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Partial Differential Equations — *A sharp Liouville theorem for elliptic operators*[1], by Enrico Priola and Feng-Yu Wang, communicated on 12 November 2010.

Abstract. — We introduce a new condition on elliptic operators $L = \frac{1}{2}\triangle + b \cdot \nabla$ which ensures the validity of the Liouville property, i.e., all smooth bounded solutions to $Lu = 0$ on $\mathbb{R}^d$ are constant. Such condition is sharp when $d = 1$. We extend our Liouville theorem to more general second order operators in non-divergence form assuming a Cordes type condition.

Key words: Liouville theorem, space-time harmonic functions.

AMS Subject Classification: 35J15, 47D07.

1. Introduction

Let

$$L = \frac{1}{2}\sum_{i,j=1}^{d} q_{ij}(x) D_{ij} + \sum_{i=1}^{d} b_i(x) D_i$$

be a uniformly elliptic second order differential operator on $\mathbb{R}^d$ with continuous coefficients q_{ij} and b_i (here $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ and $D_i = \frac{\partial}{\partial x_i}$, $1 \le i, j \le d$). Recall that a smooth real function u on $\mathbb{R}^d$ is called L-harmonic if $Lu = 0$ holds on $\mathbb{R}^d$. An operator L is said to possess the Liouville property when all bounded L-harmonic functions are constant (or, equivalently, when a two-sided Liouville theorem holds for L). Such property is also of interest for the study of non-linear PDEs of the form $\triangle u + F(u) = 0$ (see e.g. [1, 2]).

There are a plenty of results on the Liouville property. Let $\lambda_0 > 0$ be the ellipticity constant of L. A typical condition implying the Liouville property is the following (see e.g. [3, 6, 7]):

$$\frac{1}{2\lambda_0}\|q(x) - q(x+h)\|^2 + 2\langle b(x+h) - b(x), h\rangle \le 0, \quad x, h \in \mathbb{R}^d \tag{1.1}$$

(given a $d \times d$ real matrix A, we denote by $\|A\|$ its Hilbert-Schmidt norm; moreover $\langle \cdot , \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^d$). However this is not com-

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pletely satisfactory for two reasons. The first one is that when $b(x)$ is constant the matrix $q(x)$ must be constant as well. This is a restriction since it is known that the Liouville property holds when b is constant and $q(x)$ is variable (this is a standard consequence of [5, Corollary 4.1]).

The second weak point of (1.1) is that when $q(x)$ is the identity, i.e., we are considering $L_0 = \frac{1}{2}\Delta + b \cdot \nabla$, such hypothesis is not optimal even when $d = 1$. The aim of this note is to find out a sharp and easy to check criterion ensuring the Liouville property for L_0. Our condition is sharp when $d = 1$; indeed if this does not hold one can construct counterexamples of operators L_0 without the Liouville property.

We prove our Liouville type theorem in the more general setting of elliptic operators L, with $q(x)$ variable, imposing an additional Cordes type condition (see [4]). Our proof requires the coupling method of [6] (possible extensions of this method are given in [8] and [9]).

To explain the motivation of our desired condition for the Liouville property, let us start with a one-dimensional example

$$L_0 = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x}{2+x^2}\left(\delta + \frac{2}{\log(2+x^2)}\right)\frac{\mathrm{d}}{\mathrm{d}x},$$

where δ is a constant. It is easy to see that a harmonic function of L_0 has the form

$$u(x) = c_1 + c_2 \int_0^x \frac{\mathrm{d}r}{(2+r^2)^\delta \log^2(2+r^2)}, \qquad x \in \mathbb{R},$$

where c_1, c_2 are constants. Thus, all bounded harmonic functions are constant if any only if $\delta < 1/2$. In order to reduce this condition to a usual monotonicity condition on the drift $b(x) = \frac{x}{2+x^2}\left(\delta + \frac{2}{\log(2+x^2)}\right)$, we note that (using also that b is odd)

$$\lim_{s\to\infty} \sup_{|x-y|=s} (x-y)(b(x)-b(y)) = \lim_{s\to\infty} s(b(s/2) - b(-s/2)) = 4\delta.$$

Then the statement can be reformulated as all bounded L_0-harmonic functions are constant if and only if

$$\lim_{s\to\infty} \sup_{|x-y|=s} (x-y)(b(x)-b(y)) < 2.$$

In general, let e.g. $L_0 = \frac{1}{2}\Delta + b \cdot \nabla$ on $\mathbb{R}^d$, we may wish to prove the Liouville property of L_0 under the following hypothesis

$$\limsup_{s\to\infty} \sup_{|x-y|=s} \langle x-y, b(x)-b(y)\rangle < 2. \tag{1.2}$$

This follows immediately from our main result.

2. Main theorem

We prove a Liouville type theorem for bounded space-time harmonic functions. Recall that a smooth function u on $[0,\infty)\times\mathbb{R}^d$ is called space-time harmonic for L, if $\partial_t u + Lu = 0$ holds. To state our main result, we make the following assumptions.

