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Unitary Processes with Independent Increments and Representations of Hilbert Tensor Algebras

By

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Abstract

The aim of this article is to characterize unitary increment process by a quantum stochastic integral representation on symmetric Fock space. Under certain assumptions we have proved its unitary equivalence to a Hudson-Parthasarathy flow.

§1. Introduction

In the framework of the theory of quantum stochastic calculus developed by pioneering work of Hudson and Parthasarathy [6], quantum stochastic differential equations (qsde) of the form

$$dV_t = \sum_{\mu,\nu\geq 0} V_t L^\mu_\nu \Lambda^\nu_\mu(dt),\ V_0 = 1_{\mathbf{h}\otimes\Gamma}, \tag{1.1}$$

(where the coefficients $L^\mu_\nu\ :\ \mu,\ \nu\ \geq 0$ are operators in the initial Hilbert space $\mathbf{h}$ and Λ^ν_μ are fundamental processes in the symmetric Fock space $\Gamma = \Gamma_{sym}(L^2(\mathbb{R}_+,\mathbf{k}))$ with respect to a fixed orthonormal basis (in short 'ONB')

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$\{E_j : j \geq 1\}$ of the noise Hilbert space $\mathbf{k}$) have been formulated. Conditions for existence and uniqueness of a solution $\{V_t\}$ are studied by Hudson and Parthasarathy and many other authors. In particular when the coefficients $L^\mu_\nu : \mu, \nu \geq 0$ are bounded operators satisfying some conditions it is observed that the solution $\{V_t : t \geq 0\}$ is a unitary process.

In [4], using the integral representation of regular quantum martingales in symmetric Fock space [17], the authors show that any covariant Fock adapted unitary evolution $\{V_{s,t} : 0 \leq s \leq t < \infty\}$ (with norm-continuous expectation semigroup) satisfies a quantum stochastic differential equation (1.1) with constant coefficients $L^\mu_\nu \in \mathcal{B}(\mathbf{h})$. For situations where the expectation semigroup is not norm continuous, the characterization problem is discussed in [5, 1]. In [10, 11], by extended semigroup methods, Lindsay and Wills have studied such problems for Fock adapted contractive operator cocycles and completely positive cocyles.

In this article we are interested in the characterization of unitary evolutions with stationary and independent increments on $\mathbf{h} \otimes \mathcal{H}$, where $\mathbf{h}$ and $\mathcal{H}$ are separable Hilbert spaces. In [18, 19], by a co-algebraic treatment, the second author has proved that any weakly continuous unitary stationary independent increment process on $\mathbf{h} \otimes \mathcal{H}, \mathbf{h}$ finite dimensional, is unitarily equivalent to a Hudson-Parthasarathy flow with constant operator coefficients; see also [8, 9]. In this present paper we treat the case of a unitary stationary independent increment process on $\mathbf{h} \otimes \mathcal{H}, \mathbf{h}$ not necessarily finite dimensional, with norm-continuous expectation semigroup. By a GNS type construction we are able to get the noise space $\mathbf{k}$ and the bounded operator coefficients L^μ_ν such that the Hudson-Parthasarathy flow equation (1.1) admits a unique unitary solution and is unitarily equivalent to the unitary process we started with.

The article is organized as follows: Section 2 is meant for recalling some preliminary ideas and fixing some notations on linear operators on Hilbert spaces and quantum stochastic flows on Fock space. In the next Section an algebra structure is given on tensor product of Hilbert space which we are calling as Hilbert tensor algebra. The unitary processes with stationary and independent increments are described in Section 4 and filtration property of these processes is seen in Section 5. In Section 6 various semigroups associated with above mentioned unitary processes are studied and using them a Hilbert space, called noise space and structure maps are constructed from the Hilbert tensor algebra in Section 7. Associated Hudson-Parthasarathy flow is studied in Section 8 and its minimality is discussed in Section 9. In the last Section unitary equivalence to Hudson-Parthasarathy flow is established.

§2. Notation and Preliminaries

We assume that all the Hilbert spaces appearing in this article are complex separable with inner product anti-linear in the first variable. For any Hilbert spaces $\mathcal{H}, \mathcal{K}$ $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{B}_1(\mathcal{H})$ denote the Banach space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and trace class operators on $\mathcal{H}$ respectively. For a linear (not necessarily bounded) map T we write its domain as $\mathcal{D}(T)$. We denote the trace on $\mathcal{B}_1(\mathcal{H})$ by $Tr_{\mathcal{H}}$ or simply Tr. The von Neumann algebra of bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$. The Banach space $\mathcal{B}_1(\mathcal{H}, \mathcal{K}) \equiv \{\rho \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : |\rho| := \sqrt{\rho^* \rho} \in \mathcal{B}_1(\mathcal{H})\}$ with norm (Ref. Page no. 47 in [2])

$$\|\rho\|_1 = \||\rho|\|_{\mathcal{B}_1(\mathcal{H})} = \sup\{\sum_{k,l} |\langle \phi_k, \rho \psi_l \rangle| : \{\phi_k\}, \{\psi_l\} \ are\ ONB\ of\ \mathcal{K}\ and\ \mathcal{H}\ resp.\}$$

is the predual of $\mathcal{B}(\mathcal{K}, \mathcal{H})$. For an element $x \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $\mathcal{B}_1(\mathcal{H}, \mathcal{K}) \ni \rho \mapsto Tr_{\mathcal{H}}(x\rho)$ defines an element of the dual Banach space $\mathcal{B}_1(\mathcal{H}, \mathcal{K})^*$. For a linear map T on the Banach space $\mathcal{B}_1(\mathcal{H}, \mathcal{K})$ the adjoint T^* on the dual $\mathcal{B}(\mathcal{K}, \mathcal{H})$ is given by $Tr_{\mathcal{H}}(T^*(x)\rho) := Tr_{\mathcal{H}}(xT(\rho)),\ \forall x \in \mathcal{B}(\mathcal{K}, \mathcal{H}),\ \rho \in \mathcal{B}_1(\mathcal{H}, \mathcal{K})$.

For any $\xi \in \mathcal{H} \otimes \mathcal{K}, h \in \mathcal{H}$ the map

$$\mathcal{K} \ni k \mapsto \langle \xi, h \otimes k \rangle$$

defines a bounded linear functional on $\mathcal{K}$ and thus by Riesz's theorem there exists a unique vector $\langle\langle \xi, h \rangle\rangle$ in $\mathcal{K}$ such that

$$\langle\ \langle\langle \xi, h \rangle\rangle,\ k \rangle = \langle \xi, h \otimes k \rangle, \forall k \in \mathcal{K}. \tag{2.1}$$

In other words $\langle\langle \xi, h \rangle\rangle = F_h^* \xi$ where $F_h \in \mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$ is given by $F_h k = h \otimes k$.

Let $\mathbf{h}$ and $\mathcal{H}$ be two Hilbert spaces with some orthonormal bases $\{e_j : j \geq 1\}$ and $\{\zeta_n : n \geq 1\}$ respectively. For $A \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ and $u, v \in \mathbf{h}$ we define a linear operator $A(u, v) \in \mathcal{B}(\mathcal{H})$ by

$$\langle \xi_1, A(u, v)\xi_2 \rangle = \langle u \otimes \xi_1, A\ v \otimes \xi_2 \rangle,\ \forall \xi_1, \xi_2 \in \mathcal{H}$$

and read off the following properties:

Lemma 2.1. *Let $A, B \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ then for any u, v, u_i and $v_i, i = 1, 2$ in $\mathbf{h}$*

(i) *$A(u, v) \in \mathcal{B}(\mathcal{H})$ with $\|A(u, v)\| \leq \|A\|\ \|u\|\ \|v\|$ and $A(u, v)^* = A^*(v, u)$.*

(ii) *$\mathbf{h} \times \mathbf{h} \mapsto A(\cdot\ , \cdot)$ is $1-1$, i.e. if $A(u, v) = B(u, v),\ \forall u, v \in \mathbf{h}$ then $A = B$.*

(iii) $A(u_1, v_1)B(u_2, v_2) = [A(|v_1 >< u_2| \otimes 1_{\mathcal{H}})B](u_1, v_2)$

(iv) $AB(u, v) = \sum_{j\geq 1} A(u, e_j)B(e_j, v)$ (*strongly*)

(v) $0 \leq A(u, v)^* A(u, v) \leq \|u\|^2 A^* A(v, v)$

(vi) $\langle A(u, v)\xi_1, B(p, w)\xi_2\rangle = \sum_{n\geq 1}\langle p\otimes\zeta_n, [B(|w >< v|\otimes|\xi_2 >< \xi_1|)A^* u\otimes\zeta_n\rangle$
$= \langle v \otimes \xi_1,\ [A^*(|u >< p| \otimes 1_{\mathcal{H}})Bw \otimes \xi_2\rangle.$

Proof. We are omitting the proof of (i), (ii).
(iii) For any $\xi, \zeta \in \mathcal{H}$ we have

$$\begin{aligned}\langle \xi, A(u_1, v_1)B(u_2, v_2)\zeta\rangle &= \langle u_1 \otimes \xi, Av_1 \otimes B(u_2, v_2)\zeta\rangle \\ &= \langle A^* u_1 \otimes \xi, v_1 \otimes B(u_2, v_2)\zeta\rangle \\ &= \sum_{n\geq 1}\langle A^* u_1 \otimes \xi, v_1 \otimes \zeta_n\rangle\langle \zeta_n, B(u_2, v_2)\zeta\rangle \\ &= \sum_{n\geq 1}\langle A^* u_1 \otimes \xi, v_1 \otimes \zeta_n\rangle\langle u_2 \otimes \zeta_n, Bv_2 \otimes \zeta\rangle \\ &= \sum_{n\geq 1}\langle A^* u_1 \otimes \xi, (|v_1 >< u_2| \otimes |\zeta_n >< \zeta_n|)Bv_2 \otimes \zeta\rangle \\ &= \langle u_1 \otimes \xi, A(|v_1 >< u_2| \otimes 1_{\mathcal{H}})Bv_2 \otimes \zeta\rangle.\end{aligned}$$

Thus it follows that

$$A(u_1, v_1)B(u_2, v_2) = [A(|v_1 >< u_2| \otimes 1_{\mathcal{H}})B](u_1, v_2).$$

(iv) By part (iii)

$$\begin{aligned}&\sum_{j=1}^{N}\|A(e_j, u)\xi\|^2 \\ &= \sum_{j=1}^{N}\langle \xi, A^*(u, e_j)A(e_j, u)\xi\rangle \\ &= \langle \xi, [A^*(P_N \otimes 1_{\mathcal{H}})A](u, u)\xi\rangle,\end{aligned}$$

where P_N is the finite rank projection $\sum_{j=1}^{N} |e_j >< e_j|$ on $\mathbf{h}$. Since $\{[A^*(P_N \otimes 1_{\mathcal{H}})A](u, u)\}$ is an increasing sequence of positive operators and $0 \leq P_N \otimes 1_{\mathcal{H}}$ converges strongly to $1_{\mathbf{h}\otimes\mathcal{H}}$ as N tends to ∞, $[A^*(P_N \otimes 1_{\mathcal{H}})A](u, u)$ converges strongly to $[A^*A](u, u)$ as N tends to ∞. Thus

$$\lim_{N\to\infty}\sum_{j=1}^{N}\|A(e_j, u)\xi\|^2 = \langle \xi, [A^*A](u, u)\xi\rangle$$

and

$$\sum_{j=1}^{N} \|A(e_j,u)\xi\|^2 \leq \|A\ u\otimes\xi\|^2 \leq \|A\|^2\|u\|^2\|\xi\|^2, \forall N\geq 1.$$

Now let us consider the following, for $\xi,\zeta\in\mathcal{H}$

$$\begin{aligned}
&|\langle\xi,\sum_{j=1}^{N} A(u,e_j)B(e_j,v)\zeta\rangle|^2 = |\sum_{j=1}^{N}\langle A^*(e_j,u)\xi, B(e_j,v)\zeta\rangle|^2\\
&\leq \sum_{j=1}^{N}\|A^*(e_j,u)\xi\|^2\sum_{j=1}^{N}\|B(e_j,v)\zeta\|^2\\
&\leq \|A\|^2\|u\|^2\|\xi\|^2\|B\|^2\|v\|^2\|\zeta\|^2.
\end{aligned}$$

So

$$|\langle\xi,\sum_{j=1}^{N} A(u,e_j)B(e_j,v)\zeta\rangle| \leq \|A\|\|B\|\|u\|\|v\|\|\xi\|\|\zeta\|$$

and strong convergence of $\sum_{j\geq 1} A(u,e_j)B(e_j,v)$ follows.

(v) We have

$$\begin{aligned}
&\langle\xi, A(u,v)^*A(u,v)\xi\rangle = \sum_{n\geq 1}\langle\xi, A^*(v,u)\zeta_n\rangle\langle\zeta_n, A(u,v)\xi\rangle\\
&=\sum_{n\geq 1}\langle v\otimes\xi, A^*u\otimes\zeta_n\rangle\langle u\otimes\zeta_n, Av\otimes\xi\rangle\\
&=\langle v\otimes\xi, A^*\{|u><u|\otimes\sum_{n\geq 1}|\zeta_n><\zeta_n|\}Av\otimes\xi\rangle.
\end{aligned}$$

Since $\sum_{n\geq 1}|\zeta_n><\zeta_n|$ converges strongly to the identity operator

$$\langle\xi, A(u,v)^*A(u,v)\xi\rangle \leq \|u\|^2\langle v\otimes\xi, A^*Av\otimes\xi\rangle$$

and this proves the result.

(vi) We have

$$\begin{aligned}
&\langle A(u,v)\xi_1, B(p,w)\xi_2\rangle\\
&=\sum_{n\geq 1}\langle A(u,v)\xi_1,\zeta_n\rangle\langle\zeta_n, B(p,w)\xi_2\rangle\\
&=\sum_{n\geq 1}\langle Av\otimes\xi_1, u\otimes\zeta_n\rangle\langle p\otimes\zeta_n, Bw\otimes\xi_2\rangle\\
&=\sum_{n\geq 1}\langle B^*p\otimes\zeta_n, w\otimes\xi_2\rangle\langle v\otimes\xi_1, A^*u\otimes\zeta_n\rangle\\
&=\sum_{n\geq 1}\langle p\otimes\zeta_n, B(|w><v|\otimes|\xi_2><\xi_1|)A^*u\otimes\zeta_n\rangle.
\end{aligned}$$

This proves the first part. The other part follows from

$$\begin{aligned}&\sum_{n\geq 1}\langle p\otimes\zeta_n, B(|w><v|\otimes|\xi_2><\xi_1|)A^*u\otimes\zeta_n\rangle\\&=Tr_{\mathbf{h}\otimes\mathcal{H}}[(|u><p|\otimes 1_{\mathcal{H}})B(|w><v|\otimes|\xi_2><\xi_1|)A^*]\\&=Tr_{\mathbf{h}\otimes\mathcal{H}}[(|w><v|\otimes|\xi_2><\xi_1|)A^*(|u><p|\otimes 1_{\mathcal{H}})B]\\&=\langle v\otimes\xi_1,\ [A^*(|u><p|\otimes 1_{\mathcal{H}})Bw\otimes\xi_2\rangle\end{aligned}$$

□

Let us briefly recall the fundamental integrator processes of quantum stochastic calculus and the flow equation, introduced by Hudson and Parthasarathy [6]. For a Hilbert space $\mathbf{k}$ let us consider the symmetric Fock space $\Gamma=\Gamma(L^2(\mathbb{R}_+,\mathbf{k}))$. The exponential vector in the Fock space, associated with a vector $f\in L^2(\mathbb{R}_+,\mathbf{k})$ is given by

$$\mathbf{e}(f)=\bigoplus_{n\geq 0}\frac{1}{\sqrt{n!}}f^{(n)},$$

where $f^{(n)}=\underbrace{f\otimes f\otimes\cdots\otimes f}_{n-copies}$ for $n>0$ and by convention $f^{(0)}=1$. The exponential vector $\mathbf{e}(0)$ is called the vacuum vector.

Let us consider the Hudson-Parthasarathy (HP) flow equation on $\mathbf{h}\otimes\Gamma(L^2(\mathbb{R}_+,\mathbf{k}))$:

$$V_{s,t}=1_{\mathbf{h}\otimes\Gamma}+\sum_{\mu,\nu\geq 0}\int_s^t V_{s,\tau}L^\mu_\nu\Lambda^\nu_\mu(d\tau). \tag{2.2}$$

Here the coefficients $L^\mu_\nu:\mu,\ \nu\geq 0$ are operators in $\mathbf{h}$ and Λ^ν_μ are fundamental processes with respect to a fixed orthonormal basis $\{E_j:j\geq 1\}$ of $\mathbf{k}$:

$$\Lambda^\mu_\nu(t)=\begin{cases}t\ 1_{\mathbf{h}\otimes\Gamma} & \text{for } (\mu,\nu)=(0,0)\\ a(1_{[0,t]}\otimes E_j) & \text{for } (\mu,\nu)=(j,0)\\ a^\dagger(1_{[0,t]}\otimes E_k) & \text{for } (\mu,\nu)=(0,k)\\ \Lambda(1_{[0,t]}\otimes|E_k><E_j|) & \text{for } (\mu,\nu)=(j,k).\end{cases} \tag{2.3}$$

These operators act non-trivially on $\Gamma_{[0,t]}$ and as identity on $\Gamma_{(t,\infty)}$, where for any interval $I\subseteq[0,\infty),\Gamma_I=\Gamma_{sym}(L^2(I))$. For any $0\leq s\leq t<\infty,\Gamma=\Gamma_{[0,s]}\otimes\Gamma_{(s,t]}\otimes\Gamma_{(t,\infty)}$.

Theorem 2.2 ([7, 14, 16, 3]). *Let $H\in\mathcal{B}(\mathbf{h})$ be self-adjoint, $\{L_k,\ W^j_k:j,k\geq 1\}$ be a family of bounded linear operators in $\mathbf{h}$ such that $W=\sum_{j,k\geq 1}W^j_k\otimes$*

$|E_j>< E_k|$ *is an isometry* (*respectively co-isometry*) *operator in* $\mathbf{h}\otimes\mathbf{k}$ *and for some constant* $c \geq 0$,

$$\sum_{k\geq 1} \|L_k u\|^2 \leq c\|u\|^2, \ \forall u \in \mathbf{h}.$$

Let the coefficients L^μ_ν *be as follows,*

$$L^\mu_\nu = \begin{cases} iH - \frac{1}{2}\sum_{k\geq 1} L^*_k L_k & \text{for } (\mu,\nu) = (0,0) \\ L_j & \text{for } (\mu,\nu) = (j,0) \\ -\sum_{j\geq 1} L^*_j W^j_k & \text{for } (\mu,\nu) = (0,k) \\ W^j_k - \delta^j_k & \text{for } (\mu,\nu) = (j,k). \end{cases} \tag{2.4}$$

Then there exists a unique isometry (*respectively co-isometry*) *operator valued process* $V_{s,t}$ *satisfying* (2.2).

