che tendono ad  $A$ , pur non contenendo  $A$  all'interno, allora la derivata è continua.<sup>58</sup>

The notion of derivative of set function is also exposed in the textbooks *Lezioni di analisi infinitesimale* of PICONE [60, (1923) vol. II, p. 465–506], in *Lezioni di analisi matematica* of ZWIRNER [70, (1969), pp. 327–335] and in *Advanced Calculus* of R. C. and E. F. BUCK [4, (1965)]. In the book of PICONE, a definition of derivative *à la Cauchy* of “additive” set functions is given;<sup>59</sup> it represents an improvement of CAUCHY, FUBINI and VITALI definitions. Of course, his derivative is not necessarily a continuous function. Whenever the derivative is continuous, PICONE states a fundamental theorem of calculus, and applies it to the change of variables in multiple integrals. In the book of ZWIRNER the notion of derivative *à la Peano* of set functions is introduced, without mentioning PEANO and, unfortunately, without providing any application. In the third book, R. C. and E. F. BUCK introduce in a clear way a simplified notion of the uniform-derivative of LEBESGUE (without mentioning him), and they apply it to obtain the basic formula for the change of variables in multiple integrals.

---

In our opinion PEANO's influence on LEBESGUE is relevant but sporadic. After a reading of LEBESGUE's works, we have got the feeling that his knowledge of PEANO's contributions was restricted to the two papers on Peano's curve [50, (1890)] and on the definition of area [51, (1890)]. Surprisingly enough, LEBESGUE ignores the geometric calculus of GRASSMANN (and its revisitation operated by PEANO [49, (1888)]) and the PEANO's achievements on Peano-Jordan measure theory [48, (1887)].

<sup>58</sup> [It would be important to cite the paper of Peano, saying that, whenever the derivative exists and its evaluation is performed by considering domains that approach  $A$ , without requiring that the point  $A$  belongs to the domains themselves, then the derivative is continuous.]

<sup>59</sup> Significant instances of additive set functions in the sense of PICONE are outer measure of Peano-Jordan on all subsets of  $\mathbb{R}^n$  and lower/upper integrals of functions with respect to arbitrary domain of integration [60, (1923) vol. II, p. 356–357, 370–371]. The family of decompositions that leads to the notion of additive set function in the sense of PICONE is clearly defined on page 356–357 of his book [60] and includes the family of decompositions (5.12) and (5.13).

## 9. APPENDIX

All articles of PEANO are collected in *Opera omnia* [58], a CD-ROM edited by C. S. ROERO. Selected works of PEANO were assembled and commented in *Opere scelte* [56] by CASSINA, a pupil of PEANO. For a few works there are English translations in *Selected Works* [57]. Regrettably, fewer PEANO's papers have a public URL and are freely downloadable.

For reader's convenience, we provide a chronological list of some mathematicians mentioned in the paper, together with biographical sources.

Html files with biographies of mathematicians listed below with an asterisk can be attained at University of St Andrews's web-page

<http://www-history.mcs.st-and.ac.uk/history/{Name}.html>

- KEPLER, Johannes (1571–1630)\*
- CAVALIERI, Bonaventura (1598–1647)\*
- NEWTON, Isaac (1643–1727)\*
- MASCHERONI, Lorenzo (1750–1800)\*
- CAUCHY, Augustin L. (1789–1857)\*
- LOBACHEVSKY, Nikolai I. (1792–1856)\*
- MOIGNO François N. M. (1804–1884), see *Enc. Italiana*, Treccani, Roma, 1934
- GRASSMANN, Hermann (1809–1877)\*
- SERRET, Joseph A. (1819–1885)\*
- RIEMANN, Bernhard (1826–1866)\*
- JORDAN, Camille (1838–1922)\*
- DARBOUX, Gaston (1842–1917)\*
- STOLZ, Otto (1842–1905)\*
- SCHWARZ, Hermann A. (1843–1921)\*
- CANTOR, Georg (1845–1918)\*
- TANNERY, Jules (1848–1910)\*
- HARNACK, Carl (1851–1888), see May [41, (1973) p. 186]
- STIELTJES, Thomas J. (1856–1894)\*
- PEANO, Giuseppe (1858–1932)\*, see [26]
- YOUNG, William H. (1863–1942)\*
- SEGRE, Corrado (1863–1924)\*
- VALLÉE POUSSIN (de la), Charles (1866–1962)\*
- HAUSDORFF, Felix (1868–1942)\*
- BOREL, Emile (1871–1956)\*
- VACCA, Giovanni (1872–1953)\*
- CARATHÉODORY, Constantin (1873–1950)\*
- LEBESGUE, Henri (1875–1941)\*
- VITALI, Giuseppe (1875–1932)\*
- FRÉCHET, Maurice (1878–1973)\*
- FUBINI, Guido (1879–1943)\*
- RIESZ, Frigyes (1880–1956)\*