(H) (i) The coefficients $b(x)$ and $q(x)$ are continuous, and, for any $\lambda > 0$, $\omega(s) := \sup_{|x-y|\le s}\{\lambda\|q(x)-q(y)\|^2 + 2\langle x-y, b(x)-b(y)\rangle\}$ satisfies

$$\int_0^1 \frac{\omega(s)}{s}\,ds < \infty;$$

(ii) there exist two constants $0 < \lambda_0 < \Lambda_0$ such that

$$\lambda_0|h|^2 \le \sum_{i,j=1}^{n} q_{ij}(x)h_ih_j \le \Lambda_0|h|^2, \quad x,h\in\mathbb{R}^d.$$

Theorem 2.1. *Assume* **(H)**. *If*

$$\limsup_{s\to\infty}\ \sup_{|x-y|=s}\ \langle x-y, b(x)-b(y)\rangle < 2\lambda_0 - \frac{d}{2}(\Lambda_0-\lambda_0), \tag{2.1}$$

then any bounded space-time harmonic function for L is constant.

Proof. We will suitably apply [6, Theorem 3.6]. To this purpose, we have to consider a coupling for L. By (2.1) we may take constants $\mu, s_0 > 0$ and $s_1\in\mathbb{R}$ such that $\mu < \lambda_0$ and

$$\sup_{|x-y|=s}\ \langle x-y, b(x)-b(y)\rangle \le s_1 < 2\mu - \frac{1}{2}d(\Lambda_0-\mu), \quad s\ge s_0. \tag{2.2}$$

Define a symmetric positive definite matrix $\sigma(x)$, such that $\sigma(x)^2 + \mu I = q(x)$, $x\in\mathbb{R}^d$. Clearly we have $\sigma^2(x)\ge(\lambda_0-\mu)I$. We construct a coupling as in Section 3.1 of [6], replacing the ellipticity constant λ_0 with μ (note that under our assumptions the associated diffusion process does not explode). Applying [6, Lemma 3.3] we deduce that

$$\|\sigma(x)-\sigma(y)\|^2 \le \frac{1}{4(\lambda_0-\mu)}\|q(x)-q(y)\|^2, \quad x,y\in\mathbb{R}^d.$$

Combining this with **(H)**(i) for $\lambda = \frac{1}{4(\lambda_0-\mu)}$, we obtain

$$\|\sigma(x)-\sigma(y)\|^2 + 2\langle x-y, b(x)-b(y)\rangle \le \omega(|x-y|) \text{ for } x,y\in\mathbb{R}^d, \quad \text{and} \tag{2.3}$$

$$\int_0^{s_0}\frac{\omega(s)}{s}\,\mathrm{d}s < \infty.$$

On the other hand, since $\sigma(x)^2 \le (\Lambda_0 - \mu)I$, we have $0 \le \sigma(x) \le (\Lambda_0 - \mu)^{1/2} I$, for any $x \in \mathbb{R}^d$. Thus

$$-(\Lambda_0 - \mu)^{1/2} I \le \sigma(x) - \sigma(y) \le (\Lambda_0 - \mu)^{1/2} I, \quad x, y \in \mathbb{R}^d.$$

We deduce that $0 \le (\sigma(x) - \sigma(y))^2 \le (\Lambda_0 - \mu)I$ and so

$$\|\sigma(x) - \sigma(y)\|^2 = \mathrm{Tr}[(\sigma(x) - \sigma(y))^2] \le d(\Lambda_0 - \mu), \quad x, y \in \mathbb{R}^d.$$

Combining this with (2.2) we obtain

$$\|\sigma(x) - \sigma(y)\|^2 + 2\langle x - y, b(x) - b(y)\rangle \le 2s_1 + d(\Lambda_0 - \mu) =: s_2 < 4\mu,$$
$$|x - y| \ge s_0.$$

From this and (2.3) we conclude that

$$\|\sigma(x) - \sigma(y)\|^2 + 2\langle x - y, b(x) - b(y)\rangle \le |x - y| g(|x - y|), \quad x, y \in \mathbb{R}^d$$

holds for

$$g(s) := \frac{\omega(s)}{s} 1_{[0, s_0]}(s) + \frac{s_2}{s} 1_{(s_0, \infty)}, \quad s > 0.$$

Since by **(H)**

$$c := \int_0^{s_0} g(s)\, \mathrm{d}s < \infty,$$

we have

$$\int_0^\infty \exp\Big(-\frac{1}{4\mu}\int_0^r g(s)\, ds\Big)\, dr$$
$$\ge \int_1^\infty \exp\Big(-\frac{1}{4\mu}\int_0^{s_0} g(s)\, ds\Big) \exp\Big(-\frac{1}{4\mu}\int_{s_0}^r g(s)\, ds\Big)\, dr$$
$$\ge \mathrm{e}^{-c_1} \int_1^\infty s^{-s_2/[4\mu]}\, \mathrm{d}s = \infty$$

since $s_2 < 4\mu$. Applying [6, Theorem 3.6], we get the assertion. □

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