Rend. Lincei Mat. Appl. 21 (2010), 405–414
DOI 10.4171/RLM/580

Probability Theory — *BV functions in a Hilbert space with respect to a Gaussian measure*, by LUIGI AMBROSIO, GIUSEPPE DA PRATO and DIEGO PALLARA, communicated on 24 June 2010.

ABSTRACT. — Functions of bounded variation in Hilbert spaces endowed with a Gaussian measure γ are studied, mainly in connection with Ornstein-Uhlenbeck semigroups for which γ is invariant.

KEY WORDS: Gaussian measures, BV functions, Ornstein-Uhlenbecks semigroups.

AMS SUBJECT CLASSIFICATION: 26A45, 28C20, 46E35, 60H07.

1. Introduction

Functions of bounded variation, whose introduction in [13] was based on the heat semigroup, are by now a well-established tool in Euclidean spaces, and more generally in metric spaces endowed with a doubling measure, see e.g. [6] and the references there. Applications run from variational problems with possibly discontinuous solutions along surfaces and geometric measure theory (see [3] and the references there) to renormalized solutions of ODEs without uniqueness (see [1]). More recently, the theory has been extended to infinite dimensional settings (see [16, 17, 4, 5], aiming to apply the theory to variational problems (see [14, 18]), infinite dimensional geometric measure theory (see [15]), ODEs (see [2] for the Sobolev case), as well as stochastic differential equations (see [11, 12]).

If the ambient space is a Hilbert space X endowed with a Gaussian measure γ, then, beside the Malliavin calculus, on which the above quoted papers are based, an approach based on the infinite dimensional analysis as presented in [10] is possible. As in the case of Sobolev spaces, this approach turns out to be similar but not equivalent to the other, and a smaller class of BV functions is obtained. The aim of this paper is to deepen this analysis, mainly in connection with the Ornstein-Uhlenbeck semigroup R_t studied in [10] whose invariant measure is γ, which enjoys stronger regularizing properties compared to the operator P_t of the Malliavin calculus. We prove that, for $u \in L^1(X, \gamma)$, the property of having measure derivatives in a weak sense (i.e., of being BV) is equivalent to the boundedness of a (slightly enforced) Sobolev norm of the gradient of $R_t u$. This regularity result on $R_t u$, for $u \in BV$, is used as a tool, but can be interesting on its own.

2. Notation and preliminaries

Let X be a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, and let us denote by $\mathcal{B}(X)$ the Borel σ-algebra and by $B_b(X)$ the space of bounded Borel functions; since X is separable, $\mathcal{B}(X)$ is generated by the cylindrical sets, that is by the sets of the form $E = \Pi_m^{-1} B$ with $B \in \mathcal{B}(\mathbb{R}^m)$, where $\Pi_m : X \to \mathbb{R}^m$ is orthogonal (see [19, Theorem I.2.2]). The symbol $C_b^k(X)$ denotes the space of k times continuously Fréchet differentiable functions with bounded derivatives up to the order k, and the symbol $\mathcal{F}C_b^k(X)$ that of cylindrical $C_b^k(X)$ functions, that is, $u \in \mathcal{F}C_b^k(X)$ if $u(x) = v(\Pi_m x)$ for some $v \in C_b^k(\mathbb{R}^m)$. We also denote by $\mathscr{M}(X, Y)$ the set of countably additive measures on X with finite total variation with values in a separable Hilbert space Y, $\mathscr{M}(X)$ if $Y = \mathbb{R}$. We denote by $|\mu|$ the total variation measure of μ, defined by

$$|\mu|(B) := \sup\left\{ \sum_{h=1}^{\infty} |\mu(B_h)|_Y : B = \bigcup_{h=1}^{\infty} B_h \right\}, \tag{2.1}$$

for every $B \in \mathcal{B}(X)$, where the supremum runs along all the countable disjoint unions. Notice that, using the polar decomposition, there is a unit $|\mu|$-measurable vector field $\sigma : X \to Y$ such that $\mu = \sigma|\mu|$, and then the equality

$$|\mu|(X) = \sup\left\{ \int_X \langle \sigma, \phi \rangle d|\mu|, \phi \in C_b(X, Y), |\phi(x)|_Y \le 1 \ \forall x \in X \right\}$$