- TONELLI, Leonida (1885–1946)\*  
 PICONE, Mauro (1885–1977), see <http://web.math.unifi.it>  
 ASCOLI, Guido (1887–1957), see May [41, (1973) p. 63]  
 RADON, Johann (1887–1956)\*  
 NIKODYM, Otton (1887–1974)\*  
 BOULIGAND, George (1889–1979), see <http://catalogue.bnf.fr>  
 BANACH, Stefan (1892–1945)\*  
 KURATOWSKI, Kazimierz (1896–1980)\*  
 CASSINA, Ugo (1897–1964), see Kennedy [26, (1980)]  
 CARTAN, Henri (1904–2008)\*  
 DIEUDONNÉ, Jean A. E. (1906–1992)\*  
 CHOQUET, Gustave (1915–2006), see *Gazette des Math.* v111:74–76, 2007  
 MAY Kenneth O. (1915–1977), see [11, p. 479]  
 MEDVEDEV Fëdor A. (1923–1993), see [11, p. 482]

## REFERENCES

- [1] G. ASCOLI. *I motivi fondamentali dell'opera di Giuseppe Peano*. In In memoria di Giuseppe Peano (A. Terracini, ed.), Liceo Sc. Cuneo, 1955, pp. 23–30. Cuneo.  
 [2] S. BANACH. *Sur une classe de fonctions d'ensemble*. *Fund. Math.*, 6:170–188, 1924.  
 [3] E. BOREL. *Leçons sur la théorie des fonctions*. Gauthier-Villars, Paris, 1898.  
 [4] R. C. BUCK - E. F. BUCK. *Advanced Calculus*. McGraw-Hill, New York, 1965 (2nd ed.).  
 [5] G. CANTOR. *Über unendliche, lineare Punktmannigfaltigkeiten*. *Mathematische Annalen*, 24:454–488, 1884.  
 [6] C. CARATHÉODORY. *Über das lineare Mass von Punktmengen- eine Verallgemeinerung des Längenbegriffs*. *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, Math.-phys. Kl.*, pages 404–426, 1914.  
 [7] H. CARTAN. *Théorie des filtres*. C.R.A.Sc. Paris, 205:595–598, 1937.  
 [8] A. L. CAUCHY. *Cours d'analyse de l'école royale polytechnique, I<sup>re</sup> partie: Analyse algébrique*. Paris, 1821.  
 [9] A. L. CAUCHY. *Mémoire sur le rapport différentiel de deux grandeurs qui varient simultanément*. *Exercices d'analyse et de physique mathématique*, 2:188–229, 1841.  
 [10] G. CHOQUET. *Sur les notions de filtre et de grille*. C.R.A.Sc. Paris, 224:171–173, 1947.  
 [11] J. W. DAUBEN - C. J. SCRIBA. *Writing the history of Mathematics: Its historical development*. Birkhäuser, 2002.  
 [12] J. DIEUDONNÉ. *Review of "F. A. Medvedev, Set functions according to G. Peano"*. A.M.S. *Mathematical Reviews*, MR801411 (86h:01046).  
 [13] S. DOLECKI - G. H. GRECO. *Towards historical roots of necessary conditions of optimality: Regula of Peano*. *Control and Cybernetics*, 36:491–518, 2007.  
 [14] G. A. EDGAR. *Classics on fractals*. Westview Press, 2004.  
 [15] M. FRÉCHET. *Sur l'intégrale d'une fonctionnelle étendue à un ensemble abstrait*. *Bulletin de la Société Mathématique de France*, 43:248–265, 1915.  
 [16] G. FUBINI. *Esiste un corpo pesante a densità sempre nulla*. *Atti della R. Accademia delle Scienze di Torino*, 50:293–296, 1915.