§3. Hilbert Tensor Algebra

For a product vector $\underline{u} = u_1 \otimes u_2 \otimes \cdots \otimes u_n \in \mathbf{h}^{\otimes n}$ we shall denote the product vector $u_n \otimes u_{n-1} \otimes \cdots \otimes u_1$ by $\overleftarrow{\underline{u}}$. For the null vector in $\mathbf{h}^{\otimes n}$ we shall write $\underline{0}$. If $\{f_j\}_{j=1}^\infty$ is an ONB for $\mathbf{h}$, then we have a product ONB $\{f_{\underline{j}} = f_{j_1} \otimes \cdots \otimes f_{j_n} : \underline{j} = (j_1, j_2, \cdots, j_n), j_k \geq 1\}$ for the Hilbert space $\mathbf{h}^{\otimes n}$.

Consider $\mathbb{Z}_2 = \{0,1\}$, the finite field with addition modulo 2. For $n \geq 1$, let $\mathbb{Z}_2^n$ denote the n-fold direct sum of $\mathbb{Z}_2$. For $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ and $\underline{\epsilon}' = (\epsilon'_1, \epsilon'_2, \cdots, \epsilon'_m)$ we define

$$\underline{\epsilon}\oplus\underline{\epsilon}' = (\epsilon_1, \cdots, \epsilon_n, \epsilon'_1, \cdots, \epsilon'_m) \in \mathbb{Z}_2^{n+m} \text{ and } \underline{\epsilon}^* = (1+\epsilon_n, 1+\epsilon_{n-1}, \cdots, 1+\epsilon_1) \in \mathbb{Z}_2^n.$$

Let $A \in \mathcal{B}(\mathbf{h}\otimes\mathcal{H}), \epsilon \in \mathbb{Z}_2 = \{0,1\}$. We define operators $A^{(\epsilon)} \in \mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ by $A^{(\epsilon)} := A$ if $\epsilon = 0$ and $A^{(\epsilon)} := A^*$ if $\epsilon = 1$. For $1 \leq k \leq n$, we define a unitary exchange map $P_{k,n} : \mathbf{h}^{\otimes n}\otimes\mathcal{H} \to \mathbf{h}^{\otimes n}\otimes\mathcal{H}$ by putting

$$P_{k,n}(\underline{u}\otimes\xi) := u_1 \otimes \cdots \otimes u_{k-1} \otimes u_{k+1} \cdots \otimes u_n \otimes (u_k \otimes \xi)$$

on product vectors. Let $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in \mathbb{Z}_2^n$. Consider the ampliation of the operator $A^{(\epsilon_k)}$ in $\mathcal{B}(\mathbf{h}^{\otimes n}\otimes\mathcal{H})$ given by

$$A^{(n,\epsilon_k)} := P^*_{k,n}(1_{\mathbf{h}^{\otimes(n-1)}} \otimes A^{(\epsilon_k)})P_{k,n}.$$

By definition of unitary operators $P_{k,n}$ for product vectors $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$,

$$\langle \xi, A^{(n,\epsilon_k)}(\underline{u}, \underline{v})\xi' \rangle = \langle \xi, A^{(\epsilon_k)}(u_k, v_k)\xi' \rangle \prod_{j\neq k}^{n} \langle u_j, v_j \rangle, \forall \xi, \xi' \in \mathbf{H}.$$

Now we define the operator $A^{(\underline{\epsilon})} := \prod_{k=1}^n A^{(n,\epsilon_k)} := A^{(1,\epsilon_1)} \cdots A^{(n,\epsilon_n)}$ in $\mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$. Please note that as here, through out this article, the product symbol $\prod_{k=1}^n$ stands for product from left to right as k increases. We have the following preliminary observation.

Lemma 3.1. **(i)** *For product vectors* $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$

$$\prod_{i=1}^m A^{(n,\epsilon_i)}(\underline{u},\underline{v}) = \prod_{i=1}^m A^{\epsilon_i}(u_i,v_i) \prod_{i=m+1}^n \langle u_i, v_i\rangle \in \mathcal{B}(\mathcal{H}).$$

(ii) *For* $\xi, \zeta \in \mathcal{H}$

$$\prod_{i=1}^m A^{(\epsilon_i)}(\xi,\zeta) = A^{(\underline{\epsilon}^{(m)})}(\xi,\zeta) \otimes 1_{\mathbf{h}^{\otimes n-m}} \in B(\mathbf{h}^{\otimes n}).$$

(iii) *If* A *is an isometry* (*respectively unitary*) *then* $A^{(n,\epsilon_k)}$ *and* $A^{(\underline{\epsilon})}$ *are isometries* (*respectively unitaries*).

The proof is obvious and is omitted.

We note that part (i) of this Lemma in particular gives

$$A^{(\underline{\epsilon})}(\underline{u},\underline{v}) = \prod_{i=1}^n A^{(\epsilon_i)}(u_i,v_i) \tag{3.1}$$

Let $M_0 := \{(\underline{u},\underline{v},\underline{\epsilon}) : \underline{u} = \otimes_{i=1}^n u_i,\ \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}, \underline{\epsilon} = (\epsilon_1,\epsilon_2,\cdots,\epsilon_n) \in \mathbb{Z}_2^n,\ n \geq 1\}$. In M_0, we introduce an equivalence relation '$\sim$' : $(\underline{u},\underline{v},\underline{\epsilon}) \sim (\underline{p},\underline{w},\underline{\epsilon}')$ if $\underline{\epsilon} = \underline{\epsilon}'$ and $|\underline{u} >< \underline{v}| = |\underline{p} >< \underline{w}| \in \mathcal{B}(\mathbf{h}^{\otimes n})$. Expanding the vectors in term of the ONB $\{e_{\underline{j}} = e_{j_1} \otimes \cdots \otimes e_{j_n} : \underline{j} = (j_1, j_2, \cdots, j_n), j_k \geq 1\}$, from $|\underline{u} >< \underline{v}| = |\underline{p} >< \underline{w}|$ we get $\underline{u}_{\underline{\mathrm{j}}}\overline{\underline{v}}_{\underline{\mathrm{k}}} = \underline{p}_{\underline{\mathrm{j}}}\overline{\underline{w}}_{\underline{\mathrm{k}}}$ for each multi-indices $\underline{\mathrm{j}}, \underline{\mathrm{k}}$. Thus in particular when $(u,v,0) \sim (p,w,0)$, for any $\xi_1, \xi_2 \in \mathcal{H}$ we have

$$\begin{aligned}
&\langle \xi_1, A(u,v)\xi_2\rangle \\
&= \sum_{j,k\geq 1} \overline{u_j} v_k \langle e_j \otimes \xi_1, Ae_k \otimes \xi_2\rangle \\
&= \sum_{j,k\geq 1} \overline{p_j} w_k \langle e_j \otimes \xi_1, Ae_k \otimes \xi_2\rangle \\
&= \langle \xi_1, A(p,w)\xi_2\rangle.
\end{aligned}$$

In fact $A(u,v) = A(p,w)$ iff $(u,v,0) \sim (p,w,0)$ and more generally $A^{(\underline{\epsilon})}(\underline{u},\underline{v}) = A^{(\underline{\epsilon}')}(\underline{p},\underline{w})$ iff $(\underline{u},\underline{v},\underline{\epsilon}) \sim (\underline{p},\underline{w},\underline{\epsilon}')$. It is easy to see that $(\underline{0},\underline{v},\underline{\epsilon}) \sim (\underline{u},\underline{0},\underline{\epsilon}) \sim (\underline{0},\underline{0},\underline{\epsilon})$ and we call this class the 0 of the quotient set $M_0/\sim$. We first introduce the following operations on M_0 :

Vector multiplication: $(\underline{u},\underline{v},\underline{\epsilon}).(\underline{p},\underline{w},\underline{\epsilon}') = (\underline{u}\otimes\underline{p},\underline{v}\otimes\underline{w},\underline{\epsilon}\oplus\underline{\epsilon}')$ and
Involution: $(\underline{u},\underline{v},\underline{\epsilon})^* = (\underleftarrow{v},\underleftarrow{u},\underline{\epsilon}^*)$.

Since $\underleftarrow{(\underline{u}\otimes\underline{p})} = p_m\otimes\cdots\otimes p_1\otimes u_n\otimes\cdots\otimes u_1 = (\underleftarrow{p}\otimes\underleftarrow{u})$ and $(\underline{\epsilon}\oplus\underline{\epsilon}')^* = (\underline{\epsilon}')^*\oplus\underline{\epsilon}^*$

$$\begin{aligned}
&[(\underline{u},\underline{v},\underline{\epsilon}).(\underline{p},\underline{w},\underline{\epsilon}')]^* = (\underline{u}\otimes\underline{p},\underline{v}\otimes\underline{w},\underline{\epsilon}\oplus\underline{\epsilon}')^*\\
&= (\underleftarrow{\underline{v}\otimes\underline{w}},\underleftarrow{\underline{u}\otimes\underline{p}},(\underline{\epsilon}\oplus\underline{\epsilon}')^*)\\
&= (\underleftarrow{w}\otimes\underleftarrow{v},\underleftarrow{p}\otimes\underleftarrow{u},(\underline{\epsilon}')^*\oplus\underline{\epsilon}^*)\\
&= (\underline{p},\underline{w},\underline{\epsilon}')^*.(\underline{u},\underline{v},\underline{\epsilon})^*.
\end{aligned}$$

It is clear that $\underline{\epsilon} = \underline{\epsilon}' \implies \underline{\epsilon}^* = (\underline{\epsilon}')^*$ and $|\underline{u} >< \underline{v}| = |\underline{p} >< \underline{w}|$ implies $|\underleftarrow{v} >< \underleftarrow{u}| = |\underleftarrow{w} >< \underleftarrow{p}|$. Thus $(\underline{u},\underline{v},\underline{\epsilon}) \sim (\underline{p},\underline{w},\underline{\epsilon}')$ implies $(\underline{u},\underline{v},\underline{\epsilon})^* \sim (\underline{p},\underline{w},\underline{\epsilon}')^*$. Moreover, $(\underline{u},\underline{v},\underline{\epsilon}) \sim (\underline{u}',\underline{v}',\underline{\epsilon}')$ and $(\underline{p},\underline{w},\underline{\alpha}) \sim (\underline{p}',\underline{w}',\underline{\alpha}')$ implies $\underline{\epsilon}\oplus\underline{\alpha} = \underline{\epsilon}'\oplus\underline{\alpha}'$ and $|\underline{u}\otimes\underline{p} >< \underline{v}\otimes\underline{w}| = |\underline{u} >< \underline{v}|\otimes|\underline{p} >< \underline{w}| = |\underline{u}' >< \underline{v}'|\otimes|\underline{p}' >< \underline{w}'| = |\underline{u}'\otimes\underline{p}' >< \underline{v}'\otimes\underline{w}'|$. Thus involution and vector multiplication respect $\sim$.

Let M be the complex vector space spanned by $M_0/\sim$. The elements of M are formal finite linear combinations of elements of $M_0/\sim$. With the above multiplication and involution M is a $*$-algebra.

§4. Unitary Processes with Stationary and Independent Increment

Let $\{U_{s,t} : 0 \le s \le t < \infty\}$ be a family of unitary operators in $\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ and Ω be a fixed unit vector in $\mathcal{H}$. We shall also set $U_t := U_{0,t}$ for simplicity. As we discussed in the previous section, let us consider the family of operators $\{U^{(\epsilon)}_{s,t}\}$ in $\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ for $\epsilon\in\mathbb{Z}_2$ given by $U^{(\epsilon)}_{s,t} = U_{s,t}$ if $\epsilon = 0, U^{(\epsilon)}_{s,t} = U^*_{s,t}$ if $\epsilon = 1$. Furthermore for $n \ge 1, \underline{\epsilon}\in\mathbb{Z}_2^n$ fixed, $1 \le k \le n$, we consider the families of operators $\{U^{(\epsilon_k)}_{s,t}\}$ and $\{U^{(\underline{\epsilon})}_{s,t}\}$ in $\mathcal{B}(\mathbf{h}^{\otimes n}\otimes\mathcal{H})$. By Lemma 3.1 we observe that

$$U^{(\underline{\epsilon})}_{s,t}(\underline{u},\underline{v}) = \prod_{i=1}^n U^{(\epsilon_i)}_{s,t}(u_i,v_i).$$

For $\underline{\epsilon} = (0,0,\cdots,0)\in\mathbb{Z}_2^n$ and $1 \le k \le n$, we shall write $U^{(n,k)}_{s,t}$ for the unitary operator $U^{(n,\epsilon_k)}_{s,t}$ and $U^{(n)}_{s,t}$ for the unitary $U^{(\underline{\epsilon})}_{s,t}$ on $\mathbf{h}^{\otimes n}\otimes\mathcal{H}$. For $n \ge 1, \underline{s} = (s_1,s_2,\cdots,s_n), \underline{t} = (t_1,t_2,\cdots,t_n) :\ 0 \le s_1 \le t_1 \le s_2 \le \ldots \le s_n \le t_n < \infty$, $\underline{\epsilon}_k = (\alpha^{(k)}_1,\alpha^{(k)}_2,\cdots,\alpha^{(k)}_{m_k}) \in \mathbb{Z}_2^{m_k} : 1 \le k \le n, m = m_1+m_2+\cdots+m_n$ $\underline{\epsilon} = \underline{\epsilon}_1\oplus\underline{\epsilon}_2\oplus\cdots\oplus\underline{\epsilon}_n\in\mathbb{Z}_2^m$, we define $U^{(\underline{\epsilon})}_{\underline{s},\underline{t}}\in\mathcal{B}(\mathbf{h}^{\otimes m}\otimes\mathcal{H})$ by setting

$$U^{(\underline{\epsilon})}_{\underline{s},\underline{t}} := \prod_{k=1}^n U^{(\underline{\epsilon}_k)}_{s_k,t_k}. \tag{4.1}$$

Here $U^{(\epsilon_k)}_{s_k,t_k}$ is looked upon as an operator in $\mathcal{B}(\mathbf{h}^{\otimes m} \otimes \mathcal{H})$ by ampliation and appropriate tensor flip. So for $\underline{u} = \otimes_{k=1}^n \underline{u}_k, \underline{v} = \otimes_{k=1}^n \underline{v}_k \in \mathbf{h}^{\otimes m}$ we have

$$U^{(\underline{\epsilon})}_{\underline{s},\underline{t}}(\underline{u}, \underline{v}) = \prod_{k=1}^{n} U^{(\epsilon_k)}_{s_k,t_k}(\underline{u}_k, \underline{v}_k).$$

When there can be no confusion, for $\underline{\epsilon} = (0, 0, \cdots, 0)$ we write $U_{\underline{s},\underline{t}}$ for $U^{(\underline{\epsilon})}_{\underline{s},\underline{t}}$. For $a, b \geq 0, \underline{s} = (s_1, s_2, \cdots, s_n), \underline{t} = (t_1, t_2, \cdots, t_n)$ we write $a \leq \underline{s}, \underline{t} \leq b$ if $a \leq s_1 \leq t_1 \leq s_2 \leq \ldots \leq s_n \leq t_n \leq b$.

Let us assume the following properties on the unitary family $U_{s,t}$ to establish unitary equivalence of $U_{s,t}$ with an HP flow.

Assumption A

A1 (Evolution) For any $0 \leq r \leq s \leq t < \infty$, $U_{r,s}U_{s,t} = U_{r,t}$.

A2 (Independence of increments) For any $0 \leq s_i \leq t_i < \infty \;:\; i = 1, 2$ such that $[s_1, t_1) \cap [s_2, t_2) = \varnothing$

(a) For every $u_i, v_i \in \mathbf{h}$, $U_{s_1,t_1}(u_1, v_1)$ commutes with $U_{s_2,t_2}(u_2, v_2)$ and $U^*_{s_2,t_2}(u_2, v_2)$.

(b) For $s_1 \leq \underline{a}, \underline{b} \leq t_1$, $s_2 \leq \underline{q}, \underline{r} \leq t_2$ and $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$, $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes m}$, $\underline{\epsilon} \in \mathbb{Z}_2^n, \underline{\epsilon}' \in \mathbb{Z}_2^m$

$$\langle \Omega, U^{(\underline{\epsilon})}_{\underline{a},\underline{b}}(\underline{u}, \underline{v})U^{(\underline{\epsilon}')}_{\underline{q},\underline{r}}(\underline{p}, \underline{w})\Omega\rangle = \langle \Omega, U^{(\underline{\epsilon})}_{\underline{a},\underline{b}}(\underline{u}, \underline{v})\Omega\rangle\langle \Omega, U^{(\underline{\epsilon}')}_{\underline{q},\underline{r}}(\underline{p}, \underline{w})\Omega\rangle.$$

A3 (Stationarity) For any $0 \leq s \leq t < \infty$ and $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}, \underline{\epsilon} \in \mathbb{Z}_2^n$

$$\langle \Omega, U^{(\underline{\epsilon})}_{s,t}(\underline{u}, \underline{v})\Omega\rangle = \langle \Omega, U^{(\underline{\epsilon})}_{t-s}(\underline{u}, \underline{v})\Omega\rangle.$$

Assumption B (Uniform continuity)

$$\lim_{t\to 0} \; \sup\{|\langle \Omega, (U_t - 1)(u, v)\Omega\rangle| : \|u\|, \; \|v\| = 1\} = 0.$$

Assumption C (Gaussian Condition) For any $u_i, v_i \in \mathbf{h}, \epsilon_i \in \mathbb{Z}_2 : i = 1, 2, 3$

$$\lim_{t\to 0} \frac{1}{t}\langle \Omega, \; (U^{(\epsilon_1)}_t - 1)(u_1, v_1)(U^{(\epsilon_2)}_t - 1)(u_2, v_2)(U^{(\epsilon_3)}_t - 1)(u_3, v_3)\; \Omega\rangle = 0. \tag{4.2}$$

Assumption D (Minimality)

The set $\mathcal{S} = \{U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega := U_{s_1,t_1}(u_1, v_1) \cdots U_{s_n,t_n}(u_n, v_n)\Omega : \underline{s} = (s_1, s_2, \cdots, s_n), \; \underline{t} = (t_1, t_2, \cdots, t_n) : \; 0 \leq s_1 \leq t_1 \leq s_2 \leq \ldots \leq s_n \leq t_n < \infty, n \geq 1, \underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}\}$ is total in $\mathcal{H}$.

