



Algebraic Geometry — *Involutions, Humbert surfaces, and divisors on a moduli space*, by STEVEN H. WEINTRAUB, communicated on 26 July 2010.

ABSTRACT. — Let \mathfrak{M}_2^* be the Igusa compactification of the Siegel modular variety of degree 2 and level 2. In earlier work with R. Lee, we carefully investigated this variety. Subvarieties D_ℓ (compactification divisors) and H_Δ (Humbert surface of discriminant 1) play a prominent role in its structure; in particular their fundamental classes span $H_4(\mathfrak{M}_2^*; \mathbf{Z})$. We return to this variety and consider another class of subvarieties K_h (Humbert surfaces of degree 4), which we investigate with the help of involutions on \mathfrak{M}_2^* . We carefully describe these subvarieties and consider the representations of their fundamental classes in terms of the fundamental classes of the subvarieties D_ℓ and H_Δ . The space \mathfrak{M}_2^* is also known in a different context. It can also be described as the space $\overline{M}_{0,6}$ of stable curves of genus 2 with ordered Weierstrass points. In this context the divisors K_h are what have come to be known as Keel-Vermeire divisors.

KEY WORDS: Siegel modular varieties, Humbert surfaces, Keel-Vermeire divisors.

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Let \mathcal{S}_2 denote Siegel space of degree two, i.e. the space of symmetric 2-by-2 complex matrices with positive definite imaginary part. Let $\mathrm{Sp}(4, \mathbf{Z})$ be the group of 4-by-4 matrices that preserves the usual symplectic form on \mathbf{Z}^4 . Then $\mathrm{Sp}(4, \mathbf{Z})$ acts on \mathcal{S}_2 on the left by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = \left(A \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} + B \right) \left(C \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} + D \right)^{-1}.$$

Note that $-I$ acts trivially so that this action factors through the projective group $\mathrm{P}\mathrm{Sp}(4, \mathbf{Z})$. For any subgroup Γ of finite index of $\mathrm{Sp}(4, \mathbf{Z})$ we may consider the restriction to Γ of this action, and in particular we may consider $\Gamma = \Gamma(n)$, the principal congruence subgroup of level n . Then the quotient $\mathfrak{M}_\Gamma = \Gamma \backslash \mathcal{S}_2$, which for $\Gamma = \Gamma(n)$ we denote by \mathfrak{M}_n , is a moduli space of principally polarized Abelian surfaces with a level Γ structure (for $\Gamma = \Gamma(n)$ this is a level n structure). This space has a compactification \mathfrak{M}_Γ^* (or \mathfrak{M}_n^*) first constructed by Igusa [3], though nowadays best understood as an example toroidal compactification. For $n \geq 2$ \mathfrak{M}_n^* is a nonsingular projective variety.

In a series of papers, [5, 6, 7, 8], R. Lee and the author investigated the space \mathfrak{M}_2^* and considered various related matters. We will not restate our results here, but rather restate them when we need them below.

We now return to this space with another objective in mind. Our objective here is to investigate a family of divisors on this threefold that we de-

note by K_h , and our approach to these divisors is by considering involutions on \mathfrak{M}_2^* .

We thus begin this paper in Section 1 by giving enough background on \mathfrak{M}_2^* to get us started. We then, in Section 2, have an algebraic interlude in which we consider involutions in $\mathrm{PSp}(4, \mathbf{Z})$. In Section 3 we return to geometry and carefully describe the complex surfaces K_h in Theorem 3.2. In Section 4, and in particular in Theorem 4.12, we show how to express the homology class represented by K_h in $H_4(\mathfrak{M}_2^*; \mathbf{Z})$ in terms of the homology classes represented by complex surfaces H_Δ and D_ℓ that we showed in our earlier work span $H_2(\mathfrak{M}_2^*; \mathbf{Z})$. Here h, Δ , and ℓ run over indexing sets that we will describe below.

As a consequence, we show in Corollary 4.14 that K_h cannot be represented as a nonnegative integral linear combination of the classes H_Δ and D_ℓ , and in Theorem 4.15 we strengthen that to show that K_h cannot be represented as a nonnegative rational linear combination of these classes.

Finally, In Section 5 we discuss the relationship between \mathfrak{M}_2^* and another space $\overline{M}_{0,6}$, the moduli space of stable curves of genus two with ordered Weierstrass points. When our results are carried over into $\overline{M}_{0,6}$, our divisors K_h are what have come to be known as Keel-Vermeire divisors, and we are happy to acknowledge that in this context Corollary 4.14 is due to them. In view of the considerable interest in doing computations in this space, we have tried in this paper to give explicit methods and results useful for computation rather than the minimum we need to get by.

A few words about notation and terminology: We will denote subvarieties and the homology classes they represent by the same symbol. We will denote complex curves/2-dimensional homology classes by lower case letters and complex surfaces/4-dimensional homology classes by upper case letters. Also, we will use $t_\ell \cap H_\Delta$, for example, to denote the geometric intersection of these two varieties and $t_\ell \cdot H_\Delta$ for their intersection number, and we will feel free to pass between the two.