holds. Note that, by the Stone-Weierstrass theorem, the algebra $\mathcal{F}C_b^1(X)$ of C^1 cylindrical functions is dense in $C(K)$ in sup norm, since it separates points, for all compact sets $K \subset X$. Since $|\mu|$ is tight, it follows that $\mathcal{F}C_b^1(X)$ is dense in $L^1(X, |\mu|)$. Arguing componentwise, it follows that also the space $\mathcal{F}C_b^1(X, Y)$ of cylindrical functions with a finite-dimensional range is dense in $L^1(X, |\mu|, Y)$. As a consequence, σ can be approximated in $L^1(X, |\mu|, Y)$ by a uniformly bounded sequence of functions in $\mathcal{F}C_b^1(X, Y)$, and we may restrict the supremum above to these functions only to get

$$|\mu|(X) = \sup\left\{ \int_X \langle \sigma, \phi \rangle d|\mu|, \phi \in \mathcal{F}C_b^1(X, Y), |\phi(x)|_Y \le 1 \ \forall x \in X \right\}. \tag{2.2}$$

We recall the following well-known result (see for instance [5]): given a sequence of real measures (μ_j) on X and an orthonormal basis (e_j), if

$$\sup_m |(\mu_1, \ldots, \mu_m)|(X) < \infty, \tag{2.3}$$

then the measure $\mu = \sum_j \mu_j e_j$ belongs to $\mathscr{M}(X, X)$.

Let us come to a description of the differential structure in X. We refer to [10] for more details and the missing proofs. By $N_{a,Q}$ we denote a non degenerate Gaussian measure on $(X, \mathcal{B}(X))$ of mean a and trace class covariance operator Q (we also use the simpler notation $N_Q = N_{0,Q}$). Let us fix $\gamma = N_Q$, and let (e_k) be an orthonormal basis in X such that

$$Qe_k = \lambda_k e_k, \quad \forall k \ge 1,$$

with λ_k a nonincreasing sequence of strictly positive numbers such that $\sum_k \lambda_k < \infty$. Set $x_k = \langle x, e_k \rangle$ and for all $k \geq 1$, $f \in C_b(X)$, define the partial derivatives

$$D_k f(x) = \lim_{t \to 0} \frac{f(x + te_k) - f(x)}{t} \tag{2.4}$$

(provided that the limit exists) and, by linearity, the gradient operator $D : \mathcal{F}C_b^1(X) \to \mathcal{F}C_b(X, X)$. The gradient turns out to be a closable operator with respect to the topologies $L^p(X, \gamma)$ and $L^p(X, \gamma, X)$ for every $p \geq 1$, and we denote by $W^{1,p}(X, \gamma)$ the domain of the closure in $L^p(X, \gamma)$, endowed with the norm

$$\|u\|_{1,p} = \left(\int_X |u(x)|^p \, d\gamma + \int_X \left(\sum_{k=1}^{\infty} |D_k u(x)|^2 \right)^{p/2} d\gamma \right)^{1/p},$$

where we keep the notation D_k also for the closure of the partial derivative operator. For all φ, $\psi \in C_b^1(X)$ we have

$$\int_X \psi D_k \varphi \, d\gamma = -\int_X \varphi D_k \psi \, d\gamma + \frac{1}{\lambda_k} \int_X x_k \varphi \psi \, d\gamma.$$

and this formula, setting $D_k^* \varphi = D_k \varphi - \frac{x_k}{\lambda_k} \varphi$, reads

$$\int_X \psi D_k \varphi \, d\gamma = -\int_X \varphi D_k^* \psi \, d\gamma. \tag{2.5}$$

Notice that $Q^{1/2}$ is still a compact operator on X, and define the Cameron-Martin space

$$H = Q^{1/2} X = \{x \in X : \exists y \in X \text{ with } x = Q^{1/2} y\} = \left\{ x \in X : \sum_{k=1}^{\infty} \frac{|x_k|^2}{\lambda_k} < \infty \right\},$$

endowed with the orthonormal basis $\varepsilon_k = \lambda_k^{1/2} e_k$ relative to the norm $|x|_H := \left(\sum_k \frac{|x_k|^2}{\lambda_k} \right)^{1/2}$. The Malliavin derivative of $f \in C_b^1(X)$ is defined by

$$\partial_{\varepsilon_k} f(x) = \lim_{t \to 0} \frac{f(x + t\varepsilon_k) - f(x)}{t} \tag{2.6}$$