- [17] G. FUBINI. *Il teorema del valor medio*. Atti della Reale Accademia dei Lincei, Rendiconti, 24:691–693, 1915.
- [18] G. FUBINI. *Lezioni di analisi matematica*. S.T.E.N. Editrice, Torino, 1913 (1<sup>st</sup> ed.), 1915 (2<sup>nd</sup> ed.), 1920 (4<sup>th</sup> ed.), 1925 (5<sup>th</sup> ed.).
- [19] G. H. GRECO - S. MAZZUCCHI - E. M. PAGANI. *Peano on definition of surface area*. forthcoming, 2010.
- [20] A. HARNACK. *Über den Inhalt von Punktmengen*. Mathematische Annalen, 25:241–250, 1885.
- [21] F. HAUSDORFF. *Dimension und äusseres Mass*. Mathematische Annalen, 79:157–179, 1919.
- [22] T. HAWKINS. *Lebesgue's theory of integration: its origins and development*. Chelsea Publ. Comp., New York, 1975.
- [23] T. HOCHKIRCHEN. *Theory of measure and integration from Riemann to Lebesgue*. In H. N. Jahnke, editor, *A History of Analysis*, chapter 9, pages 261–290. American Mathematical Society, 2003.
- [24] C. JORDAN. *Remarques sur les intégrales définies*. J. Math. Pures Appl., 8:69–99, 1892.
- [25] C. JORDAN. *Cours d'analyse de l'école polytechnique*. Gauthier-Villars, Paris, 1893 (2<sup>nd</sup> ed.).
- [26] H. C. KENNEDY. *Life and works of Giuseppe Peano*. D. Reidel Publ. Co, Dodrecht, 1980.
- [27] H. LEBESGUE. *Intégrale, longueur, aire*. Annali di Matematica Pura e Applicata, 7:231–359, 1902.
- [28] H. LEBESGUE. *Leçons sur l'intégration et la recherche des fonctions primitives*. Gauthier-Villars, Paris, 1904.
- [29] H. LEBESGUE. *Sur l'intégration des fonctions discontinues*. Ann. École Normale, 27:361–450, 1910.
- [30] H. LEBESGUE. *Sur le développement de la notion d'intégrale*. Mat. Tidsskrift, B:54–74, 1926.
- [31] H. LEBESGUE. *Evolución de la noción de integral*. Revista Matemática Hispano-Americana, 2:65–75, 92–106, 1927.
- [32] H. LEBESGUE. *Leçons sur l'intégration et la recherche des fonctions primitives*. Gauthier-Villars, Paris, 1928 (2<sup>nd</sup> ed.).
- [33] H. LEBESGUE. *Sur la mesure des grandeurs* (sec. I, II). Enseignement Mathématique, 31:173–206, 1931.
- [34] H. LEBESGUE. *Sur la mesure des grandeurs* (sec. VI). Enseignement Mathématique, 33:270–284, 1934.
- [35] H. LEBESGUE. *Sur la mesure des grandeurs* (sec. VII). Enseignement Mathématique, 34:176–219, 1935.
- [36] H. LEBESGUE. *Measure and the Integral* (ed. Kenneth O. May). Holden-Day, San Francisco, 1966.
- [37] H. LEBESGUE. *Oeuvres scientifiques* (vols. I–V). L'Enseignement Mathématique, 1972.
- [38] G. LETTA. *Misura*. Enciclopedia Europea Garzanti, 7:641–644, 1978.
- [39] G. LETTA. *Le condizioni di Riemann per l'integrabilità e il loro influsso sulla nascita del concetto di misura*. Rendiconti Accademia Nazionale delle Scienze dei XL, Memorie di Matematica, 18:143–169, 1994.
- [40] N. I. LOBACHEVSKI. *Geometrical researches on the theory of parallels*. Open Court Publ. Comp., Chicago, 1914.