Remark 4.1. (a) The **Assumptions A, B** and **C** hold in many situations, for example for unitary solutions of the Hudson-Parthasarathy flow (2.2) with bounded operator coefficients and having no Poisson terms. We will see (Lemma 6.6) that **Assumption C** means that expressions of the form (4.2) vanish for arbitrary length $n \geq 3$ not only for length 3. This corresponds to the fact that the increments of order $n \geq 3$ of a Gaussian distribution vanish. Moreover, this property characterizes Gaussian distributions.

In the case of $dim(\mathbf{h}) < \infty$ **Assumption C** is equivalent to the condition that the generator of the quantum Lévy process associated with $\{U_{s,t}\}$ vanishes on products of length 3 of elements of the kernel of the counit. This again is equivalent to the condition that the preservation term in the corresponding quantum stochastic differential equation does not appear, Ref. Chapter 5 of [19].

Remark 4.2. The **Assumption D** is not really a restriction, since one can as well work with replacing $\mathcal{H}$ by the span closure of S. Taking $0 \leq s_1 \leq t_1 \leq s_2 \leq \ldots \leq s_n \leq t_n < \infty$ in the definition of $S \subseteq \mathcal{H}$ is enough for totality of the set S because : for $0 \leq r \leq s \leq t \leq \infty$, we have $U_{r,t}(p,w)) = \sum_j U_{r,s}(p,e_j)U_{s,t}(e_j,w)$. So if there are overlapping intervals $[s_k,t_k) \cap [s_{k+1}, t_{k+1}) \neq \varnothing$ then the vector $\xi = U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega$ in $\mathcal{H}$ can be obtained as a vector in the closure of the linear span of S.

For any $n \geq 1$ we have the following useful observations.

Lemma 4.3. **(i)** *For any* $0 \leq r \leq s \leq t < \infty$,

$$U^{(n,k)}_{r,t} = U^{(n,k)}_{r,s} U^{(n,k)}_{s,t}. \tag{4.3}$$

(ii) *For any* $1 \leq k_1, k_2, \cdots, k_m \leq n \ : k_i \neq k_j$ *for* $i \neq j$ *and* $0 \leq s_i \leq t_i < \infty \ : i = 1, 2, \cdots, n$

$$\prod_{i=1}^{m} U^{(n,\epsilon_{k_i})}_{s_i,t_i}(\underline{u},\underline{v}) = \prod_{i=1}^{m} U^{(n,\epsilon_{k_i})}_{s_i,t_i}(u_{k_i}, v_{k_i}) \prod_{j \neq k_i} \langle u_j, v_j \rangle \tag{4.4}$$

for every $\underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}$ *and* $\underline{\epsilon} \in \mathbb{Z}_2^n$.
(iii)

$$U^{(n)}_{r,t} = U^{(n)}_{r,s} U^{(n)}_{s,t}. \tag{4.5}$$

Proof. (i) It follows from the definition and **Assumptions A1, A2**.
(ii) For $\underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}$ and $\underline{\epsilon} \in \mathbb{Z}_2^n$ we can see by induction that

$$\langle \xi, \prod_{i=1}^m U_{s_i,t_i}^{(n,\epsilon_{k_i})}(\underline{u}, \underline{v})\xi' \rangle = \langle \xi, \prod_{i=1}^m U_{s_i,t_i}^{(n,\epsilon_{k_i})}(u_{k_i}, v_{k_i}) \prod_{j \neq k_i} \xi' \rangle \prod_{j \neq k}^n \langle u_j, v_j \rangle, \forall \xi, \xi' \in \mathbf{H}.$$

Thus (4.4) follows.
(iii) Since $U_{r,t}^{(n)}$ is a product of $U_{r,t}^{(n,k)}$: $k = 1, 2, \ldots, n$ and we have

$$U_{r,t}^{(n,k)} = U_{r,s}^{(n,k)} U_{s,t}^{(n,k)},$$

it is enough to prove that the unitary operators $U_{r,s}^{(n,k)}$ and $U_{s,t}^{(n,l)}$ commute for $k \neq l$. To see this let us consider the following. By part (ii) and the fact that $U_{r,s}(u_k, v_k)$ and $U_{s,t}(u_l, v_l)$ commute by **Assumption A2**, we get

$$\begin{aligned} U_{r,s}^{(n,k)} U_{s,t}^{(n,l)}(\underline{u}, \underline{v}) &= U_{r,s}(u_k, v_k) U_{s,t}(u_l, v_l) \prod_{i \neq k,l} \langle u_i, v_i \rangle \\ &= U_{s,t}(u_l, v_l) U_{r,s}(u_k, v_k) \prod_{i \neq k,l} \langle u_i, v_i \rangle = U_{s,t}^{(n,l)} U_{r,s}^{(n,k)}(\underline{u}, \underline{v}). \end{aligned}$$

As all the operators U appear here are bounded this implies

$$U_{r,s}^{(n,k)} U_{s,t}^{(n,l)} = U_{s,t}^{(n,l)} U_{r,s}^{(n,k)}.$$

□

§5. Filtration

For any $0 \leq q \leq t < \infty$, let $\mathcal{H}_{[q,t]} = \overline{Span}\ \mathcal{S}_{[q,t]}$, where $\mathcal{S}_{[q,t]} \subseteq \mathcal{H}$ is given by $\{\xi_{[q,t]} = U_{\underline{r},\underline{s}}^{(n)}(\underline{u}, \underline{v})\Omega = U_{r_1,s_1}(u_1, v_1) \cdots U_{r_n,s_n}(u_n, v_n)\Omega \in \mathcal{S} : q \leq \underline{r}, \underline{s} < t, n \geq 1, \underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}\}$. We shall denote the Hilbert spaces $\mathcal{H}_{[0,t]}$ and $\mathcal{H}_{[t,\infty)}$ by $\mathcal{H}_{t]}$ and $\mathcal{H}_{[t}$ respectively.

Lemma 5.1. *For $0 \leq t \leq T \leq \infty$, there exist a unitary isomorphism $\Xi_t : \mathcal{H}_{t]} \otimes \mathcal{H}_{(t,T]} \to \mathcal{H}_{T]}$ such that*

$$U_t(u, v) = \Xi_t^* U_t(u, v) \otimes 1_{\mathcal{H}_{(t,T]}} \Xi_t. \tag{5.1}$$

Proof. Let us define a map $\Xi_t : \mathcal{H}_{t]} \otimes \mathcal{H}_{[t,T]} \to \mathcal{H}_{T]}$ by

$$\Xi_t(\xi_{[0,t]} \otimes \zeta_{[t,T]}) = U_{\underline{r},\underline{s}}^{(n)}(\underline{u}, \underline{v}) U_{\underline{r}',\underline{s}'}^{(n)}(\underline{p}, \underline{w})\Omega$$

for $\xi_{[0,t]} = U_{\underline{r},\underline{s}}^{(n)}(\underline{u}, \underline{v})\Omega \in \mathcal{S}_{t]}$ and $\zeta_{[t,T]} = U_{\underline{r}',\underline{s}'}^{(n)}(\underline{p}, \underline{w})\Omega \in \mathcal{S}_{[t,T]}$, then extending linearly.

Now let us consider the following. By **Assumption A**, for $\xi_{[0,t]}$ and $\zeta_{[t,T]}$ as above and $\eta_{[0,t]} = U^{(n)}_{\underline{a},\underline{b}}(\underline{x},\underline{y})\Omega \in \mathcal{S}_{t]}$ and $\gamma_{[t,T]} = U^{(n)}_{\underline{a}',\underline{b}'}(\underline{g},\underline{h})\Omega \in \mathcal{S}_{[t,T]}$, we have

$$\begin{aligned}
&\langle \Xi_t(\xi_{[0,t]} \otimes \zeta_{[t,T]}), \Xi_t(\eta_{[0,t]} \otimes \gamma_{[t,T]})\rangle \\
&= \langle U^{(n)}_{\underline{r},\underline{s}}(\underline{u},\underline{v}) U^{(n)}_{\underline{r}',\underline{s}'}(\underline{p},\underline{w})\Omega, U^{(n)}_{\underline{a},\underline{b}}(\underline{x},\underline{y}) U^{(n)}_{\underline{a}',\underline{b}'}(\underline{g},\underline{h})\Omega\rangle \\
&= \langle \Omega, \left[U^{(n)}_{\underline{r},\underline{s}}(\underline{u},\underline{v}) U^{(n)}_{\underline{r}',\underline{s}'}(\underline{p}^{(n)},\underline{w}^{(n)})\right]^* U^{(n)}_{\underline{a},\underline{b}}(\underline{x},\underline{y}) U^{(n)}_{\underline{a}',\underline{b}'}(\underline{g},\underline{h})\Omega\rangle \\
&= \langle \Omega, \left[U^{(n)}_{\underline{r},\underline{s}}(\underline{u},\underline{v})\right]^* U^{(n)}_{\underline{a},\underline{b}}(\underline{x},\underline{y})\Omega\rangle \\
&\quad \langle \Omega, \left[U^{(n)}_{\underline{r}',\underline{s}'}(\underline{p},\underline{w})\right]^* U^{(n)}_{\underline{a}',\underline{b}'}(\underline{g},\underline{h})\Omega\rangle \\
&= \langle \xi_{[0,t]}, \eta_{[0,t]}\rangle \langle \zeta_{[t,T]}, \gamma_{[t,T]}\rangle.
\end{aligned}$$

Thus we get $\langle \Xi_t(\xi_{[0,t]} \otimes \zeta_{[t,T]}), \Xi_t(\eta_{[0,t]} \otimes \gamma_{[t,T]})\rangle = \langle \xi_{[0,t]} \otimes \zeta_{[t,T]}\ ,\ \eta_{[0,t]} \otimes \gamma_{[t,T]}\rangle$. Since by definition the range of Ξ_t is dense in $\mathcal{H}_{T]}$, this proves Ξ_t is a unitary operator.

Again by similar arguments to those above, for any $u, v \in \mathbf{h}$, we have

$$\begin{aligned}
&\langle \Xi_t\ \xi_{[0,t]} \otimes \zeta_{[t,T]},\ U_t(u,v)\ \Xi_t\ \eta_{[0,t]} \otimes \gamma_{[t,T]}\rangle \\
&= \langle U^{(n)}_{\underline{r},\underline{s}}(\underline{u},\underline{v})\Omega, U_t(u,v) U^{(n)}_{\underline{a},\underline{b}}(\underline{x},\underline{y})\Omega\rangle \\
&\quad \langle U^{(n)}_{\underline{r}',\underline{s}'}(\underline{p},\underline{w})\Omega, U^{(n)}_{\underline{a}',\underline{b}'}(\underline{g},\underline{h})\Omega\rangle \\
&= \langle \xi_{[0,t]}\ ,\ U_t(u,v)\eta_{[0,t]}\rangle\ \ \langle \zeta_{[t,T]}, \gamma_{[t,T]}\rangle.
\end{aligned}$$

This proves (5.1). □

§6. Expectation Semigroups

Let us look at the various semigroups associated with the unitary evolution $\{U_{s,t}\}$. For any fixed $n \geq 1$, we define a family of operators $\{T^{(n)}_t\}$ on $\mathbf{h}^{\otimes n}$ by setting

$$\langle \phi, T^{(n)}_t\ \psi\rangle := \langle \Omega, U^{(n)}_t(\phi,\psi)\ \Omega\rangle,\ \forall \phi, \psi \in \mathbf{h}^{\otimes n}.$$

Then in particular for product vectors $\underline{u} = \otimes^n_{i=1} u_i,\ \underline{v} = \otimes^n_{i=1} v_i \in \mathbf{h}^{\otimes n}$

$$\langle \underline{u}, T^{(n)}_t\ \underline{v}\rangle = \langle \Omega, U^{(n)}_t(\underline{u},\underline{v})\ \Omega\rangle = \langle \Omega, U_t(u_1,v_1) U_t(u_2,v_2) \cdots U_t(u_n,v_n)\ \Omega\rangle.$$

For $n = 1$, we shall write T_t for the family $T^{(1)}_t$.

Lemma 6.1. *The above family of operators $\{T^{(n)}_t\}$ is a semigroup of contractions on $\mathbf{h}^{\otimes n}$.*

Proof. Since $U_t^{(n)}$ is in particular contractive, for any $\phi, \psi \in \mathbf{h}^{\otimes n}$

$$|\langle \phi, T_t^{(n)} \ \psi \rangle| = |\langle \phi \ \Omega, U_t^{(n)} \psi \ \Omega \rangle| \leq \|\phi\| \ \|\psi\|$$

and contractivity of $T_t^{(n)}$ follows.

In order to prove that this family of contractions $T_t^{(n)}$ is a semigroup it is enough to show that for any $0 \leq s \leq t$ and product vectors $\underline{u} = \otimes_{i=1}^n u_i, \ \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}$,

$$\langle \underline{u}, T_t^{(n)} \ \underline{v} \rangle = \langle \underline{u}, T_s^{(n)} T_{t-s}^{(n)} \underline{v} \rangle.$$

Consider the product orthonormal basis $\{e_{\underline{j}} = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} : \ \underline{j} = (j_1, j_2, \cdots, j_n) : j_1, j_2, \cdots, j_n \geq 1\}$ of $\mathbf{h}^{\otimes n}$. By part (iii) of Lemma 2.1 and the evolution property (4.5) of $U_t^{(n)}$,

$$\begin{aligned}
&\langle \underline{u}, T_t^{(n)} \ \underline{v} \rangle = \langle \Omega, U_t^{(n)} (\underline{u}, \underline{v}) \ \Omega \rangle \\
&= \sum_{\underline{j}} \langle \Omega, \ U_s^{(n)} (\underline{u}, e_{\underline{j}}) U_{s,t}^{(n)} (e_{\underline{j}}, \underline{v}) \Omega \rangle \\
&= \sum_{\underline{j}} \langle \Omega, \ U_s^{(n)} (\underline{u}, e_{\underline{j}}) \ \Omega \rangle \langle \Omega \ , U_{t-s}^{(n)} (e_{\underline{j}}, \underline{v}) \Omega \rangle \\
&= \sum_{\underline{j}} \langle \underline{u}, T_s^{(n)} e_{\underline{j}} \rangle \langle e_{\underline{j}}, T_{t-s}^{(n)} \underline{v} \rangle = \langle \underline{u}, T_s^{(n)} T_{t-s}^{(n)} \underline{v} \rangle.
\end{aligned}$$

□

The following Lemma will be needed in the sequel

Lemma 6.2. **(i)** *For* $1 \leq k \leq n$,

$$\langle \Omega, U_t^{(n,k)} (\underline{p}, \underline{w}) \Omega \rangle = \langle \underline{p}, T_t^{(n,k)} \underline{w} \rangle, \ \forall \underline{p}, \underline{w} \in \mathbf{h}^{\otimes n} \tag{6.1}$$

where $T_t^{(n,k)} = 1_{\mathbf{h}^{(\otimes k-1)}} \otimes T_t \otimes 1_{\mathbf{h}^{(\otimes n-k)}}$.

(ii) *For any* $1 \leq m \leq n, \ \underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}$,

$$\langle \Omega, (\prod_{k=1}^m U_t^{(n,k)}) (\underline{p}, \underline{w}) \Omega \rangle = \langle \underline{p}, T_t^{(m)} \otimes 1_{\mathbf{h}^{(\otimes n-m)}} \ \underline{w} \rangle.$$

(iii) *For any* $\phi \in \mathbf{h}^{\otimes n}$,

$$\begin{aligned}
&\|(U_t^{(n,k)} - 1) \phi \otimes \Omega\|^2 \\
&= \langle (1 - T_t^{(n,k)}) \phi, \phi \rangle + \langle \phi, (1 - T_t^{(n,k)}) \phi \rangle \\
&\leq 2 \|1 - T_t\| \ \|\phi\|^2.
\end{aligned}$$

(iv)

$$\begin{aligned}
&\|(U_t^{(n)}-1)\phi\otimes\Omega\|^2\\
&=\langle(1-T_t^{(n)})\phi,\phi\rangle+\langle\phi,(1-T_t^{(n)})\phi\rangle\\
&\le 2\|(1-T_t^{(n)})\|\ \|\phi\|^2.
\end{aligned}$$

(v) *For any* $v\in\mathbf{h}$

$$\sum_{m\ge 1}\|(U_t-1)(e_m,v)\Omega\|^2=2Re\langle v,(1-T_t)v\rangle\le 2\|v\|^2\|T_t-1\|. \tag{6.2}$$

Proof. (i) It follows from the fact that for product vectors $\underline{p},\underline{w}$

$$\langle\Omega,U_t^{(n,k)}(\underline{p},\underline{w})\Omega\rangle=\langle p_k,T_t^{(n,k)}w_k\rangle\prod_{i\ne k}\langle p_i,w_i\rangle. \tag{6.3}$$

Part (ii) follows from Lemma 4.3 (ii).

The proofs of (iii) and (iv) are similar so we give the proof only for $U_t^{(n,k)}$. We have

$$\begin{aligned}
&\|(U_t^{(n,k)}-1)\ \phi\Omega\|^2\\
&=\langle\phi\Omega,[(U_t^{(n,k)}-1)^*(U_t^{(n,k)}-1)]\phi\Omega\rangle\\
&=\langle\phi\Omega,[2-(U_t^{(n,k)})^*-U_t^{(n,k)}]\phi\Omega\rangle\quad\text{(since } U_t^{(n,k)} \text{ is an unitary operator)}\\
&=\langle(1-T_t^{(n,k)})\phi,\phi\rangle+\langle\phi,(1-T_t^{(n,k)})\phi\rangle
\end{aligned}$$

Thus the statement follows.