(provided that the limit exists) and turns out to be a closable operator as well (see [7] or apply (2.8) below) with respect to the topology $L^p(X, \gamma)$ for every $p \geq 1$. We denote by $\nabla_H f$ the gradient and by $\mathbb{D}^{1,p}(X, \gamma)$ the domain of its closure in $L^p(X, \gamma)$, endowed with the obvious norm. As a consequence of the relation $\varepsilon_k = \lambda_k^{1/2} e_k$ we have also

$$\partial_{\varepsilon_k} = \lambda_k^{1/2} D_k, \tag{2.7}$$

so that $W^{1,p}(X,\gamma) \subset \mathbb{D}^{1,p}(X,\gamma)$, since $|\nabla_H f|_H = \left(\sum_k \lambda_k |D_k f|^2\right)^{1/2}$. By (2.7) and (2.5) the integration by parts formula corresponding to the Malliavin calculus reads

$$\int_X \psi \partial_k \varphi \, d\gamma = -\int_X \varphi \partial_k \psi \, d\gamma + \int_X \frac{1}{\sqrt{\lambda_k}} x_k \varphi \psi \, d\gamma. \tag{2.8}$$

There exist infinitely many Ornstein-Uhlenbeck semigroups having γ as invariant measure. Let us choose the one corresponding to the stochastic evolution equation

$$dX = AX \, dt + dW(t), \quad X(0) = x \in X \tag{2.9}$$

where $A := -\frac{1}{2} Q^{-1}$ is selfadjoint and

$$\langle W(t), z \rangle = \sum_{k=1}^{\infty} W_k(t) z_k, \quad z \in X,$$

with $(W_k)_{k \in \mathbb{N}}$ sequence of independent real Brownian motions. We have $Ae_k = -\alpha_k e_k$, where

$$\alpha_k = \frac{1}{2\lambda_k}.$$

The transition semigroup corresponding to (2.9) is given by

$$R_t f(x) = \int_X f(y) \, dN_{e^{tA}x, Q_t}(y) = \int_X f(e^{tA}x + y) \, dN_{Q_t}(y), \quad f \in B_b(X), \tag{2.10}$$

where

$$Q_t = \int_0^t e^{2sA} \, ds = -\frac{1}{2} A^{-1}(1 - e^{2tA}).$$

Therefore $N_{Q_t} \to N_Q = \gamma$ weakly as $t \to \infty$, so that γ is invariant for R_t. Moreover, for every $k \geq 1$, $v \in C_b^1(X)$, from (2.10) we get

$$D_k R_t v(x) = e^{-\alpha_k t} \int_X D_k v(e^{tA}x + y) \, dN_{Q_t}(y) = e^{-\alpha_k t} R_t D_k v(x),$$

whence, since R_t is symmetric, we deduce that for every $u \in L^1(X,\gamma)$ and $\varphi \in \mathcal{F}C_b^1(X)$ the equality

$$\int_X R_t u D_k^* \varphi \, d\gamma = e^{-\alpha_k t} \int_X u D_k^* R_t \varphi \, d\gamma \tag{2.11}$$

holds. In fact, if u is bounded, by [10, Theorem 8.16] we know that $R_t u \in C_b^\infty(X)$ for every $t > 0$, and then for every $\varphi \in C_b^1(X)$ we have

$$\begin{aligned}\int_X R_t u D_k^* \varphi \, d\gamma &= -\int D_k(R_t u)\varphi \, d\gamma = -e^{-\alpha_k t} \int_X R_t D_k u \varphi \, d\gamma \\ &= -e^{-\alpha_k t} \int_X D_k u R_t \varphi \, d\gamma = e^{-\alpha_k t} \int_X u D_k^* R_t \varphi \, d\gamma.\end{aligned}$$

In the general case $u \in L^1(X,\gamma)$ we use the density of $C_b^1(X)$ in $L^1(X,\gamma)$, as both sides in (2.11) are continuous with respect to $L^1(X,\gamma)$ convergence in u.