- [41] K. O. MAY. *Bibliography and research manual of the history of Mathematics*. University Toronto Press, 1973.
- [42] F. A. MEDVEDEV. *Le funzioni d'insieme secondo G. Peano*. Archives Internationales d'Histoire des Sciences, 33:112–117, 1983.
- [43] MOIGNO, L'ABBÉ. *Leçons de mécanique analytique, d'après les méthodes d'Augustin Cauchy*. Statique. Gauthier-Villars, Paris, 1868.
- [44] O. NIKODYM. *Sur une généralisation des intégrales de M. J. Radon*. Fundamenta Mathematicae, 15:131–179, 1930.
- [45] G. PEANO. *Sull'integrabilità delle funzioni*. Atti R. Acad. Sci. Torino, 18:439–446, 1883.
- [46] G. PEANO. *Extrait d'une lettre [sur un théorème de M. Camille Jordan]*. Nouvelles Annales de Mathématiques, 3:45–47, 1884.
- [47] G. PEANO. *Réponse à Ph. Gilbert*. Nouvelles Annales de Mathématiques, 3:252–256, 1884.
- [48] G. PEANO. *Applicazioni geometriche del calcolo infinitesimale*. Fratelli Bocca Editori, 1887.
- [49] G. PEANO. *Calcolo geometrico secondo Ausdehnungslehre di H. Grassmann*. Fratelli Bocca, Torino, 1888.
- [50] G. PEANO. *Sur une courbe, qui remplit toute une aire plane*. Mathematische Annalen, 36:157–160, 1890.
- [51] G. PEANO. *Sulla definizione dell'area d'una superficie*. Atti della Reale Accademia dei Lincei, Rendiconti, 6 (1st sem.):54–57, 1890.
- [52] G. PEANO. *Démonstration de l'intégrabilité des équations différentielles ordinaires*. Mathematische Annalen, 37:182–228, 1890.
- [53] G. PEANO. *Sur la définition de la dérivée*. Mathesis, 2:12–14, 1892.
- [54] G. PEANO. *Lezioni di analisi infinitesimale*. Candeletti, Torino, 1893.
- [55] G. PEANO. *Le grandezze coesistenti di Cauchy*. Atti della Reale Accademia delle Scienze di Torino, 50:1146–1155, 1914–15.
- [56] G. PEANO. *Opere scelte*. Edizioni Cremonese, Roma, 1957–59. 3 volumes.
- [57] G. PEANO. *Selected papers* (H. C. Kennedy, ed.). University Toronto Press, 1973.
- [58] G. PEANO. *Opera omnia* (1 CD-ROM edited by C. S. Roero). Dipartimento di Matematica, Università, Torino, 2002.
- [59] I. N. PESIN. *Classical and modern integration theory*. Academic Press, 1970.
- [60] M. PICONE. *Lezioni di analisi infinitesimale* (vol. I and II). Circolo Matematico di Catania, Catania, 1923.
- [61] J. RADON. *Theorie und Anwendungen der absolut additiven Mengenfunktionen*. Sitzungsberichte der Akademie der Wissenschaften in Wien. Mathematisch-naturwissenschaftliche Klasse. Abteilung IIa., 112:1295–1438, 1913.
- [62] J. SERRET. *Cours de calcul différentiel et intégral*. Paris, 1879.
- [63] T. J. STIELTJES. *Recherches sur les fractions continues*. Ann. Fac. Sci. Toulouse, 8:1–122, 1894.
- [64] O. STOLZ. *Über einen zu einer unendlichen Punktmenge gehörigen Grenzwert*. Mathematische Annalen, 24:152–156, 1884.
- [65] J. TANNERY. *Review of "Peano G.: Applicazioni geometriche del calcolo infinitesimale"*. Bull. des sciences mathématiques, 11:237–239, 1887.
- [66] L. TONELLI. *Sulla rettificazione delle curve*. Atti R. Acc. Scienze Torino, 43:783–800, 1908.

- [67] G. VITALI. *I teoremi della media e di Rolle* (da una lettera al prof. Guido Fubini). Atti della R. Accademia delle Scienze di Torino, 51:143–147, 1915.
- [68] G. VITALI. *Sui teoremi di Rolle e della media per le funzioni additive*. Atti della Reale Accademia dei Lincei, Rendiconti, 25:684–688, 1916.
- [69] G. VITALI. *Opere sull'analisi reale e complessa e carteggio*. Cremonese, Roma, 1984.
- [70] G. ZWIRNER. *Lezioni di analisi matematica* (part II). CEDAM, Padova, 1969 (3<sup>rd</sup> ed.).

---

Received 15 March 2010,  
and in revised form 7 April 2010.

Gabriele H. Greco  
Dipartimento di Matematica  
Università di Trento—38050 Povo (TN) (Italy)  
greco@science.unitn.it

Sonia Mazzucchi  
Dipartimento di Matematica  
Università di Trento—38050 Povo (TN) (Italy)  
mazzucch@science.unitn.it

Enrico M. Pagani  
Dipartimento di Matematica  
Università di Trento—38050 Povo (TN) (Italy)  
pagani@science.unitn.it