(v) For any $v\in\mathbf{h}$

$$\begin{aligned}
&\sum_{m\ge 1}\|(U_t-1)(e_m,v)\Omega\|^2\\
&=\sum_{m\ge 1}\langle\Omega,(U_t-1)^*(v,e_m))(U_t-1)(e_m,v)\Omega\rangle\\
&=\langle\Omega,[(U_t-1)^*(U_t-1)](v,v)\Omega\rangle\\
&=\langle\Omega,[2-U_t^*-U_t](v,v)\Omega\rangle\\
&=\langle v,[2-T_t^*-T_t]v\rangle=2Re\langle v,(1-T_t)v\rangle=2\|v\|^2\|T_t-1\|.
\end{aligned}$$

□

Now we are ready to prove

Proposition 6.3. *Under the* **Assumption B** *the semigroup* $\{T_t^{(n)}\}$ *is uniformly continuous.*

Proof. **Assumption B** on the family of unitary operators $\{U_{s,t}\}$ implies that the semigroup of contractions $\{T_t\}$ on $\mathbf{h}$ is uniformly continuous. To apply induction let us assume that for some $m \geq 1$, the contractive semigroups $\{T_t^{(n)}\}$ are uniformly continuous for all $1 \leq n \leq m-1$. Now, for any $\phi, \psi \in \mathbf{h}^{\otimes m}$

$$\begin{aligned}
&\langle \phi \otimes \Omega, (U_t^{(m)} - 1)\psi \otimes \Omega\rangle \\
&= \langle \phi \otimes \Omega, \left([\prod_{k=1}^{m-1} U_t^{(m,k)}][U_t^{(m,m)}] - 1 \right) \psi \otimes \Omega\rangle \\
&= \langle [\prod_{k=1}^{m-1} U_t^{(m,k)}]^* \phi \otimes \Omega, \left([U_t^{(m,m)}] - 1 \right) \psi \otimes \Omega\rangle \\
&\quad + \langle \phi \otimes \Omega, \left([\prod_{k=1}^{m-1} U_t^{(m,k)}] - 1 \right) \psi \otimes \Omega\rangle.
\end{aligned}$$

Taking absolute values, by Lemma 6.2 we get

$$\begin{aligned}
&|\langle \phi,\ (T_t^{(m)} - 1_{\mathbf{h}^{\otimes m}})\psi\rangle| \\
&\leq \|\phi\|\ \|\psi\| \sqrt{2\|T_t^{(m,m)} - 1_{\mathbf{h}^{\otimes m}}\|} + |\langle \phi, \left([T_t^{(m-1)} \otimes 1_{\mathbf{h}}] - 1_{\mathbf{h}^{\otimes m}} \right) \psi\rangle| \\
&\leq \|\phi\|\|\psi\| \left[\sqrt{2\|T_t - 1\|} + \|T_t^{(m-1)} - 1\| \right].
\end{aligned}$$

So uniform continuity of $T_t^{(m-1)}$ and T_t implies that $T_t^{(m)}$ is uniformly continuous. □

Let us denote the bounded generator of the uniformly continuous semigroup $T_t^{(n)}$ on $\mathbf{h}^{\otimes n}$ by $G^{(n)}$ and for $n = 1$ by G.

For $n \geq 1$, we define a family of operators $\{Z_t^{(n)} : t \geq 0\}$ on the Banach space $\mathcal{B}_1(\mathbf{h}^{\otimes n})$ by

$$Z_t^{(n)}\rho = Tr_{\mathcal{H}}[U_t^{(n)}(\rho \otimes |\Omega >< \Omega|)(U_t^{(n)})^*],\ \rho \in \mathcal{B}_1(\mathbf{h}^{\mathbf{h}^{\otimes n}}).$$

Then in particular for product vectors $\underline{u}, \underline{v}, \underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}$.

$$\langle \underline{p}, Z_t^{(n)}(|\underline{w} >< \underline{v}|)\underline{u}\rangle := \langle U_t^{(n)}(\underline{u}, \underline{v})\Omega,\ U_t^{(n)}(\underline{p}, \underline{w})\ \Omega\rangle. \tag{6.4}$$

Lemma 6.4. *The above family* $\{Z_t^{(n)}\}$ *is a semigroup of contractive maps on* $\mathcal{B}_1(\mathbf{h}^{\otimes n})$*. Furthermore,* **Assumption B** *implies that* $\{Z_t^{(n)}\}$ *is uniformly continuous.*

Proof. For $\rho \in \mathcal{B}_1(\mathbf{h}^{\otimes n})$

$$\begin{aligned}
&\|Z_t^{(n)}\rho\|_1 = \|Tr_{\mathcal{H}}[U_t^{(n)}(\rho \otimes |\Omega >< \Omega|)(U_t^{(n)})^*]\|_1 \\
&= \sup_{\phi^{(n)}}\{\sum_{k\geq 1} |\langle \phi_k^{(n)}, Tr_{\mathcal{H}}[U_t^{(n)}(\rho \otimes |\Omega >< \Omega|)(U_t^{(n)})^*]\phi_k^{(n)}\rangle|, \\
&\text{(where the supremum is over all ONB } \phi^{(n)} \equiv \{\phi_k^{(n)}\} \text{ of } \mathbf{h}^{\otimes n}\text{)} \\
&\leq \sup_{\phi^{(n)}} \sum_{j,k\geq 1} |\langle \phi_k^{(n)} \otimes \zeta_j, U_t^{(n)}(\rho \otimes |\Omega >< \Omega|)(U_t^{(n)})^* \phi_k^{(n)} \otimes \zeta_j\rangle| \\
&\leq \|U_t^{(n)}(\rho \otimes |\Omega >< \Omega|)(U_t^{(n)})^*\|_1.
\end{aligned}$$

Since $\{U_t^{(n)}\}$ is in particular a contractive family of operators

$$\|Z_t^{(n)}\rho\|_1 \leq \|\rho \otimes |\Omega >< \Omega|\|_1 = \|\rho\|_1.$$

In order to prove that the family of contractions $\{Z_t^{(n)}\}$ is a semigroup it is enough to verify that property for the rank one operator $\rho = |\underline{w} >< \underline{v}| \ : \ \underline{v} = \otimes_{i=1}^n v_i, \underline{w} = \otimes_{i=1}^n w_i \in \mathbf{h}^{\otimes n}$. Therefore, it suffices to prove that

$$\langle \underline{p}, Z_t^{(n)}(\rho)\underline{u}\rangle = \langle \underline{p}, Z_s^{(n)} Z_{t-s}^{(n)}(\rho)\underline{u}\rangle, \forall \underline{p} = \otimes_{i=1}^n p_i, \underline{u} = \otimes_{i=1}^n u_i \in \mathbf{h}^{\otimes n}.$$

By Lemma 4.3, part (iv) of Lemma 2.1 and **Assumption A**, for $0 \leq s \leq t$, $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$ and product ONB $\{e_{\underline{k}}^{(n)} = e_{k_1} \otimes \cdots \otimes e_{k_n}\}$ of $\mathbf{h}^{\otimes n}$

$$\begin{aligned}
&\langle U_t^{(n)}(\underline{u}, \underline{v})\Omega,\ U_t^{(n)}(\underline{p}, \underline{w})\ \Omega\rangle \\
&= \sum_{\underline{j},\underline{k}} \langle U_s^{(n)}(\underline{u}, e_{\underline{j}}^{(n)})\Omega,\ U_s^{(n)}(\underline{p}, e_{\underline{k}}^{(n)})\ \Omega\rangle \langle U_{t-s}^{(n)}(e_{\underline{j}}^{(n)}, \underline{v})\Omega,\ U_{t-s}^{(n)}(e_{\underline{k}}^{(n)}, \underline{p})\ \Omega\rangle.
\end{aligned}$$

This gives

$$\begin{aligned}
&\langle \underline{p}, Z_t^{(n)}(\rho)\underline{u}\rangle \\
&= \sum_{\underline{j},\underline{k}} \langle \underline{p}, Z_s^{(n)}(|e_{\underline{k}}^{(n)} >< e_{\underline{j}}^{(n)}|)\underline{u}\rangle \langle e_{\underline{k}}^{(n)}, Z_{t-s}^{(n)}(\rho) e_{\underline{j}}^{(n)}\rangle \\
&= \sum_{\underline{j},\underline{k}} \langle e_{\underline{j}}^{(n)}, (Z_s^{(n)})^*(|\underline{u} >< \underline{p}|) e_{\underline{k}}^{(n)}\rangle \langle e_{\underline{k}}^{(n)}, Z_{t-s}^{(n)}(\rho) e_{\underline{j}}^{(n)}\rangle \\
&= \sum_{\underline{j}} \langle e_{\underline{j}}^{(n)}, (Z_s^{(n)})^*(|\underline{u} >< \underline{p}|) Z_{t-s}^{(n)}(\rho) e_{\underline{j}}^{(n)}\rangle \\
&= Tr[(Z_s^{(n)})^*(|\underline{u} >< \underline{p}|) Z_{t-s}^{(n)}(\rho)] \\
&= Tr[|\underline{u} >< \underline{p}| Z_s^{(n)} Z_{t-s}^{(n)}(\rho)] \\
&= \langle \underline{p}, Z_s^{(n)} Z_{t-s}^{(n)}(\rho)\underline{u}\rangle.
\end{aligned}$$

In order to prove uniform continuity of $Z_t^{(n)}$ we consider

$$\begin{aligned}
&\|(Z_t^{(n)}-1)(|\underline{w}><\underline{v}|)\|_1\\
&=\sup_{\phi^{(n)}}\{\sum_{k\geq 1}|\langle\phi_k^{(n)},(Z_t^{(n)}-1)(|\underline{w}><\underline{v}|)\phi_k^{(n)}\rangle|\}\\
&=\sup_{\phi^{(n)}}\sum_{k\geq 1}|\langle U_t^{(n)}(\phi_k^{(n)},\underline{v})\Omega,U_t^{(n)}(\phi_k^{(n)},\underline{w})\Omega\rangle-\overline{\langle\phi_k^{(n)},\underline{v}\rangle}\langle\phi_k^{(n)},\underline{w}\rangle|\\
&\leq\sup_{\phi^{(n)}}\sum_{k\geq 1}|\langle(U_t^{(n)}-1)(\phi_k^{(n)},\underline{v})\Omega,U_t^{(n)}(\phi_k^{(n)},\underline{w})\Omega\rangle|\\
&+\sup_{\phi^{(n)}}\sum_{k\geq 1}|\overline{\langle\phi_k^{(n)},\underline{v}\rangle}\langle\Omega,(U_t^{(n)}-1)(\phi_k^{(n)},\underline{w})\Omega\rangle|\\
&\leq\sup_{\phi^{(n)}}\left[\sum_{k\geq 1}\|(U_t^{(n)}-1)(\phi_k^{(n)},\underline{v})\Omega\|^2\right]^{\frac{1}{2}}\left[\sum_{k\geq 1}\|U_t^{(n)}(\phi_k^{(n)},\underline{w})\Omega\|^2\right]^{\frac{1}{2}}\\
&+\sup_{\phi^{(n)}}\left[\sum_{k\geq 1}|\langle\phi_k^{(n)},\underline{v}\rangle|^2\right]^{\frac{1}{2}}\left[\sum_{k\geq 1}\|(U_t^{(n)}-1)(\phi_k^{(n)},\underline{w})\Omega\|^2\right]^{\frac{1}{2}}
\end{aligned}$$

So by Lemma 6.2

$$\begin{aligned}
&\|(Z_t^{(n)}-1)(|\underline{w}><\underline{v}|)\|_1\\
&\leq 2\sqrt{2}\|\underline{v}\|\;\|\underline{w}\|\left(\sqrt{\|T_t^{(n)}-1\|}\right).
\end{aligned}$$

Now for any $\rho=\sum_k\lambda_k|\phi_k^{(n)}><\phi_k^{(n)}|\in\mathcal{B}_1(\mathbf{h}^{\otimes n})$ we have

$$\begin{aligned}
&\|Z_t^{(n)}(\rho)-\rho\|_1\\
&\leq 2\sqrt{2}\sum_k|\lambda_k|\left(\sqrt{\|T_t^{(n)}-1\|}\right)\\
&\leq 2\sqrt{2}\|\rho\|_1\;\left(\sqrt{\|T_t^{(n)}-1\|}\right).
\end{aligned}$$

Thus by uniform continuity of the semigroup $T_t^{(n)}$ it follows that the semigroup $Z_t^{(n)}$ is uniformly continuous on $\mathcal{B}_1(\mathbf{h}^{\otimes n})$. □

We shall denote the bounded generator of the semi-group $Z_t^{(n)}$ by $\mathcal{L}^{(n)}$ and for $n=1$ we simply write Z_t and $\mathcal{L}$ for the semigroup $Z_t^{(1)}$ and its generator $\mathcal{L}^{(1)}$ respectively.

Lemma 6.5. *For any $n \geq 1$, $Z_t^{(n)}$ is a positive trace preserving semigroup on $\mathcal{B}_1(\mathbf{h}^{\otimes n})$.*

Proof. Positivity follows from

$$\langle \underline{u}, Z_t^{(n)}(|\underline{v} >< \underline{v}|)\underline{u}\rangle = \|U_t^{(n)}(\underline{u}, \underline{v})\Omega\|^2 \geq 0 \ \forall \ \ \underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}.$$

To prove that $Z_t^{(n)}$ is trace preserving it is enough to show that

$$Tr[Z_t^{(n)}(|\underline{u} >< \underline{v}|)] = \langle \underline{v}, \underline{u}\rangle.$$

By definition and Lemma 2.1

$$\begin{aligned} Tr[Z_t^{(n)}(|\underline{u} >< \underline{v}|)] &= \sum_k \langle \underline{e}_k, Z_t^{(n)}(|\underline{u} >< \underline{v}|)\underline{e}_k\rangle \\ &= \sum_k \langle U_t^{(n)}(\underline{e}_k, \underline{v})\Omega, U_t^{(n)}(\underline{e}_k, \underline{u})\Omega\rangle \\ &= \langle \Omega, (U_t^{(n)})^* U_t^{(n)}(\underline{v}, \underline{u})\Omega\rangle. \end{aligned}$$

Since $U_t^{(n)}$ is unitary, we get

$$Tr[Z_t^{(n)}(|\underline{u} >< \underline{v}|)] = \langle \underline{v}, \underline{u}\rangle.$$

□

The above Lemma yields

$$Tr(\mathcal{L}^{(n)}\rho) = 0, \ \forall \rho \in \mathcal{B}_1(\mathbf{h}^{\otimes n}). \tag{6.5}$$

We conclude this section by the following useful observations.

Lemma 6.6. *Under the* **Assumption C**, *for any $n \geq 3$, $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$, $\underline{\epsilon} \in \mathbb{Z}_2^n$*

$$\lim_{t \to 0} \frac{1}{t}\langle \Omega, \ (U_t^{(\epsilon_1)} - 1)(u_1, v_1) \cdots (U_t^{(\epsilon_n)} - 1)(u_n, v_n) \ \Omega\rangle = 0. \tag{6.6}$$

Proof. We have

$$\begin{aligned} &|\frac{1}{t}\langle [(U_t^{(\epsilon_1)} - 1)(u_1, v_1)(U_t^{(\epsilon_2)} - 1)(u_2, v_2)]^* \ \Omega, \\ &\qquad (U_t^{(\epsilon_1)} - 3)(u_3, v_3) \cdots (U_t^{(\epsilon_n)} - 1)(u_n, v_n) \ \Omega\rangle\Big|^2 \\ &\leq \frac{1}{t^2}\|[(U_t^{(\epsilon_1)} - 1)(u_1, v_1)(U_t^{(\epsilon_2)} - 1)(u_2, v_2)]^* \ \Omega\|^2 \\ &\qquad \|(U_t^{\epsilon_3} - 1)(u_3, v_3) \cdots (U_t^{(\epsilon_n)} - 1)(u_n, v_n) \ \Omega\|^2 \\ &\leq C_{\underline{u}, \underline{v}} \ \ \frac{1}{t}\|[(U_t^{(\epsilon_1)} - 1)(u_1, v_1)(U_t^{(\epsilon_2)} - 1)(u_2, v_2)]^* \ \Omega\|^2 \\ &\qquad \frac{1}{t}\|(U_t^{(\epsilon_{n-1})} - 1)(u_{n-1}, v_{n-1})(U_t^{(\epsilon_n)} - 1)(u_n, v_n) \ \Omega\|^2 \end{aligned}$$

for some constant $C_{\underline{u},\underline{v}}$ independent of t. So to prove (6.6) it is enough to show that for any $u, v, p, w \in \mathbf{h}$ and $\epsilon, \epsilon' \in \mathbb{Z}_2$

$$\lim_{t\to 0} \frac{1}{t} \|(U_t^{(\epsilon)} - 1)(u, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\|^2 = 0. \tag{6.7}$$

So let us look at the following

$$\begin{aligned}
&\|(U_t^{(\epsilon)} - 1)(u, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\|^2 \\
&= \langle (U_t^{(\epsilon)} - 1)(u, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega, (U_t^{(\epsilon)} - 1)(u, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\rangle \\
&= \langle (U_t^{(\epsilon')} - 1)(p, w)\ \Omega, [(U_t^{(\epsilon)} - 1)(u, v)]^*(U_t^{(\epsilon)} - 1)(u, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\rangle.
\end{aligned}$$

By part (v) of Lemma 2.1 the above quantity is $\le \|u\|^2\langle (U_t^{(\epsilon')} - 1)(p, w)\ \Omega, [(U_t^{(\epsilon)} - 1)^*(U_t^{(\epsilon)} - 1)](v, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\rangle$. Since by contractivity of the family $U_t^{(\epsilon)}$, $(U_t^{(\epsilon)})^* U_t^{(\epsilon)} \le 1$, we get

$$\begin{aligned}
&\|(U_t^{(\epsilon)} - 1)(u, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\|^2 \\
&\le \|u\|^2\langle (U_t^{(\epsilon')} - 1)(p, w)\ \Omega, [1 - (U_t^{(\epsilon)})^* + 1 - U_t^{(\epsilon)}](v, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\rangle \\
&= -\|u\|^2\langle (U_t^{(\epsilon')} - 1)(p, w)\ \Omega, [U_t^{(1+\epsilon)} - 1](v, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\rangle \\
&\quad - \|u\|^2\langle (U_t^{(\epsilon')} - 1)(p, w)\ \Omega, (U_t^{(\epsilon)} - 1)(v, v)(U_t^{(\epsilon')} - 1)(p, w)\ \Omega\rangle.
\end{aligned}$$

Thus by **Assumption C** we get (6.7) and the proof is complete. □

Lemma 6.7. *For vectors $u, v \in \mathbf{h}$, product vectors $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}$ and $\epsilon \in \mathbb{Z}_2, \underline{\epsilon'} \in \mathbb{Z}_2^n$*

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}\langle (U_t - 1)^{(\epsilon)}(u, v)\ \Omega, (U_t^{(\underline{\epsilon'})} - 1)(\underline{p}, \underline{w})\ \Omega\rangle \\
&\qquad = (-1)^\epsilon \lim_{t\to 0} \frac{1}{t}\langle (U_t - 1)(u, v)\ \Omega, (U_t^{(\underline{\epsilon'})} - 1)(\underline{p}, \underline{w})\ \Omega\rangle.
\end{aligned} \tag{6.8}$$

Proof. For $\epsilon = 0$ there is nothing to prove. To see this for $\epsilon = 1$ consider the following

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}\langle (U_t + U_t^* - 2)(u, v)\Omega, (U_t^{(\underline{\epsilon'})} - 1)(\underline{p}, \underline{w})\ \Omega\rangle \\
&= -\lim_{t\to 0} \frac{1}{t}\langle [(U_t^* - 1)(U_t - 1)](u, v)\Omega, (U_t^{(\underline{\epsilon'})} - 1)(\underline{p}, \underline{w})\ \Omega\rangle \\
&= -\lim_{t\to 0} \frac{1}{t} \sum_{m\ge 1} \langle (U_t - 1)(e_m, v)\Omega, (U_t - 1)(e_m, u)(U_t^{(\underline{\epsilon'})} - 1)(\underline{p}, \underline{w})\ \Omega\rangle.
\end{aligned} \tag{6.9}$$