By a standard duality argument we can define a linear contraction operator $R_t^* : \mathscr{M}(X) \to L^1(X,\gamma)$ characterized by:

$$\int_X R_t^* \mu \varphi \, d\gamma = \int_X R_t \varphi \, d\mu, \quad \varphi \in B_b(X). \tag{2.12}$$

To see that this is a good definition, using Hahn decomposition we may assume with no loss of generality that μ is nonnegative. Under this assumption, we notice that $(\varphi_i) \subset B_b(X)$ equibounded and $\varphi_i \uparrow \varphi$, with $\varphi \in B_b(X)$, implies $\displaystyle\int_X R_t \varphi_i \, d\mu \uparrow \int_X R_t \varphi \, d\mu$, hence Daniell's theorem (see e.g. [8, Theorem 7.8.1]) shows that $\varphi \mapsto \displaystyle\int_X R_t \varphi \, d\mu$ is the restriction to $B_b(X)$ of $\varphi \mapsto \displaystyle\int_X \varphi \, d\mu^*$ for a suitable (unique) nonnegative $\mu^* \in \mathscr{M}(X)$. In order to show that $R_t^* \mu \ll \gamma$, take a Borel set B with $\gamma(B) = 0$. Then

$$(R_t^*)\mu(B) = \int_X \chi_B \, dR_t^* \mu = \int_X R_t \chi_B \, d\mu,$$

but $R_t \chi_B(x) = N_{e^{tA}x, Q_t}(B)$ and since $N_{e^{tA}x, Q_t} \ll \gamma$ (see [12, Lemma 10.3.3]) we have $R_t \chi_B(x) = 0$ for all x and the claim follows. Finally, since $R_t 1 = 1$ we obtain that $\mu^*(X) = \mu(X)$, hence R_t^* is a contraction. It is also useful to notice that R_t^* is contractive on vector measures as well. In fact, R_t is a contraction in C_b, hence $|\langle R_t^* \mu, \phi \rangle| = |\langle \mu, R_t \phi \rangle| \le \langle |\mu|, |\phi| \rangle$ for every $\varphi \in C_b(X)$. Since for every vector measure v the minimal positive measure σ such that $|\langle v, \phi \rangle| \le \langle \sigma, |\phi| \rangle$ for all φ is $|v|$, taking $v = R_t^* \mu$ we conclude.

3. Functions of bounded variation

In the present context it is possible to define functions of bounded variation, as it has been done, using the Malliavin derivative, in [16], [17] and [4], [5], and to relate BV functions to the Ornstein-Uhlenbeck semigroup R_t. According to [5], in order to distinguish the two notions of BV functions, we keep the notation $BV(X,\gamma)$ for the functions coming from the ∇_H operator and use the notation $BV_X(X,\gamma)$ for those coming from D.

Definition 3.1. *A function $u \in L^1(X,\gamma)$ belongs to $BV_X(X,\gamma)$ if there exists $\nu^u \in \mathscr{M}(X,X)$ such that for any $k \geq 1$ we have*

$$\int_X u(x)D_k\varphi(x)\,d\gamma = -\int_X \varphi(x)\,d\nu_k^u + \frac{1}{\lambda_k}\int_X x_k u(x)\varphi(x)\,d\gamma, \quad \varphi \in \mathcal{F}C_b^1(X),$$

with $\nu_k^u = \langle \nu^u, e_k\rangle_X$. If $u \in BV_X(X,\gamma)$, we denote by Du the measure ν^u, and by $|Du|$ its total variation.

According to (2.2), for $u \in BV_X(X,\gamma)$ the total variation of Du is given by

$$|Du|(X) = \sup\left\{\int_X u\left[\sum_k D_k^*\phi_k\right]d\gamma, \phi \in \mathcal{F}C_b^1(X,X), |\phi(x)| \leq 1\ \forall x \in X\right\}. \tag{3.1}$$

Obviously, if $u \in W^{1,1}(X,\gamma)$ then $u \in BV_X(X,\gamma)$ and $|Du|(X) = \int_X |Du|\,d\gamma$.

Recalling that $u \in BV(X,\gamma)$ if there is a finite measure $D_\gamma u = (D_\gamma^k u)_k \in \mathscr{M}(X,X)$ such that

$$\int_X u(x)\partial_k\varphi(x)\,d\gamma = -\int_X \varphi(x)\,dD_\gamma^k u + \frac{1}{\sqrt{\lambda_k}}\int_X x_k u(x)\varphi(x)\,d\gamma,$$
$$\varphi \in \mathcal{F}C_b^1(X),\ k \geq 1,$$

it is immediate to check that $BV_X(X,\gamma)$ is contained in $BV(X,\gamma)$ and that

$$D_\gamma^k u = \lambda_k^{1/2}\nu_k^u, \quad \forall k \geq 1. \tag{3.2}$$

The next proposition provides a simple criterion, analogous to the finite-dimensional one, for the verification of the BV_X property.