That this limit vanishes can be seen from the following

$$|\frac{1}{t}\sum_{m\geq 1}\langle (U_t-1)(e_m,v)\Omega,(U_t-1)(e_m,u)(U_t^{(\underline{\epsilon})}-1)(\underline{p},\underline{w})\ \Omega\rangle|^2$$
$$\leq \sum_{m\geq 1}\frac{1}{t}\|(U_t-1)(e_m,v)\Omega\|^2\quad \sum_{m\geq 1}\frac{1}{t}\|(U_t-1)(e_m,u)(U_t^{(\underline{\epsilon})}-1)(\underline{p},\underline{w})\ \Omega\|^2.$$

By Lemma 6.2 (v) and Lemma 2.1 (iv) the above quantity is equal to

$$2Re\langle v,\frac{1-T_t}{t}v\rangle\frac{1}{t}\langle (U_t^{(\underline{\epsilon})}-1)(\underline{p},\underline{w})\ \Omega,[(U_t^*-1)(U_t-1)](u,u)(U_t^{(\underline{\epsilon})}-1)(\underline{p},\underline{w})\ \Omega\rangle$$
$$\leq 2Re\langle v,\frac{1-T_t}{t}v\rangle\frac{1}{t}\langle (U_t^{(\underline{\epsilon})}-1)(\underline{p},\underline{w})\ \Omega,(2-U_t^*-U_t)(u,u)(U_t^{(\underline{\epsilon})}-1)(\underline{p},\underline{w})\ \Omega\rangle$$

Therefore, since T_t is continuous, by **Assumption C**

$$\lim_{t\to 0}\frac{1}{t}\sum_{m\geq 1}\langle (U_t-1)(e_m,v)\Omega,(U_t-1)(e_m,u)(U_t^{(\underline{\epsilon})}-1)(\underline{p},\underline{w})\ \Omega\rangle=0.$$

□

In particular for vectors $u, v, p, w \in \mathbf{h}$ the identity (6.8) gives

$$\lim_{t\to 0}\frac{1}{t}\langle (U_t-1)^{(\epsilon)}(u,v)\ \Omega,(U_t-1)^{\epsilon'}(p,w)\ \Omega\rangle \tag{6.10}$$
$$=(-1)^{\epsilon+\epsilon'}\lim_{t\to 0}\frac{1}{t}\langle (U_t-1)(u,v)\ \Omega,(U_t-1)(p,w)\ \Omega\rangle.$$

§7. Representation of Hilbert Tensor Algebra and Construction of Noise Space

We define a scalar valued map K on $M\times M$ by setting, for $(\underline{u},\underline{v},\underline{\epsilon}),(\underline{p},\underline{w},\underline{\epsilon}')\in M_0$,

$$K\left((\underline{u},\underline{v},\underline{\epsilon}),(\underline{p},\underline{w},\underline{\epsilon}')\right):=\lim_{t\to 0}\frac{1}{t}\langle (U_t^{(\underline{\epsilon})}-1)(\underline{u},\underline{v})\Omega,\ (U_t^{\underline{\epsilon}'}-1)(\underline{p},\underline{w})\Omega\rangle,\ \text{when it exists.}$$

Lemma 7.1. **(i)** *The map K is a well defined positive definite kernel on M.*
(ii) *Up to unitary equivalence there exists a unique separable Hilbert space* $\mathbf{k}$, *an embedding $\eta: M\to\mathbf{k}$ and a $*$-representation π of M, $\pi: M\to\mathcal{B}(\mathbf{k})$ such that*

$$\{\eta(\underline{u},\underline{v},\underline{\epsilon}):(\underline{u},\underline{v},\underline{\epsilon})\in M_0\}\quad \textit{is total in } \mathbf{k}, \tag{7.1}$$

$$\langle\eta(\underline{u},\underline{v},\underline{\epsilon}),\eta(\underline{p},\underline{w},\underline{\epsilon}')\rangle = K\left((\underline{u},\underline{v},\underline{\epsilon}),(\underline{p},\underline{w},\underline{\epsilon}')\right) \tag{7.2}$$

and

$$\pi(\underline{u},\underline{v},\underline{\epsilon})\eta(\underline{p},\underline{w},\underline{\epsilon}') = \eta(\underline{u}\otimes\underline{p},\underline{v}\otimes\underline{w},\underline{\epsilon}\oplus\underline{\epsilon}') - \langle\underline{p},\underline{w}\rangle\eta(\underline{u},\underline{v},\underline{\epsilon}). \tag{7.3}$$

Proof. (i) First note that for any $(\underline{u},\underline{v},\underline{\epsilon})\in M_0$, $\underline{u}=\otimes_{i=1}^n u_i$, $\underline{v}=\otimes_{i=1}^n v_i$, $\underline{\epsilon}=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)$ we can write

$$\begin{aligned}(U_t^{(\underline{\epsilon})}-1)(\underline{u},\underline{v}) &= \prod_{i=1}^n U_t^{(\epsilon_i)}(u_i,v_i) - \prod_{i=1}^n\langle u_i,v_i\rangle \\ &= \sum_{1\le i\le n}(U_t-1)^{(\epsilon_i)}(u_i,v_i)\prod_{j\neq i}\langle u_j,v_j\rangle \\ &+ \sum_{2\le l\le n}\ \sum_{1\le i_1<\ldots<i_m\le n}\prod_{k=1}^l (U_t-1)^{\epsilon_{i_k}}(u_{i_k},v_{i_k})\prod_{j\neq i_k}\langle u_j,v_j\rangle.\end{aligned} \tag{7.4}$$

Now by Lemma 6.6, for elements $(\underline{u},\underline{v},\underline{\epsilon}),(\underline{p},\underline{w},\underline{\epsilon}')\in M_0$, $\underline{\epsilon}\in\mathbb{Z}_2^m$ and $\underline{\epsilon}'\in\mathbb{Z}_2^n$, we have

$$\begin{aligned}K\left((\underline{u},\underline{v},\underline{\epsilon}),(\underline{p},\underline{w},\underline{\epsilon}')\right) &= \lim_{t\to 0}\frac{1}{t}\langle(U_t^{(\underline{\epsilon})}-1)(\underline{u},\underline{v})\Omega,\ (U_t^{\underline{\epsilon}'}-1)(\underline{p},\underline{w})\ \Omega\rangle \\ = \sum_{1\le i\le m,\ 1\le j\le n}&\Bigg(\prod_{k\neq i}\overline{\langle u_k,v_k\rangle}\prod_{l\neq j}\langle p_l,w_l\rangle \\ &\times\lim_{t\to 0}\frac{1}{t}\langle(U_t-1)^{(\epsilon_i)}(u_i,v_i)\ \Omega,(U_t-1)^{\epsilon_j'}(p_j,w_j)\ \Omega\rangle\Bigg).\end{aligned}$$

Hence by (6.10) $K((\underline{u},\underline{v},\underline{\epsilon}),(\underline{p},\underline{w},\underline{\epsilon}'))$ is given by

$$\begin{aligned}\sum_{1\le i\le m,\ 1\le j\le n}(-1)^{\epsilon_i+\epsilon_j'}&\Bigg(\prod_{k\neq i}\overline{\langle u_k,v_k\rangle}\prod_{l\neq j}\langle p_l,w_l\rangle \\ &\times\lim_{t\to 0}\frac{1}{t}\langle(U_t-1)(u_i,v_i)\ \Omega,(U_t-1)(p_j,w_j)\ \Omega\rangle\Bigg).\end{aligned}$$

Since T_t and Z_t are uniformly continuous semigroup existence of the above limit follows from the following

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}\langle(U_t-1)(u,v)\Omega,\ (U_t-1)(p,w)\ \Omega\rangle \\ &= \lim_{t\to 0}\frac{1}{t}\{\langle U_t(u,v)\Omega,\ U_t(p,w)\ \Omega\rangle - \overline{\langle u,v\rangle}\langle p,w\rangle\} \\ &- \lim_{t\to 0}\frac{1}{t}\overline{\langle u,v\rangle}\langle\Omega,\ [(U_t-1)(p,w)]\ \Omega\rangle\end{aligned}$$

$$-\lim_{t\to 0}\frac{1}{t}\overline{\langle\Omega,\ [(U_t-1)(u,v)]\Omega\rangle}\langle p,w\rangle$$
$$=\lim_{t\to 0}\{\langle p,\frac{Z_t-1}{t}(|w><v|)u\rangle-\overline{\langle u,v\rangle}\ \langle p,\frac{T_t-1}{t}w\rangle-\overline{\langle u,\frac{T_t-1}{t}v\rangle}\ \langle p,w\rangle\}$$

Thus K is well defined on M_0. Now extend this to the $*$-algebra M sesquilinearly.

In particular we have

$$K((u,v,0),(p,w,0))=\langle p,\mathcal{L}(|w><v|)u\rangle-\overline{\langle u,v\rangle}\langle p,G\ w\rangle \\ -\overline{\langle u,G\ v\rangle}\langle p,w\rangle \tag{7.5}$$

where $\mathcal{L}$ and G are generators of Z_t and T_t respectively. Positive definiteness of K follows from the fact that, defining $\xi_i(t)=[U_t^{(\underline{\epsilon}_i)}(\underline{u}_i,\underline{v}_i)-\langle\underline{u}_i,\underline{v}_i\rangle]\Omega$,

$$\sum_{i,j=1}^{N}\bar{c}_ic_jK\left((\underline{u}_i,\underline{v}_i,\underline{\epsilon}_i),(\underline{u}_j,\underline{v}_j,\underline{\epsilon}_j)\right)=\lim_{t\to 0}\frac{1}{t}\|\sum_{i=1}^{N}c_i\xi_i(t)\|^2\geq 0.$$

(ii) Kolmogorov's construction (Ref. [16]) to the pair (M,K) gives the Hilbert space $\mathbf{k}$ and embedding η satisfying (7.1). The separability of $\mathbf{k}$ follows from (7.1) and the verifiable fact $\|\eta(\underline{u},\underline{v},\underline{\epsilon})-\eta(\underline{p},\underline{w},\underline{\epsilon})\|_{\mathbf{k}}\to 0$ as $\|\underline{u}-\underline{p}\|$ and $\|\underline{v}-\underline{w}\|\to 0$.

Defining π by (7.3) we show that the map $\pi(\underline{u},\underline{v},\underline{\epsilon})$ extends to a bounded linear operator on $\mathbf{k}$ with $\|\pi(\underline{u},\underline{v},\underline{\epsilon})\|\leq\|\underline{u}\|\ \|\underline{v}\|$. For any $\xi=\sum_{i=1}^{N}c_i\eta(\underline{u}_i,\underline{v}_i,\underline{\epsilon}_i)\in\mathbf{k}$ let us consider

$$\begin{aligned}
&\|\pi(\underline{u},\underline{v},\underline{\epsilon})\xi\|^2\\
&=\sum_{i,j=1}^{N}\bar{c}_ic_j\langle\pi(\underline{u},\underline{v},\underline{\epsilon})\eta(\underline{u}_i,\underline{v}_i,\underline{\epsilon}_i),\pi(\underline{u},\underline{v},\underline{\epsilon})\eta(\underline{u}_j,\underline{v}_j,\underline{\epsilon}_j)\rangle\\
&=\sum_{i,j=1}^{N}\bar{c}_ic_j\langle[\eta(\underline{u}\otimes\underline{u}_i,\underline{v}\otimes\underline{v}_i,\underline{\epsilon}\oplus\underline{\epsilon}_i)-\langle\underline{u}_i^{(\underline{\epsilon}_i)},\underline{v}_i^{(\underline{\epsilon}_i)}\rangle\eta(\underline{u},\underline{v},\underline{\epsilon})],\\
&\qquad[\eta(\underline{u}\otimes\underline{u}_j,\underline{v}\otimes\underline{v}_j,\underline{\epsilon}\oplus\underline{\epsilon}_j)-\langle\underline{u}_j^{(\underline{\epsilon}_j)},\underline{v}_j^{(\underline{\epsilon}_j)}\rangle\eta(\underline{u},\underline{v},\underline{\epsilon})]\rangle\\
&=\lim_{t\to 0}\frac{1}{t}\sum_{i,j=1}^{N}\bar{c}_ic_j\langle U_t^{(\underline{\epsilon})}(\underline{u}^{(\underline{\epsilon})},\underline{v}^{(\underline{\epsilon})})[U_t^{(\underline{\epsilon}_i)}-1]\\
&\qquad\times(\underline{u}_i,\underline{v}_i)\Omega,U_t^{(\underline{\epsilon})}(\underline{u}^{(\underline{\epsilon})},\underline{v}^{(\underline{\epsilon})})[U_t^{(\underline{\epsilon}_j)}-1](\underline{u}_j^{(\underline{\epsilon}_j)},\underline{v}_j^{(\underline{\epsilon}_j)})\Omega\rangle.
\end{aligned}$$

In the above identity we have used the fact that for any $\underline{\epsilon}\in\mathbb{Z}_2^m,\underline{\alpha}\in\mathbb{Z}_2^n$ and product vectors $\underline{p}^{(\underline{\epsilon})},\underline{w}^{(\underline{\epsilon})}\in\mathbf{h}^{(\underline{\epsilon})},\underline{\mathrm{x}}^{(\underline{\alpha})},\ \underline{\mathrm{y}}^{(\underline{\alpha})}\in\mathbf{h}^{(\underline{\alpha})}$

$$[U_t^{\underline{\epsilon}\oplus\underline{\alpha}}-1](\underline{p}^{(\underline{\epsilon})}\otimes\underline{\mathrm{x}}^{(\underline{\alpha})},\underline{w}^{(\underline{\epsilon})}\otimes\underline{\mathrm{y}}^{(\underline{\alpha})})-\langle\underline{\mathrm{x}}^{(\underline{\alpha})},\underline{\mathrm{y}}^{(\underline{\alpha})}\rangle[U_t^{(\underline{\epsilon})}-1](\underline{p}^{(\underline{\epsilon})},\underline{w}^{(\underline{\epsilon})}) \\ =U_t^{(\underline{\epsilon})}(\underline{p}^{(\underline{\epsilon})},\underline{w}^{(\underline{\epsilon})})[U_t^{(\underline{\alpha})}-1](\underline{\mathrm{x}}^{(\underline{\alpha})},\underline{\mathrm{y}}^{(\underline{\alpha})}). \tag{7.6}$$

Now setting $\phi(t) := \sum_{i,=1}^{N} c_i [U_t^{(\epsilon_i)} - 1](\underline{u}_i^{(\epsilon_i)}, \underline{v}_i^{(\epsilon_i)})\Omega \in \mathcal{H}$. we get

$$\|\pi(\underline{u}, \underline{v}, \underline{\epsilon})\xi\|^2 = \lim_{t\to 0} \frac{1}{t} \|U_t^{(\epsilon)}(\underline{u}^{(\epsilon)}, \underline{v}^{(\epsilon)})\phi(t)\|^2.$$

Since $U_t^{(\epsilon)}(\underline{u}^{(\epsilon)}, \underline{v}^{(\epsilon)})$ has its norm bounded by $\|\underline{u}\|^2\ \|\underline{v}\|^2$ we get

$$\begin{aligned}
&\|\pi(\underline{u}, \underline{v}, \underline{\epsilon})\xi\|^2 \le \|\underline{u}\|^2\ \|\underline{v}\|^2 \lim_{t\to 0} \frac{1}{t} \|\phi(t)\|^2 \\
&= \sum_{i,j=1}^{N} \bar{c}_i c_j \lim_{t\to 0} \frac{1}{t} \langle [U_t^{(\epsilon_i)} - 1](\underline{u}_i^{(\epsilon_i)}, \underline{v}_i^{(\epsilon_i)})\Omega, [U_t^{(\epsilon_j)} - 1](\underline{u}_j^{(\epsilon_j)}, \underline{v}_j^{(\epsilon_j)})\Omega\rangle \\
&= \|\underline{u}\|^2\ \|\underline{v}\|^2 \|\xi\|^2
\end{aligned}$$

which proves that $\pi(\underline{u}, \underline{v}, \underline{\epsilon})$ extends to a bounded operator on $\mathbf{k}$ with $\|\pi(\underline{u}, \underline{v}, \underline{\epsilon})\| \le \|\underline{u}\|\ \|\underline{v}\|$.

In order to prove that π is a $*$-representation of the algebra M it is enough to show that for any $\underline{\epsilon} \in \mathbb{Z}_2^m, \underline{\epsilon}' \in \mathbb{Z}_2^n, \underline{\epsilon}'' \in \mathbb{Z}_2^q$ and product vectors $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes m}, \underline{p}', \underline{w}' \in \mathbf{h}^{\otimes n}, \underline{\mathrm{x}}, \underline{\mathrm{y}} \in \mathbf{h}^{\otimes q}$,

$$\pi(\underline{u}, \underline{v}, \underline{\epsilon})\pi(\underline{p}, \underline{w}, \underline{\epsilon}')\eta(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \underline{\epsilon}'') = \pi(\underline{u} \otimes \underline{p}, \underline{v} \otimes \underline{w}, \underline{\epsilon} \oplus \underline{\epsilon}')\eta(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \underline{\epsilon}''), \tag{7.7}$$

$$\langle \pi(\underline{u}, \underline{v}, \underline{\epsilon})\eta(\underline{p}, \underline{w}, \underline{\epsilon}'),\ \eta(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \underline{\epsilon}'')\rangle = \langle \eta(\underline{p}, \underline{w}, \underline{\epsilon}'),\ \pi(\underleftarrow{u}, \underleftarrow{v}, \underline{\epsilon}^*)\eta(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \underline{\epsilon}'')\rangle. \tag{7.8}$$

By the definition of π

$$\begin{aligned}
&\pi(\underline{u}, \underline{v}, \underline{\epsilon})\pi(\underline{p}, \underline{w}, \underline{\epsilon}')\eta(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \underline{\epsilon}'') \\
&= \pi(\underline{u}, \underline{v}, \underline{\epsilon})[\eta(\underline{p} \otimes \underline{\mathrm{x}}, \underline{w} \otimes \underline{\mathrm{y}}, \underline{\epsilon}' \oplus \underline{\epsilon}'') - \langle \underline{\mathrm{x}}, \underline{\mathrm{y}}\rangle \eta(\underline{p}, \underline{w}, \underline{\epsilon}')] \\
&= \eta(\underline{u} \otimes \underline{p} \otimes \underline{\mathrm{x}}, \underline{v} \otimes \underline{w} \otimes \underline{\mathrm{y}}, \underline{\epsilon} \oplus \underline{\epsilon}' \oplus \underline{\epsilon}'') - \langle \underline{p} \otimes \underline{\mathrm{x}}, \underline{w} \otimes \underline{\mathrm{y}}\rangle \eta(\underline{u}, \underline{v}, \underline{\epsilon}) \\
&- \langle \underline{\mathrm{x}}, \underline{\mathrm{y}}\rangle [\eta(\underline{u} \otimes \underline{p}, \underline{v} \otimes \underline{w}, \underline{\epsilon} \oplus \underline{\epsilon}') - \langle \underline{p}, \underline{w}\rangle \eta(\underline{u}, \underline{v}, \underline{\epsilon})] \\
&= \eta(\underline{u} \otimes \underline{p} \otimes \underline{\mathrm{x}}, \underline{v} \otimes \underline{w} \otimes \underline{\mathrm{y}}, \underline{\epsilon} \oplus \underline{\epsilon}' \oplus \underline{\epsilon}'') - \langle \underline{\mathrm{x}}, \underline{\mathrm{y}}\rangle \eta(\underline{u} \otimes \underline{p}, \underline{v} \otimes \underline{w}, \underline{\epsilon} \oplus \underline{\epsilon}')
\end{aligned}$$

and (7.7) follows. To see (7.8) let us look at the left hand side. By (7.6)

$$\begin{aligned}
&\langle \pi(\underline{u}, \underline{v}, \underline{\epsilon})\eta(\underline{p}, \underline{w}, \underline{\epsilon}'),\ \eta(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \underline{\epsilon}'')\rangle \\
&= \lim_{t\to 0} \frac{1}{t} \langle U_t^{(\epsilon)}(\underline{u}, \underline{v})(U_t^{\epsilon'} - 1)(\underline{p}, \underline{w})\ \Omega, (U_t^{\epsilon''} - 1)(\underline{\mathrm{x}}, \underline{\mathrm{y}})\ \Omega\rangle \\
&= \lim_{t\to 0} \frac{1}{t} \langle (U_t^{\epsilon'} - 1)(\underline{p}, \underline{w})\ \Omega, U_t^{\epsilon^*}(\underleftarrow{v}, \underleftarrow{u})(U_t^{\epsilon''} - 1)(\underline{\mathrm{x}}, \underline{\mathrm{y}})\ \Omega\rangle \\
&= \langle \eta(\underline{p}, \underline{w}, \underline{\epsilon}'),\ \pi(\underleftarrow{v}, \underleftarrow{u}, \underline{\epsilon}^*)\eta(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \underline{\epsilon}'')\rangle = RHS.
\end{aligned}$$

Thus

$$\pi(\underline{u},\underline{v},\underline{\epsilon})\pi(\underline{p},\underline{w},\underline{\epsilon}') = \pi(\underline{u}\otimes\underline{p},\underline{v}\otimes\underline{w},\underline{\epsilon}\oplus\underline{\epsilon}')$$
$$\pi(\underline{u},\underline{v},\underline{\epsilon})^* = \pi(\underleftarrow{v},\underleftarrow{u},\underline{\epsilon}^*). \tag{7.9}$$

□

Lemma 7.2. **(i)** *For any $u,v\in\mathbf{h}$, $\eta(u,v,1)=-\eta(u,v,0)$.*

(ii) *For any $(\underline{u},\underline{v},\underline{\epsilon})\in M_0$, $\underline{u}=\otimes_{i=1}^n u_i$, $\underline{v}=\otimes_{i=1}^n v_i$ and $\underline{\epsilon}=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)$*

$$\eta(\underline{u},\underline{v},\underline{\epsilon}) = \sum_{i=1}^n(-1)^{\epsilon_i}\prod_{k\neq i}\langle u_k,v_k\rangle\eta(u_i,v_i,\epsilon_i) \tag{7.10}$$

Proof. (i) Follows from the identity (6.8).
(ii) For any $(\underline{p},\underline{w},\underline{\epsilon}')\in M_0$, by (7.4) and Lemma 6.6, we have

$$\begin{aligned}&\langle\eta(\underline{u},\underline{v},\underline{\epsilon}),\eta(\underline{p},\underline{w},\underline{\epsilon}')\rangle = K\left((\underline{u},\underline{v},\underline{\epsilon}),(\underline{p},\underline{w},\underline{\epsilon}')\right)\\ &=\lim_{t\to0}\frac{1}{t}\langle(U_t^{(\underline{\epsilon})}-1)(\underline{u},\underline{v})\Omega,\ (U_t^{\underline{\epsilon}'}-1)(\underline{p},\underline{w})\ \Omega\rangle\\ &=\sum_{i=1}^n\prod_{k\neq i}\overline{\langle u_k,v_k\rangle}\lim_{t\to0}\frac{1}{t}\langle(U_t-1)^{(\epsilon_i)}(u_i,v_i)\ \Omega,(U_t^{\underline{\epsilon}'}-1)(\underline{p},\underline{w})\ \Omega\rangle\\ &=\sum_{i=1}^n\prod_{k\neq i}\overline{\langle u_k,v_k\rangle}\langle\eta(u_i,v_i,\epsilon_i),\eta(\underline{p},\underline{w},\underline{\epsilon}')\rangle.\end{aligned}$$

Since $\{\eta(\underline{p},\underline{w},\underline{\epsilon}') : (\underline{p},\underline{w},\underline{\epsilon}')\in M_0\}$ is a total subset of $\mathbf{k}$, by the part (i) requirement follows. □

Remark 7.3. Writing $\eta(u,v)$ for the vector $\eta(u,v,0)\in\mathbf{k}$,

$$\overline{Span}\{\eta(u,v) : u,v\in\mathbf{h}\} = \mathbf{k}. \tag{7.11}$$

Remark 7.4. The $*$-representation π of M in $\mathbf{k}$ is trivial

$$\pi(\underline{u},\underline{v},\underline{\epsilon})\eta(\underline{p},\underline{w},\underline{\epsilon}') = \langle\underline{u},\underline{v}\rangle\eta(\underline{p},\underline{w},\underline{\epsilon}') \tag{7.12}$$

Now we fix an ONB $\{E_j : j\geq1\}$ for the separable Hilbert space $\mathbf{k}$. Then we have the following crucial observations.

Lemma 7.5. **(i)** *There exists a unique family $\{L_j : j\geq1\}$ in $\mathcal{B}(\mathbf{h})$ such that $\langle u,L_jv\rangle=\langle E_j,\eta(u,v)\rangle$ and $\sum_{j\geq1}\|L_ju\|^2\leq 2\|G\|\|u\|^2$, $\forall\ u\in\mathbf{h}$, so that $\sum_{j\geq1}L_j^*L_j$ converges strongly.*

(ii) *The family of operators* $\{L_j : j \geq 1\}$ *is linearly independent, i.e.* $\sum_{j\geq 1} c_j L_j = 0$ *for some* $c = (c_j) \in l^2(\mathbb{N})$ *implies* $c_j = 0,\ \forall j$.

(iii) *If we set* $iH := G + \frac{1}{2}\sum_{j\geq 1} L_j^* L_j$ *then* H *is a bounded self-adjoint operator on* $\mathbf{h}$.

Proof. (i) By (7.5), for any $u, v \in \mathbf{h}$

$$\begin{aligned}&\|\eta(u,v)\|^2\\ &= \langle u, \mathcal{L}(|v><v|)u\rangle - \overline{\langle u, v\rangle}\langle u, G\ v\rangle - \overline{\langle u, G\ v\rangle}\langle u, v\rangle\\ &\leq [\|\mathcal{L}\| + 2\|G\|]\ \ \|u\|^2\ \ \|v\|^2.\end{aligned}$$

So for each $j \geq 1$, the map $\eta_j(u,v) := \langle E_j, \eta(u,v)\rangle$, defines a bounded quadratic form on $\mathbf{h}$ and hence by Riesz's representation theorem there exists a unique bounded operator $L_j \in \mathcal{B}(\mathbf{h})$ such that $\langle u, L_j v\rangle = \eta_j(u,v)$. Now consider the following

$$\begin{aligned}&\sum_j \|L_j u\|^2 = \sum_{j,k} |\eta_j(e_k,u)|^2 = \sum_k \|\eta(e_k,u)\|^2\\ &= \sum_k \Big[\langle e_k, \mathcal{L}(|u><u|)e_k\rangle - \overline{\langle e_k, u\rangle}\langle e_k, G\ u\rangle - \overline{\langle e_k, G\ u\rangle}\langle e_k, u\rangle\ \ \Big]\\ &= Tr\mathcal{L}(|u><u|) - \langle u, G\ u\rangle - \overline{\langle u, G\ u\rangle}.\end{aligned}$$

Since Z_t is trace preserving

$$\sum_j \|L_j u\|^2 = -\langle u, G\ u\rangle - \overline{\langle u, G\ u\rangle} \leq 2\|G\|\ \|u\|^2. \tag{7.13}$$

(ii) Let $\sum_{j\geq 1} c_j L_j = 0$ for some $c = (c_j) \in l^2(\mathbb{N})$. Then for any $u, v \in \mathbf{h}$ we have

$$0 = \langle u, \sum_{j\geq 1} c_j L_j v\rangle = \sum_{j\geq 1} c_j \langle u, L_j v\rangle = \langle \sum_{j\geq 1} \overline{c}_j E_j, \eta(u,v)\rangle.$$

Since $\overline{Span}\{\eta(u,v) : u, v \in \mathbf{h}\} = \mathbf{k}$, it follows that $\sum_{j\geq 1} \overline{c}_j E_j = 0 \in \mathbf{k}$ and hence $c_j = 0,\ \forall j$.

(iii) The boundedness of G and (7.13) imply that $\sum_{j\geq 1} L_j^* L_j$ is a bounded self-adjoint operator and hence H is bounded. For any $u \in \mathbf{h}$ by the identity (7.13)

$$\begin{aligned}&\langle u, (2G + \sum_{j\geq 1} L_j^* L_j)u\rangle\\ &= \langle u, 2Gu\rangle + \sum_j \|L_j u\|^2 = \langle u, Gu\rangle - \langle Gu, u\rangle\\ &= -\langle (2G + \sum_{j\geq 1} L_j^* L_j)u, u\rangle\end{aligned}$$

Thus $\langle u, Hu\rangle = \langle Hu, u\rangle$ and by applying the polarization principle to the sesqui-linear form $(u,v) \mapsto \langle u, Hu\rangle$ it proves that H is self-adjoint. □

Lemma 7.6. *The generator $\mathcal{L}$ of the uniformly continuous semigroup Z_t on $\mathcal{B}_1(\mathbf{h})$ satisfies*

$$\mathcal{L}\rho = G\rho + \rho G^* + \sum_{j\geq 1} L_j \rho L_j^*, \quad \forall \rho \in \mathcal{B}_1(\mathbf{h}). \tag{7.14}$$

Proof. By (7.5), for any $u, v, p, w \in \mathbf{h}$ we have

$$\begin{aligned}\langle \eta(u,v), \eta(p,w)\rangle &= \sum_{j\geq 1} \overline{\langle u, L_j v\rangle}\langle p, L_j w\rangle \\ &= \langle p, \mathcal{L}(|w><v|)u\rangle - \overline{\langle u, v\rangle}\langle p, G\ w\rangle - \overline{\langle u, G\ v\rangle}\langle p, w\rangle,\end{aligned}$$

which gives

$$\begin{aligned}&\langle p, \mathcal{L}(|w><v|)\ u\rangle \\ &= \langle p, |Gw><v|\ u\rangle + \langle p, |w><Gv|\ u\rangle + \sum_{j\geq 1}\langle p, |L_j w><L_j v|\ u\rangle \\ &= \langle p, G|w><v|\ u\rangle + \langle p, |w><v|G^*\ u\rangle + \sum_{j\geq 1}\langle p, L_j|w><v|L_j^*\ u\rangle.\end{aligned}$$

Since all the operators involved are bounded (7.14) follows.

□

§8. Associated Hudson-Parthasarathy (HP) Flows

Recall from the previous section that starting from the family of unitary operators $\{U_{s,t}\}$ with **Assumption A, B** and **C** we obtained the noise Hilbert space $\mathbf{k}$ and bounded linear operators $G, L_j : j \geq 1$ on the initial Hilbert space $\mathbf{h}$. Now define a family of operators $\{L^\mu_\nu : \mu, \nu \geq 0\}$ in $\mathcal{B}(\mathbf{h})$ by

$$L^\mu_\nu = \begin{cases} G = iH - \frac{1}{2}\sum_{k\geq 1} L_k^* L_k & \text{for } (\mu,\nu) = (0,0) \\ L_j & \text{for } (\mu,\nu) = (j,0) \\ -L_k^* & \text{for } (\mu,\nu) = (0,k) \\ 0 & \text{for } (\mu,\nu) = (j,k). \end{cases} \tag{8.1}$$

Note that the indices μ, ν vary over non negative integers while j, k vary over non zero positive integers.

Let us consider the HP type quantum stochastic differential equation in $\mathbf{h} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$:

$$V_{s,t} = 1_{\mathbf{h}\otimes\Gamma} + \sum_{\mu,\nu\geq 0} \int_s^t V_{s,r} L_\nu^\mu \Lambda_\mu^\nu(dr) \tag{8.2}$$

with bounded operator coefficients L_ν^μ given by (8.1). By Theorem 2.2, there exists a unique unitary solution $\{V_{s,t}\}$ of the above HP equation. We shall write $V_t := V_{0,t}$ for simplicity. The family $\{V_{s,t}^*\}$ satisfies:

$$dV_{s,t}^* = \sum_{\mu,\nu\geq 0} (L_\mu^\nu)^* V_{s,t}^* \Lambda_\mu^\nu(dt), \ V_{s,s} = 1_{\mathbf{h}\otimes\Gamma} \tag{8.3}$$

and for any $u, v \in \mathbf{h}, V_{s,t}(u,v)$ and $V_{s,t}(u,v)^*$ satisfy the following qsde on Γ :

$$dV_{s,t}(u,v) = \sum_{\mu,\nu\geq 0} V_{s,t}(u, L_\nu^\mu v)\Lambda_\mu^\nu(dt), \ V_{s,s}(u,v) = \langle u, v\rangle 1_\Gamma. \tag{8.4}$$

$$dV_{s,t}^*(u,v) = \sum_{\mu,\nu\geq 0} V_{s,t}^*(L_\mu^\nu u, v)\Lambda_\mu^\nu(dt), \ V_{s,s}^*(u,v)] = \langle u, v\rangle 1_\Gamma. \tag{8.5}$$

As for the family of unitary operators $\{U_{s,t}\}$ on $\mathbf{h}\otimes\mathcal{H}$, for $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in \mathbb{Z}_2^n$ we define $V_{s,t}^{(\underline{\epsilon})} \in \mathcal{B}(\mathbf{h}^{\otimes n} \otimes \Gamma)$ by setting $V_{s,t}^{(\epsilon)} \in \mathcal{B}(\mathbf{h} \otimes \Gamma)$ by

$$\begin{aligned} V_{s,t}^{(\epsilon)} &= V_{s,t} \text{ for } \epsilon = 0 \\ &= V_{s,t}^* \text{ for } \epsilon = 1. \end{aligned}$$

We shall write $V_{s,t}^{(n)}$ for $V_{s,t}^{(\underline{\epsilon})}, \ \underline{\epsilon} = (0, 0, \cdots, 0) \in \mathbb{Z}_2^n$.

Lemma 8.1. *The family of unitary operators $\{V_{s,t}\}$ satisfies*

(i) *For any $0 \leq r \leq s \leq t < \infty, V_{r,t} = V_{r,s}V_{s,t}$.*

(ii) *For $[q,r) \cap [s,t) = \varnothing, V_{q,r}(u,v)$ commute with $V_{s,t}(p,w)$ and $V_{s,t}(p,w)^*$ for every $u, v, p, w \in \mathbf{h}$.*

(iii) *For any $0 \leq s \leq t < \infty$,*
$\langle \boldsymbol{e}(0), V_{s,t}(u,v)\boldsymbol{e}(0)\rangle = \langle \boldsymbol{e}(0), V_{t-s}(u,v)\boldsymbol{e}(0)\rangle = \langle u, T_{t-s}v\rangle, \ \forall u, v \in \mathbf{h}$.

Proof. (i) For fixed $0 \leq r \leq s \leq t < \infty$, we set $W_{r,t} = V_{r,s}V_{s,t}$ and $W_{r,s} = V_{r,s}$. Then by (8.2) we have

$$
\begin{aligned}
W_{r,t} &= V_{r,s} + \sum_{\mu,\nu \geq 0} \int_s^t V_{r,s} V_{s,q} L_\nu^\mu \Lambda_\mu^\nu(dq) \\
&= W_{r,s} + \sum_{\mu,\nu \geq 0} \int_s^t W_{r,q} L_\nu^\mu \Lambda_\mu^\nu(dq).
\end{aligned}
$$

Thus the family of unitary operators $\{W_{r,t}\}$ also satisfies the HP equation (8.2) and, hence by uniqueness of the solution of this qsde, $W_{r,t} = V_{r,t}, \forall t \geq s$ and the result follows.

(ii) For any $0 \leq s \leq t < \infty$ $V_{s,t} \in \mathcal{B}(\mathbf{h} \otimes \Gamma_{[s,t]})$. So for $p, w \in \mathbf{h},\ V_{s,t}(p,w) \in \mathcal{B}(\Gamma_{[s,t]})$ and the statement follows.

(iii) Let us set a family of contraction operators $\{\widetilde{S}_{s,t}\}$ on $\mathbf{h}$ by

$$
\langle u, \widetilde{S}_{s,t} v \rangle = \langle u \otimes \mathbf{e}(0), V_{s,t} v \otimes \mathbf{e}(0) \rangle,\ \forall u, v \in \mathbf{h}.
$$

Then for fixed $s \geq 0$, this family $\{\widetilde{S}_{s,t}\}$ satisfies the following differential equation

$$
\frac{d\widetilde{S}_{s,t}}{dt} = \widetilde{S}_{s,t} G
$$

where $G \ \ (= L_0^0)$ is the generator of the uniformly continuous semigroup $\{T_t\}$ so $\widetilde{S}_{s,t} = T_{t-s}$ and this proves the claim.