Proposition 3.2. *Let $u \in L^1(X,\gamma)$ and let us assume that*

$$\mathcal{R}(u) := \sup_m \sup\left\{\int_X \sum_{k=1}^m uD_k^*\varphi_k\,d\gamma : \varphi_k \in C_b^1(X), \sum_{i=1}^m \varphi_k^2 \leq 1\right\} < \infty. \tag{3.3}$$

Then $u \in BV_X(X,\gamma)$ and $|Du|(X) \leq \mathcal{R}(u)$.

Proof. Fix $k \geq 1$, set $X_k = \{x \in X : x = se_k, s \in \mathbb{R}\}$, $X_k^\perp = \{x \in X : \langle x, e_k\rangle = 0\}$, and define

$$V_k(u) := \sup\left\{\int_X u\left(\partial_k\phi - \frac{1}{\sqrt{\lambda_k}}\phi\right)d\gamma : \phi \in C_c^1(X), |\phi(x)| \leq 1\ \forall x \in X\right\},$$

$$\mathscr{V}_k(u) := \sup\left\{\int_X u\left(D_k\phi - \frac{1}{\lambda_k}\phi\right)d\gamma : \phi \in C_c^1(X), |\phi(x)| \leq 1\ \forall x \in X\right\}.$$

For $y \in X_k^\perp$, define the function $u_y(s) = u(y + se_k)$, $s \in \mathbb{R}$, and notice that $V_k(u) = \sqrt{\lambda_k}\mathscr{V}_k(u)$, so that by [5, Theorem 3.10] we have

$$\mathscr{V}_k(u) = \int_{X_k^\perp} \mathscr{V}(u_y)\, d\gamma^\perp(y),$$

where $\mathscr{V}$ denotes the 1-dimensional variation of u_y and we have used the factorization $\gamma = \gamma_1 \otimes \gamma^\perp$ induced by the orthogonal decomposition $X = X_k \oplus X_k^\perp$.

Since $\mathscr{V}_k(u) \le \mathcal{R}(u)$ we have

$$\int_{X_k^\perp} \mathscr{V}(u_y)\, d\gamma^\perp(y) < \infty.$$

It follows that for $\gamma^\perp$-a.e. $y \in X_k^\perp$ the function u_y has bounded variation in $\mathbb{R}$. By a Fubini argument, based on the factorization $\gamma = \gamma_1 \otimes \gamma^\perp$, the 1-dimensional integration by parts formula yields that the measure $D_k u$ coincides with $Du_y \otimes \gamma^\perp$, i.e.,

$$D_k u(A) = \int_{X_k^\perp} Du_y(A_y)\, d\gamma^\perp(y)$$

(where $A_y := \{s : y + se_k \in A\}$ is the y-section of a Borel set A) provides the derivative of u along e_k. Notice that $D_k u$ is well defined, since we have just proved that $\int_{X_k^\perp} |Du_y|(\mathbb{R})\, d\gamma^\perp$ is finite.

Now, setting $\mu_k = D_k u$, by the implication stated in (2.3) we obtain that $|Du|(X) \le \mathcal{R}(u)$. □

The next theorem characterizes the BV class in terms of the semigroup R_t: notice that the functions $R_t u$, for $u \in BV(X, \gamma)$, turn out to be slightly better than $W^{1,1}(X, \gamma)$, since not only $|DR_t u|$, but also $|e^{-tA} DR_t u|$ is integrable.

Theorem 3.3. *Let $u \in L^1(X, \gamma)$. Then, $u \in BV_X(X, \gamma)$ if and only if $R_t u \in W^{1,1}(X, \gamma)$, $|e^{-tA} DR_t u| \in L^1(X, \gamma)$ for all $t > 0$ and*

$$\liminf_{t \downarrow 0} \int_X |e^{-tA} DR_t u|\, d\gamma < \infty. \tag{3.4}$$

Moreover, if $u \in BV_X(X, \gamma)$ we have $DR_t u = e^{-tA} R_t^* Du$,

$$\int_X |e^{-tA} DR_t u|\, d\gamma \le |Du|(X), \quad \forall t > 0 \tag{3.5}$$

and

$$\lim_{t \downarrow 0} \int_X |e^{-tA} DR_t u|\, d\gamma = |Du|(X). \tag{3.6}$$

Proof. Let $u \in BV_X(X,\gamma)$. We use (2.11) to deduce

$$\int_X R_t u D_k^* \varphi \, d\gamma = -e^{-\alpha_k t} \int_X R_t \varphi \, dD_k u \quad \forall \varphi \in \mathcal{F}C_b^1(X), \; t > 0.$$

According to (2.12), this implies that $D_k R_t u = e^{-\alpha_k t} R_t^* D_k u \in L^1(X,\gamma)$. Therefore, as R_t^* is a contractive semigroup also on vector measures,

$$\int_X |e^{-tA} D R_t u| \, d\gamma = \int_X |R_t^* D u| \, d\gamma \le |Du|(X)$$

for every $t > 0$ and (3.5) follows.

Conversely, let us assume that $R_t u \in W^{1,1}(X,\gamma)$ for all $t > 0$ and that the lim inf in (3.4) is finite. We shall denote by $\Pi_m : X \to \mathbb{R}^m$ the canonical projection on the first m coordinates and we shall actually prove that $u \in BV_X(X,\gamma)$ and

$$|Du|(X) \le \sup_m \liminf_{t \downarrow 0} \int_X |\Pi_m D R_t u| \, d\gamma \tag{3.7}$$

under the only assumption that the right hand side of (3.7) is finite. Indeed, fix an integer m and notice that an integration by parts gives

$$\sup\left\{ \int_X \sum_{k=1}^m R_t u D_k^* \varphi_k \, d\gamma : \varphi_k \in C_b^1(X), \sum_{i=1}^m \varphi_k^2 \le 1 \right\} \le \int_X |\Pi_m D R_t u| \, d\gamma,$$

so that passing to the limit as $t \downarrow 0$ and taking the supremum over m we obtain

$$\mathcal{R}(u) \le \sup_m \liminf_{t \downarrow 0} \int_X |\Pi_m D R_t u| \, d\gamma,$$

with $\mathcal{R}$ defined as in (3.3). Therefore we obtain the inequality (3.7) by Proposition 3.2. Finally (3.6) follows combining (3.5) with (3.7). □

Remark 3.4. (1) Notice that the inclusion $BV_X(X,\gamma) \subset BV(X,\gamma)$ allows us to exploit the results in [5] in order to prove one implication in the above theorem, while the other one uses the strong regularizing properties of the semigroup R_t. Anyway, we have tried to keep the use of the results in the above quoted paper to a minimum, and in fact only Theorem 3.10 in [5] has been used in the proof of Proposition 3.2. It is most likely possible to give a proof completely independent from [5], but some of the arguments therein should be rephrased and proved again, basically along the same lines.

(2) The argument used in the proof of the theorem shows that $D_k R_t u \in L^1(X,\gamma)$ for all $t > 0$, $k \ge 1$ and finiteness of the right hand side of (3.7) suffices to conclude that $u \in BV_X(X,\gamma)$. Furthermore, combining (3.5) and (3.7) we obtain that $\int_X |D R_t u| \, d\gamma \to |Du|(X)$ as $t \downarrow 0$, as well.

(3) By the same argument as [5] one can use (2) to conclude that the measures $e^{-tA}DR_t u\gamma$ are equi-tight as $t \downarrow 0$; hence, they converge (componentwise) to Du not only on $\mathcal{F}C_b^1(X)$ but also on $C_b^0(X)$.

We recall also that both Sobolev and BV spaces in the present context are compactly embedded into the corresponding Lebesgue spaces. The following statement is proved in [5, Theorem 5.3], see also [9] for the case $1 < p < \infty$.

Theorem 3.5. *For every $p \geq 1$, the embedding of $W^{1,p}(X,\gamma)$ into $L^p(X,\gamma)$ is compact. The embedding of $BV_X(X,\gamma)$ into $L^1(X,\gamma)$ is also compact.*

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Received 30 April 2010,
and in revised form 11 June 2010.

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