□

Consider the family of maps $\widetilde{Z}_{s,t}$ defined by

$$
\widetilde{Z}_{s,t} \rho = Tr_{\mathcal{H}}[V_{s,t}(\rho \otimes |\mathbf{e}(0) >< \mathbf{e}(0)|) V_{s,t}^*],\ \forall \rho \in \mathcal{B}_1(\mathbf{h}).
$$

As for Z_t, it can be easily seen that $\widetilde{Z}_{s,t}$ is a contractive family of maps on $\mathcal{B}_1(\mathbf{h})$ and in particular, for any $u, v, p, w \in \mathbf{h}$

$$
\langle p, \widetilde{Z}_{s,t}(|w >< v|)\ u \rangle = \langle V_{s,t}(u,v)\mathbf{e}(0), V_{s,t}(p,w)\mathbf{e}(0) \rangle.
$$

Lemma 8.2. *The family $\widetilde{Z}_t := \widetilde{Z}_{0,t}$ is a uniformly continuous semigroup of contractions on $\mathcal{B}_1(\mathbf{h})$ and $\widetilde{Z}_{s,t} = \widetilde{Z}_{t-s} = Z_{t-s}$.*

Proof. By (8.4) and Ito's formula

$$
\begin{aligned}
&\langle p, [\widetilde{Z}_{s,t} - 1](|w><v|)\ u\rangle \\
&= \langle V_{s,t}(u,v)\mathbf{e}(0), V_{s,t}(p,w)\mathbf{e}(0)\rangle - \overline{\langle u,v\rangle}\langle p,w\rangle \\
&= \int_s^t \langle V_{s,\tau}(u,v)\mathbf{e}(0), V_{s,\tau}(p,Gw)\mathbf{e}(0)\rangle d\tau + \int_s^t \langle V_{s,\tau}(u,Gv)\mathbf{e}(0), V_{s,\tau}(p,w)\mathbf{e}(0)\rangle d\tau \\
&+ \int_s^t \langle V_{s,\tau}(u,L_j v)\mathbf{e}(0), V_{s,\tau}(p,L_j w)\mathbf{e}(0)\rangle d\tau \\
&= \int_s^t \langle p, \widetilde{Z}_{s,\tau}(|Gw><v|)\ u\rangle d\tau + \int_s^t \langle p, \widetilde{Z}_{s,\tau}(|w><Gv|)\ u\rangle d\tau \\
&+ \sum_{j\geq 1}\int_s^t \langle p, \widetilde{Z}_{s,\tau}(|L_j w><L_j v|)\ u\rangle d\tau \\
&= \int_s^t \langle p, \widetilde{Z}_{s,\tau}\mathcal{L}(|w><v|)\ u\rangle d\tau,
\end{aligned}
$$

where $\mathcal{L}$ is the generator of the uniformly continuous semigroup Z_t. Since the maps $\mathcal{L}$ and $\widetilde{Z}_{a,b} : 0 \leq a \leq b$ are bounded, for fixed $s \geq 0$, $\widetilde{Z}_{s,t}$ satisfies the differential equation

$$
\widetilde{Z}_{s,t}(\rho) = \rho + \int_s^t \widetilde{Z}_{s,\tau}\mathcal{L}(\rho)d\tau, \quad \rho \in \mathcal{B}_1(\mathbf{h}).
$$

Hence $\widetilde{Z}_t$ is a uniformly continuous semigroup on $\mathcal{B}_1(\mathbf{h})$ and $\widetilde{Z}_{s,t} = \widetilde{Z}_{t-s} = Z_{t-s}$. □

§9. Minimality of HP Flows

In this section we shall show the minimality of the HP flow $V_{s,t}$ discussed above. We prove that the subset $\mathcal{S}' := \{\zeta = V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\mathbf{e}(0) := V_{s_1,t_1}(u_1,v_1)\cdots V_{s_n,t_n}(u_n,v_n)\mathbf{e}(0) : \underline{s} = (s_1,s_2,\cdots,s_n), \underline{t} = (t_1,t_2,\cdots,t_n) :\ 0 \leq s_1 \leq t_1 \leq s_2 \leq \ldots \leq s_n \leq t_n < \infty, n \geq 1, \underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}\}$ is total in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+,\mathbf{k}))$.

We note that for any $0 \leq s < t \leq \tau < \infty, u,v \in \mathbf{h}$ by the HP equation (8.2)

$$
\begin{aligned}
(9.1) \qquad &\frac{1}{t-s}[V_{s,t} - 1](u,v)\mathbf{e}(0) \\
&= \frac{1}{t-s}\{\sum_{j\geq 1}\int_s^t V_{s,\lambda}(u,L_j v)a_j^\dagger(d\lambda) + \int_s^t V_{s,\lambda}(u,Gv)d\lambda\}\mathbf{e}(0) \\
&= \gamma(s,t,u,v) + \langle u,Gv\rangle\ \ \mathbf{e}(0) + \zeta(s,t,u,v) + \varsigma(s,t,u,v),
\end{aligned}
$$

where

$\gamma(s,t,u,v) := \frac{1}{t-s}\sum_{j\geq 1}\langle u, L_j v\rangle a_j^\dagger([s,t])\ \ \mathbf{e}(0)$

$\zeta(s,t,u,v) := \frac{1}{t-s}\sum_{j\geq 1}\int_s^t (V_{s,\lambda}-1)(u,L_jv)a_j^\dagger(d\lambda)\ \ \mathbf{e}(0)$

$\varsigma(s,t,u,v) := \frac{1}{t-s}\int_s^t (V_{s,\lambda}-1)(u,Gv)d\lambda\ \ \mathbf{e}(0).$

Note that any $\xi \in \Gamma$ can be written as $\xi = \xi^{(0)}\mathbf{e}(0) \oplus \xi^{(1)} \oplus \cdots$, where $\xi^{(n)}$ is in the n-fold symmetric tensor product $L^2(\mathbb{R}_+, \mathbf{k})^{\otimes n} \equiv L^2(\Sigma_n) \otimes \mathbf{k}^{\otimes n}$, where Σ_n is the n-simplex $\{\underline{t} = (t_1, t_2, \cdots, t_n) : 0 \leq t_1 < t_2 < \ldots < t_n < \infty\}$.

Lemma 9.1. *Let $\tau \geq 0$. For any $v \in \mathbf{h}, 0 \leq s \leq t \leq \tau$, define constants $C_\tau = 2e^\tau$ and $C_{\tau,v} = C_\tau\{\sum_{j\geq 1}\|L_jv\|^2 + \tau\|G\ v\|^2\}$. Then*

(i)

$$\|(V_{s,t}-1)v\boldsymbol{e}(0)\|^2 \leq C_{\tau,v}(t-s). \tag{9.2}$$

(ii) *For any $u \in \mathbf{h}$*

$$\begin{aligned}
&\|\sum_{j\geq 1}\int_s^t V_{s,\lambda}(u,L_jv)a_j^\dagger(d\lambda)\boldsymbol{e}(0)\|^2\\
&\leq C_\tau\|u\|^2\sum_{j\geq 1}\int_s^t \|V_{s,\lambda}L_jv\otimes\boldsymbol{e}(0)\|^2\ d\lambda\\
&\leq C_\tau(t-s)\|u\|^2\sum_{j\geq 1}\|L_jv\|^2.
\end{aligned}$$

Proof. (i) By estimates of quantum stochastic integration (Proposition 27.1, [16])

$$\begin{aligned}
&\|(V_{s,t}-1)v\mathbf{e}(0)\|^2\\
&= \|\sum_{j\geq 1}\int_s^t V_{s,\lambda}L_ja_j^\dagger(d\lambda)\ \ v\mathbf{e}(0) + \int_s^t V_{s,\lambda}Gd\lambda\ \ v\mathbf{e}(0)\|^2\\
&\leq C_\tau\int_s^t\{\sum_{j\geq 1}\|L_jv\|^2 + \|Gv\|^2\}d\lambda\\
&= C_{\tau,v}(t-s).
\end{aligned}$$

(ii) For any ϕ in the Fock space $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$,

$$\begin{aligned}
&\langle \phi, \sum_{j\geq 1} \int_s^t V_{s,\lambda}(u, L_j v) a_j^\dagger(d\lambda)\mathbf{e}(0)\rangle|^2 \\
&= |\langle u\otimes\phi, \{\sum_{j\geq 1}\int_s^t V_{s,\lambda} L_j a_j^\dagger(d\lambda)\} v\mathbf{e}(0)\rangle|^2 \\
&\leq \|u\otimes\phi\|^2 \|\{\sum_{j\geq 1}\int_s^t V_{s,\lambda} L_j a_j^\dagger(d\lambda)\} v\mathbf{e}(0)\|^2.
\end{aligned}$$

By estimates of quantum stochastic integration the above quantity is

$$\leq C_\tau \|u\otimes\phi\|^2 \sum_{j\geq 1}\int_s^t \|V_{s,\lambda} L_j v\mathbf{e}(0)\|^2 \; d\lambda.$$

Since ϕ is arbitrary and the $V_{s,\lambda}$'s are contractive the statement follows. □

Lemma 9.2. *Let $\tau \geq 0$. For any $u, v \in \mathbf{h}, 0 \leq s \leq t \leq \tau$*

(i) $\|(V_{s,t} - 1)(u, v) \;\; \boldsymbol{e}(0)\|^2 \leq 2C_{\tau,v}\|u\|^2(t - s)$.

(ii) $\sup\{\|\zeta(s,t,u,v)\|^2 : 0 \leq s \leq t \leq \tau\} < \infty$ *and*
$\|\varsigma(s,t,u,v)\| \leq \|u\|\sqrt{2C_{\tau,Gv}(t-s)}, \;\forall\; 0 \leq s < t \leq \tau$.

(iii) *For any $\xi \in \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$,* $\lim_{s\to t}\langle \xi, \zeta(s,t,u,v)\rangle = 0$ *and*

$$\lim_{s\to t}\langle \xi, \gamma(s,t,u,v)\rangle = \sum_{j\geq 1}\langle u, L_j v\rangle \overline{\xi_j^{(1)}}(t) = \langle \xi^{(1)}(t), \eta(u,v)\rangle, \quad a.e. \;\; t \geq 0.$$

Proof. (i) By identity (9.1) and Lemma 9.1 (ii) we have

$$\begin{aligned}
&\|(V_{s,t} - 1)(u, v) \;\; \mathbf{e}(0)\|^2 \\
&= \|\sum_{j\geq 1}\int_s^t V_{s,\alpha}(u, L_j v) a_j^\dagger(d\alpha) \; \mathbf{e}(0) + \int_s^t V_{s,\alpha}(u, Gv) \; \mathbf{e}(0) d\alpha\|^2 \\
&\leq 2\|\sum_{j\geq 1}\int_s^t V_{s,\alpha}(u, L_j v) a_j^\dagger(d\alpha) \;\; \mathbf{e}(0)\|^2 + [\int_s^t \|V_{s,\alpha}(u, Gv) \;\; \mathbf{e}(0)\| d\alpha]^2 \\
&\leq 2\|u\|^2[C_\tau(t-s)\sum_{j\geq 1}\|L_j v\|^2 + [(t-s)\|G\, v\|]^2] \\
&\leq 2C_{\tau,v}\|u\|^2(t-s).
\end{aligned}$$

(ii) To prove the first statement, as in the Lemma 9.1 (ii) we consider

$$\|\zeta(s,t,u,v)\|^2 = \frac{1}{(t-s)^2}\|\sum_{j\geq 1}\int_s^t (V_{s,\lambda}-1)(u,L_jv)a_j^\dagger(d\lambda)\ \ \mathbf{e}(0)\|^2$$
$$\leq \frac{\|u\|^2}{(t-s)^2}\|\sum_{j\geq 1}\int_s^t \|(V_{s,\lambda}-1)L_jv\ \ \mathbf{e}(0)\|^2 d\lambda.$$

Now by Lemma 9.1 (i), the above quantity is

$$\leq \frac{C_\tau\|u\|^2}{(t-s)^2}\sum_{j\geq 1} C_\tau(t-s)^2\{\sum_{i\geq 1}\|L_iL_jv\|^2+\tau\|G\ L_j\ v\|]^2\}$$
$$\leq C_\tau^2\|u\|^2\{\sum_{j\geq 1}\sum_{i\geq 1}\|L_iL_jv\|^2+\tau\sum_{j\geq 1}\|G\ L_j\ v\|^2\}.$$

Since $\sum_{j\geq 1}\|L_j\ v\|^2 = -2Re\langle v, Gv\rangle$, the above quantity is bounded and is independent of s,t.

To prove the second statement consider the following,

$$\|\varsigma(s,t,u,v)\| = \frac{1}{(t-s)}\|\int_s^t (V_{s,\lambda}-1)(u,Gv)d\lambda\ \ \mathbf{e}(0)\|$$
$$\leq \frac{1}{(t-s)}\int_s^t \|(V_{s,\lambda}-1)(u,Gv)\ \ \mathbf{e}(0)\|d\lambda.$$

By (i) the estimate follows.
(iii) To prove the first statement, let us consider the following. For any $f \in L^2(\mathbb{R}_+,\mathbf{k})$,

$$\langle \mathbf{e}(f), \zeta(s,t,u,v)\rangle = \langle \mathbf{e}(f), \frac{1}{t-s}\sum_{j\geq 1}\int_s^t (V_{s,\lambda}-1)(u,L_jv)a_j^\dagger(d\lambda)\ \ \mathbf{e}(0)\rangle$$
$$= \frac{1}{t-s}\sum_{j\geq 1}\int_s^t \overline{f_j(\lambda)}\langle \mathbf{e}(f), (V_{s,\lambda}-1)(u,L_jv)\ \ \mathbf{e}(0)\rangle d\lambda$$
$$= \frac{1}{t-s}\int_s^t G(s,\lambda)d\lambda,$$

where $G(s,\lambda) = \sum_{j\geq 1}\overline{f_j(\lambda)}\langle \mathbf{e}(f), (V_{s,\lambda}-1)(u,L_jv)\ \ \mathbf{e}(0)\rangle$. Note that the complex valued function $G(s,\lambda)$ is uniformly continuous in both the variables s,λ on $[0,\tau]$ and $G(t,t)=0$. So we get

$$\lim_{s\to t}\langle \mathbf{e}(f), \zeta(s,t,u,v)\rangle = 0.$$

Since $\zeta(s,t,u,v)$ is uniformly bounded in s,t

$$\lim_{s\to t}\langle \xi, \zeta(s,t,u,v)\rangle = 0, \forall \xi \in \Gamma.$$

To prove the second statement, we consider

$$\langle \xi, \gamma(s,t,u,v)\rangle = \frac{1}{t-s}\sum_{j\geq 1}\langle u, L_j v\rangle \int_s^t \overline{\xi_j^{(1)}}(\lambda)d\lambda. \tag{9.3}$$

Since

$$|\sum_{j\geq 1}\langle u, L_j v\rangle \overline{\xi_j^{(1)}}(t)|^2 \leq \|u\|^2 \sum_{j\geq 1}\|L_j v\|^2 \sum_{j\geq 1}|\xi_j^{(1)}(t)|^2 \leq C\|v\|^2\|\xi^{(1)}(t)\|^2,$$

the function $\sum_{j\geq 1}\langle u, L_j v\rangle \overline{\xi_j^{(1)}}(\cdot)$ is in L^2 and hence locally integrable. Thus we get

$$\lim_{s\to t}\langle \xi, \gamma(s,t,u,v)\rangle = \sum_{j\geq 1}\langle u, L_j v\rangle \overline{\xi_j^{(1)}}(t) \quad \text{a.e. } t\geq 0.$$

□

Lemma 9.3. *For $n\geq 1$, $\underline{t}\in\Sigma_n$ and $u_k, v_k \in \mathbf{h} : k = 1,2,\cdots,n, \xi \in \Gamma(L^2(\mathbb{R}_+,\mathbf{k}))$ and disjoint intervals $[s_k, t_k)$,*

(i) $\lim_{\underline{s}\to\underline{t}}\langle \xi, \prod_{k=1}^n M(s_k,t_k,u_k,v_k)\ \ \boldsymbol{e}(0)\rangle = 0,$
where $M(s_k,t_k,u_k,v_k) = \frac{(V_{s_k,t_k}-1)}{t_k-s_k}(u_k,v_k) - \langle u_k, G\ v_k\rangle - \gamma(s_k,t_k,u_k,v_k)$ and $\lim_{\underline{s}\to\underline{t}}$ means s_k tends to t_k for each k.

(ii) $\lim_{\underline{s}\to\underline{t}}\langle \xi, \otimes_{k=1}^n \gamma(s_k,t_k,u_k,v_k)\rangle = \langle \xi^{(n)}(t_1,t_2,\cdots,t_n), \eta(u_1,v_1)\otimes\cdots\otimes\eta(u_n, v_n)\rangle$.

Proof. (i) First note that $M(s,t,u,v)\mathbf{e}(0) = \zeta(s,t,u,v) + \varsigma(s,t,u,v)$. So by the above observations $\{M(s,t,u,v)\mathbf{e}(0)\}$ is uniformly bounded in s,t and $\lim_{s\to t}\langle \mathbf{e}(f), M(s,t,u,v)\mathbf{e}(0)\rangle = 0, \forall f\in L^2(\mathbb{R}_+,\mathbf{k})$. Since the intervals $[s_k,t_k)$'s are disjoint for different k's,

$$\langle \mathbf{e}(f), \prod_{k=1}^n M(s_k,t_k,u_k,v_k)\ \ \mathbf{e}(0)\rangle = \prod_{k=1}^n \langle \mathbf{e}(f_{[s_k,t_k)}), M(s_k,t_k,u_k,v_k)\ \ \mathbf{e}(0)\rangle$$

and thus

$$\lim_{\underline{s}\to\underline{t}}\langle \mathbf{e}(f), \prod_{k=1}^n M(s_k,t_k,u_k,v_k)\ \ \mathbf{e}(0)\rangle = 0.$$

By Lemma 9.2, the vector $\prod_{k=1}^{n} M(s_k, t_k, u_k, v_k)\ \ \mathbf{e}(0)$ is uniformly bounded in s_k, t_k and the convergence can be extended to Fock Space.

(ii) It can be proved similarly as part (iii) of the previous Lemma.

□

Lemma 9.4. *Let $\xi \in \Gamma$ be such that*

$$\langle \xi, \zeta \rangle = 0, \ \forall \zeta \in \mathcal{S}', \tag{9.4}$$

Then

(i) $\xi^{(0)} = 0$.

(ii) $\xi^{(1)}(t) = 0$, *for a.e.* $t \in [0, \tau]$.

(iii) *For any* $n \geq 0$, $\xi^{(n)}(\underline{t}) = 0$, *for a.e.* $\underline{t} \in \Sigma_n : t_i \leq \tau$.

(iv) *The set* $\mathcal{S}'$ *is total in the Fock space* Γ.

Proof. (i) For any $s \geq 0$, $V_{s,s} = 1_{\mathbf{h} \otimes \Gamma}$ so in particular (9.4) gives, for any $u, v \in \mathbf{h}$

$$0 = \langle \xi, V_{s,s}(u,v)\mathbf{e}(0) \rangle = \langle u, v \rangle \overline{\xi^{(0)}}$$

and hence $\xi^{(0)} = 0$.

(ii) By (9.4), $\langle \xi, [V_{s,t} - 1](u,v)\mathbf{e}(0) \rangle = 0$ for any $0 \leq s < t \leq \tau < \infty, u, v \in \mathbf{h}$. By HP equation (8.2) and Lemma 9.1 we have

$$\begin{aligned} 0 &= \lim_{s \to t} \frac{1}{t-s} \langle \xi, [V_{s,t} - 1](u,v)\mathbf{e}(0) \rangle \\ &= \sum_{j \geq 1} \langle u, L_j v \rangle \overline{\xi_j^{(1)}(t)} = \sum_{j \geq 1} \eta_j(u,v) \overline{\xi_j^{(1)}(t)}. \end{aligned}$$

So $\langle \xi^{(1)}(t), \eta(u,v) \rangle = 0, \forall u, v \in \mathbf{h}$. Since $\{\eta(u,v) : u, v \in \mathbf{h}\}$ is total in $\mathbf{k}$ it follows that $\xi^{(1)}(t) = 0$ for $0 \leq t \leq \tau$.

(iii) We prove this by induction. The result is already proved for $n = 0, 1$. For $n \geq 2$, assume as induction hypothesis that for all $m \leq n-1$, $\xi^{(m)}(\underline{t}) = 0$, for a.e. $\underline{t} \in \Sigma_m : t_i \leq \tau, i = 1, 2, \cdots, m$. We now show that $\xi^{(n)}(\underline{t}) = 0$, for a.e. $\underline{t} \in \Sigma_n : t_i \leq \tau$.

Let $0 \leq s_1 < t_1 \leq s_2 < t_2 < \ldots < s_n < t_n \leq \tau$ and $u_i, v_i \in \mathbf{h} : i = 1, 2 \cdots, n$. By (9.4) and part (i) we have

$$\langle \xi, \prod_{k=1}^{n} \frac{(V_{s_k,t_k} - 1)}{t_k - s_k}(u_k, v_k) \;\; \mathbf{e}(0)\rangle = 0.$$

Thus

$$\begin{aligned} (9.5) \qquad 0 &= \lim_{\underline{s} \to \underline{t}} \langle \xi, \prod_{k=1}^{n} \frac{(V_{s_k,t_k} - 1)}{t_k - s_k}(u_k, v_k) \;\; \mathbf{e}(0)\rangle \\ &= \lim_{\underline{s} \to \underline{t}} \langle \xi, \prod_{k=1}^{n} \{M(s_k, t_k, u_k, v_k) + \langle u_k, G\, v_k\rangle + \gamma(s_k, t_k, u_k, v_k)\} \;\; \mathbf{e}(0)\rangle. \end{aligned}$$

Let P, Q, R and P', R' be two sets of disjoint partitions of $\{1, 2, \cdots, n\}$ such that Q and R are non empty. We write $|S|$ for the cardinality of set S. Then by Lemma 9.3 (ii) the right hand side of (9.5) is equal to

$$\begin{aligned} &\sum_{P',R'} \langle \xi^{(|R'|)}(t_{r'_1}, \cdots, t_{r'_{|R'|}}), \otimes_{k \in R'} \eta(u_k, v_k)\rangle \prod_{k \in P'} \langle u_k, G\, v_k\rangle \\ &+ \lim_{\underline{s} \to \underline{t}} \sum_{P,Q,R} \langle \xi, \prod_{k \in P} \langle u_k, G\, v_k\rangle \prod_{k \in Q} \{M(s_k, t_k, u_k, v_k)\} \prod_{k \in R} \{\gamma(s_k, t_k, u_k, v_k)\} \;\; \mathbf{e}(0)\rangle. \end{aligned}$$

Thus by the induction hypothesis,

(9.6)
$$\begin{aligned} 0 &= \langle \xi^{(n)}(t_1, t_2, \cdots, t_n), \eta(u_1, v_1) \otimes \cdots \otimes \eta(u_n, v_n)\rangle \\ &+ \lim_{\underline{s} \to \underline{t}} \sum_{P,Q,R} \langle \xi, \prod_{k \in P} \langle u_k, G\, v_k\rangle \prod_{k \in Q} \{M(s_k, t_k, u_k, v_k)\} \prod_{k \in R} \{\gamma(s_k, t_k, u_k, v_k)\} \;\; \mathbf{e}(0)\rangle. \end{aligned}$$

We claim that the second term in (9.6) vanishes. To prove this claim, it is enough to show that for any two non empty disjoint subsets $Q \equiv \{q_1, q_2, \cdots, q_{|Q|}\}, R \equiv \{r_1, r_2, \cdots, r_{|R|}\}$ of $\{1, 2, \cdots, n\}$,

$$(9.7) \qquad \lim_{\underline{s} \to \underline{t}} \langle \xi, \prod_{q \in Q} \{M(s_q, t_q, u_q, v_q)\} \prod_{r \in R} \{\gamma(s_r, t_r, u_r, v_r)\} \;\; \mathbf{e}(0)\rangle = 0.$$

Writing ψ for the vector $\prod_{q\in Q}\{M(s_q,t_q,u_q,v_q)\}\mathbf{e}(0)$, we have

$$
\begin{aligned}
(9.8)\qquad &\langle \xi, \prod_{q\in Q}\{M(s_q,t_q,u_q,v_q)\}\prod_{r\in R}\{\gamma(s_r,t_r,u_r,v_r)\}\ \ \mathbf{e}(0)\rangle\\
&= \langle \xi, \psi\otimes \otimes_{r\in R}\frac{1_{[s_r,t_r]}\eta(u_r,v_r)}{t_r-s_r}\rangle\\
&= \langle \xi, \psi\otimes \otimes_{r\in R}\frac{1_{[s_r,t_r]}\eta(u_r,v_r)}{t_r-s_r}\rangle\\
&= \sum_{l\geq |R|}\langle \xi^{(l)}, \psi^{(l-|R|)}\otimes \otimes_{r\in R}\frac{1_{[s_r,t_r]}\eta(u_r,v_r)}{t_r-s_r}\rangle\\
&= \langle \sum_{l\geq |R|}\langle\langle \xi^{(l)}, \psi^{(l-|R|)}\rangle\rangle, \otimes_{r\in R}\frac{1_{[s_r,t_r]}\eta(u_r,v_r)}{t_r-s_r}\rangle.
\end{aligned}
$$

Here $\langle\langle \xi^{(l)}, \psi^{(l-|R|)}\rangle\rangle \in L^2(\mathbf{R}_+,\mathbf{k})^{\otimes |R|}$ is defined as in (2.1) by

$$
\begin{aligned}
(9.9)\qquad &\langle\ \langle\langle \xi^{(l)}, \psi^{(l-|R|)}\rangle\rangle, \rho^{(|R|)}\ \ \rangle = \langle \xi^{(l)}, \psi^{(l-|R|)}\otimes \rho^{(|R|)}\rangle\\
&= \int_{\Sigma_l}\langle \xi^{(l)}(x_1,x_2,\cdots,x_l), \psi^{(l-|R|)}(x_1,x_2,\cdots,x_{l-|R|})\\
&\quad \otimes \rho^{(|R|)}(x_{l-|R|+1},\cdots,x_l)\rangle_{\mathbf{k}^{\otimes l}}\ \ dx
\end{aligned}
$$

for any $\rho^{(|R|)} \in L^2(\mathbf{R}_+,\mathbf{k})^{\otimes |R|}$.

By Lemma 9.3 (i),

$$
(9.10)\qquad \lim_{s_q\to t_q}\langle \xi, \prod_{q\in Q}\{M(s_q,t_q,u_q,v_q)\}\prod_{r\in R}\{\gamma(s_r,t_r,u_r,v_r)\}\ \ \mathbf{e}(0)\rangle = 0.
$$

However, we need to prove (9.7) where the limit $\underline{s}\to\underline{t}$ has to be in arbitrary order. On the other hand, by (9.8) and (9.9) we get

$$
\begin{aligned}
(9.11)\qquad &\lim_{s_q\to t_q}\lim_{s_r\to t_r}\langle \xi, \prod_{q\in Q}\{M(s_q,t_q,u_q,v_q)\}\prod_{r\in R}\{\gamma(s_r,t_r,u_r,v_r)\}\ \ \mathbf{e}(0)\rangle\\
&= \lim_{s_q\to t_q}\lim_{s_r\to t_r}\langle \sum_{l\geq |R|}\langle\langle \xi^{(l)}, \psi^{(l-|R|)}\rangle\rangle, \otimes_{r\in R}\frac{1_{[s_r,t_r]}\eta(u_r,v_r)}{t_r-s_r}\rangle\\
&= \lim_{s_q\to t_q}\lim_{s_r\to t_r}\langle \int_{\Sigma_{|R|}}\langle [\sum_{l\geq |R|}\langle\langle \xi^{(l)}, \psi^{(l-|R|)}\rangle\rangle](x_1,x_2,\cdots,x_{|R|}),\\
&\qquad\qquad \otimes_{r\in R}\ \frac{1_{[s_r,t_r]}(x_r)\ \ \eta(u_r,v_r)}{t_r-s_r}\rangle dx\\
&= \lim_{s_q\to t_q}\langle \sum_{l\geq |R|}\langle\langle \xi^{(l)}, \psi^{(l-|R|)}\rangle\rangle(t_{r_1},\cdots,t_{r_{|R|}}), \otimes_{r\in R}\eta(u_r,v_r)\rangle,
\end{aligned}
$$

for almost all $\underline{t} \in \Sigma_{|R|}$. We fix $\underline{t} \in \Sigma_{|R|}$ and define families of vectors $\widetilde{\xi}^{(l)} : l \geq 0$ in $L^2(\mathbb{R}_+, \mathbf{k})^{\otimes l}$ by

$$\widetilde{\xi}^{(0)} = \langle \xi^{(|R|)}(t_{r_1}, \cdots, t_{r_{|R|}}), \otimes_{r \in R} \eta(u_r, v_r) \rangle \in \mathbb{C}$$
$$\widetilde{\xi}^{(l)}(x_1, x_2, \cdots, x_l) = \langle \langle \xi^{(|R|+l)}(x_1, \cdots, x_l, t_{r_1}, \cdots, t_{r_{|R|}}), \otimes_{r \in R} \eta(u_r, v_r) \rangle \rangle,$$

which defines a Fock space vector $\widetilde{\xi}$. Therefore, from (9.11), we get that

$$\lim_{s_q \to t_q} \lim_{s_r \to t_r} \langle \xi, \prod_{q \in Q} \{M(s_q, t_q, u_q, v_q)\} \prod_{r \in R} \{\gamma(s_r, t_r, u_r, v_r)\} \ \mathbf{e}(0) \rangle = \lim_{s_q \to t_q} \langle \widetilde{\xi}\ , \ \psi \rangle$$
$$= \lim_{s_q \to t_q} \langle \widetilde{\xi}\ , \ [\prod_{q \in Q} M(s_q, t_q, u_q, v_q)] \ \mathbf{e}(0) \rangle,$$

which is equal to 0 by Lemma 9.3 (a). Thus from (9.6) we get that

$$\langle \xi^{(n)}(t_1, t_2, \cdots, t_n), \eta(u_1, v_1) \otimes \cdots \otimes \eta(u_n, v_n) \rangle = 0.$$

Since $\{\eta(u, v) : u, v \in \mathbf{h}\}$ is total in $\mathbf{k}$, it follows that $\xi^{(n)}(t_1, t_2, \cdots, t_n) = 0$ for almost every $(t_1, t_2, \cdots, t_n) \in \Sigma_n : t_k \leq \tau$.

(iv) Since $\tau \geq 0$ is arbitrary $\xi^{(n)} = 0 \in L^2(\mathbb{R}_+, \mathbf{k})^{\otimes n} : n \geq 0$ and hence $\xi = 0$. This proves the totality of $\mathcal{S}'$ in Γ. ☐

§10. Unitary Equivalence

Here we shall show that the unitary evolution $\{U_{s,t}\}$ on $\mathbf{h} \otimes \mathcal{H}$ is unitarily equivalent to the HP flow $\{V_{s,t}\}$ on $\mathbf{h} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ discussed above. Let us recall that the subset $\mathcal{S} = \{\xi = U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega := U_{s_1,t_1}(u_1, v_1) \cdots U_{s_n,t_n}(u_n, v_n)\Omega : \underline{s} = (s_1, s_2, \cdots, s_n), \underline{t} = (t_1, t_2, \cdots, t_n) : \ 0 \leq s_1 \leq t_1 \leq s_2 \leq \ldots \leq s_n \leq t_n < \infty, n \geq 1, \underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}\}$ is total in $\mathcal{H}$ and the subset
$\mathcal{S}' := \{\zeta = V_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0) := V_{s_1,t_1}(u_1, v_1) \cdots V_{s_n,t_n}(u_n, v_n)\mathbf{e}(0) :$
$\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}, \underline{s} = (s_1, s_2, \cdots, s_n), \underline{t} = (t_1, t_2, \cdots, t_n)\}$ is total in Γ.

Lemma 10.1. *Let $U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega, \ \ U_{\underline{s}',\underline{t}'}(\underline{p}, \underline{w})\Omega \in \mathcal{S}$.*
Then there exist an integer $m \geq 1$, $\underline{a} = (a_1, a_2, \cdots, a_m), \underline{b} = (b_1, b_2, \cdots, b_m) : 0 \leq a_1 \leq b_1 \leq a_2 \leq \ldots \leq a_m \leq b_m < \infty$, an ordered partition $R_1 \cup R_2 \cup R_3 = \{1, 2, \cdots, m\}$ with $|R_i| = m_i$ and a family of vectors $x_{k_l}, y_{k_l} \in \mathbf{h}, k_l \geq 1 : l = 1, 2, \cdots, m_1 + m_2$ and $g_{k_l}, h_{k_l} \in \mathbf{h}, k_l \geq 1 : l = 1, 2, \cdots, m_2 + m_3$ such that

$$U_{\underline{s},\underline{t}}(\underline{u}, \underline{v}) = \sum_{\underline{k}} \prod_{l \in R_1 \cup R_2} U_{a_l,b_l}(x_{k_l}, y_{k_l}) \tag{10.1}$$

$$U_{\underline{s}',\underline{t}'}(\underline{p},\underline{w}) = \sum_{\underline{k}} \prod_{l\in R_2\cup R_3} U_{a_l,b_l}(g_{k_l},h_{k_l}). \tag{10.2}$$

Proof. This follows from the evolution hypothesis of the family of unitary operators $\{U_{s,t}\}$. □

Remark 10.2. Since the family of unitaries $\{V_{s,t}\}$ on $\mathbf{h}\otimes\Gamma$, enjoy all the properties satisfied by the family of unitaries $\{U_{s,t}\}$ on $\mathbf{h}\otimes\mathcal{H}$, the above Lemma also hold if we replace $U_{s,t}$ by $V_{s,t}$.

Lemma 10.3. *For* $U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega,\quad U_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\Omega\in\mathcal{S}$.

$$\langle U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega, U_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\Omega\rangle = \langle V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\boldsymbol{e}(0), V_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\boldsymbol{e}(0)\rangle. \tag{10.3}$$

Proof. We have by previous Lemma and **Assumption: A**

$$\begin{aligned}
&\langle U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega, U_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\Omega\rangle\\
&= \sum_{\underline{k}} \prod_{l\in R_1} \langle U_{b_l-a_l}(x_{k_l},y_{k_l})\Omega,\Omega\rangle \prod_{l\in R_2} \langle U_{b_l-a_l}(x_{k_l},y_{k_l})\Omega, U_{b_l-a_l}(g_{k_l},h_{k_l})\Omega\rangle\\
&\qquad \prod_{l\in R_3} \langle \Omega, U_{b_l-a_l}(g_{k_l},h_{k_l})\Omega\rangle\\
&= \sum_{\underline{k}} \prod_{l\in R_1} \langle T_{b_l-a_l} y_{k_l}, x_{k_l}\rangle \prod_{l\in R_2} \langle g_{k_l}, Z_{b_l-a_l}(|h_{k_l} >< y_{k_l}|)\ x_{k_l}\rangle\\
&\qquad \prod_{l\in R_3} \langle g_{k_l}, T_{b_l-a_l} h_{k_l}\rangle\\
&= \sum_{\underline{k}} \prod_{l\in R_1} \langle V_{b_l-a_l}(x_{k_l},y_{k_l})\mathbf{e}(0),\ \mathbf{e}(0)\rangle\\
&\qquad \prod_{l\in R_2} \langle V_{b_l-a_l}(x_{k_l},y_{k_l})\mathbf{e}(0), V_{b_l-a_l}(g_{k_l},h_{k_l})\mathbf{e}(0)\rangle\\
&\qquad \prod_{l\in R_3} \langle \mathbf{e}(0), V_{b_l-a_l}(g_{k_l},h_{k_l})\mathbf{e}(0)\rangle.
\end{aligned}$$

Now, the above quantity is equal to $\langle V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\mathbf{e}(0), V_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\mathbf{e}(0)\rangle$ by Remark (10.2).

□

Theorem 10.4. *There exists a unitary isomorphism* $\Xi : \mathbf{h}\otimes\mathcal{H} \to \mathbf{h}\otimes\Gamma$ *such that*

$$U_t = \Xi^* \; V_t \; \Xi, \; \forall \;\; t \geq 0. \tag{10.4}$$

Proof. Let us define a map $\Xi : \mathcal{H} \to \Gamma$ by setting, for any $\xi = U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega \in \mathcal{S}$, $\Xi\xi := V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\mathbf{e}(0) \in \mathcal{S}'$ and then extending linearly. So by definition and by the totality of $\mathcal{S}'$, the range of Ξ is dense in Γ. To see that Ξ is a unitary operator from $\mathcal{H}$ to Γ it is enough to note that

$$\langle \Xi \; \xi, \Xi \; \xi' \rangle = \langle \xi, \xi' \rangle, \; \forall \;\; \xi, \xi' \in \mathcal{S} \tag{10.5}$$

which is already proved in the previous Lemma.

Now consider the ampliated unitary operator $1_{\mathbf{h}} \otimes \Xi$ from $\mathbf{h}\otimes\mathcal{H}$ to $\mathbf{h}\otimes\Gamma$ and denote it by the same symbol Ξ. In order to prove (10.4) it is enough to show that

$$\langle u\otimes\xi, U_t v\otimes\xi' \rangle = \langle \Xi(u\otimes\xi), V_t\Xi(v\otimes\xi') \rangle, \forall \;\; u, v \in \mathbf{h}, \xi, \xi' \in \mathcal{S}. \tag{10.6}$$

Note that $\Xi \; U_t(u,v)\xi' = V_t(u,v) \; \Xi \; \xi'$. Now by unitarity of Ξ, we have

$$\begin{aligned} &\langle u\otimes\xi, U_t v\otimes\xi' \rangle = \langle \xi, U_t(u,v)\xi' \rangle = \langle \Xi \; \xi, \Xi \; U_t(u,v)\xi' \rangle \\ &= \langle \Xi \; \xi, V_t(u,v) \; \Xi \; \xi' \rangle = \langle u\otimes\Xi \; \xi, V_t \; v\otimes\Xi \; \xi' \rangle = \langle \Xi(u\otimes\xi), V_t \; \Xi(v\otimes\xi') \rangle. \end{aligned}$$

Thus proof complete. □

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