Finally, we will remark that because of our geometric approach, it is most natural for us to work in homology, but for \mathfrak{M}_2^* homological and algebraic equivalence of divisors are identical.

1. BACKGROUND ON \mathfrak{M}_2^*

In this section we describe salient features of \mathfrak{M}_2^* . We refer the reader to [5, 6, 8] for more details. In addition, [2] provides a careful and extensive description of the compactification procedure.

We let $V = \mathbf{Z}^4$ be the space of row vectors equipped with the nonsingular form $\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle = (x_1 y_3 - x_3 y_1) + (x_2 y_4 - x_4 y_2)$. We let $\mathrm{Sp}(4, \mathbf{Z})$ be the symplectic group, the group of 4-by-4 integral matrices preserving this form, where $\mathrm{Sp}(4, \mathbf{Z})$ acts on V on the left by $g(v) = vg^{-1}$. This descends to an action of $\mathrm{Sp}(4, \mathbf{Z}/2\mathbf{Z}) = \mathrm{Sp}(4, \mathbf{Z})/\Gamma(2)$ on $\overline{V} = V \otimes (\mathbf{Z}/2\mathbf{Z})$ preserving \langle , \rangle , which is the above form taken modulo 2.

We begin by describing the quotient Tits building $\overline{\mathcal{F}}$ of \overline{V} , which is the quotient of the Tits building \mathcal{F} of V by the action of $\Gamma(2)$. (The expert will recognize

that we are taking advantage of some simplifications due to the fact that we are working mod 2.)

\mathcal{F} is a bipartite graph, with two kinds of vertices, ℓ -vertices and h -vertices. Here ℓ is a line through the origin, so is specified by a nonzero point in \bar{V} , and h is an isotropic plane, i.e., a plane through the origin that is totally isotropic with respect to the form \langle , \rangle . Then h contains three lines, say $\ell_1, \ell_2,$ and $\ell_3,$ and we write $h = \ell_1 \wedge \ell_2$.

\mathcal{F} has 15 ℓ vertices and 15 h vertices. There is an edge joining ℓ to h if $\ell \in h$. Then every h contains 3 ℓ 's and every ℓ is contained in 3 h 's, so every vertex of \mathcal{F} has valence 3 and \mathcal{F} has a total of 45 edges. We say that two distinct vertices of \mathcal{F} are *nearby* if there is a path in \mathcal{F} of length two joining them, i.e., ℓ_1 and ℓ_2 are nearby if there is an h with $\ell_1 \in h$ and $\ell_2 \in h$, and h_1 and h_2 are nearby if there is an ℓ with $\ell \in h_1$ and $\ell \in h_2$. Given a fixed ℓ vertex, there are 6 nearby ℓ vertices, and given a fixed h vertex, there are 6 nearby h vertices. The action of $\text{Sp}(4, \mathbf{Z}/2\mathbf{Z})$ on \bar{V} induces an action on \mathcal{F} , and this action is transitive on ℓ -vertices, on h -vertices, and on edges.

In the compactification \mathfrak{M}_2^* we have 15 corank 1 boundary components $\{D_\ell\}$, and 15 corank 2 boundary components $\{C_h\}$. They are each permuted transitively by the action of $\text{Sp}(4, \mathbf{Z}/2\mathbf{Z})$, so are mutually isomorphic in each case, and so it suffices to describe one of each.

We begin with D_ℓ , and for the sake of definiteness take $\ell = (1, 0, 0, 0)$, which we henceforth abbreviate as $\ell = (1000)$ (and similarly for all ℓ and for all h). This comes from the stabilizer of the line generated by $\pm(1, 0, 0, 0)$ in V (there is an ambiguity of sign). The stabilizer $P(\pm(1, 0, 0, 0))$ in $\Gamma(2)$ consists of matrices of the form

$$\begin{pmatrix} \varepsilon & m & s & n \\ 0 & a & * & b \\ 0 & 0 & \varepsilon & 0 \\ 0 & c & * & d \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2)$, the principal congruence subgroup of level 2 in $\text{SL}(2, \mathbf{Z})$, $\varepsilon = \pm 1$, and $m, n, s \in 2\mathbf{Z}$, and the entries $*$ are determined by the condition that this matrix be symplectic.

There is a homomorphism, given by the notation, of this group to the group of matrices of the form

$$\begin{pmatrix} \varepsilon & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

satisfying the same congruence conditions.

We think of the associated corank 1 boundary component as the component given by “ $\tau_1 = i\infty$ ” (this can be made precise) and then the above element acts on $\mathbf{C} \times \mathcal{S}_1 = \left\{ \begin{pmatrix} z \\ \tau \end{pmatrix} = \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} \right\}$ by

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} z \\ \tau \end{pmatrix} &= \begin{pmatrix} z/(c\tau + d) \\ (a\tau + b)/(c\tau + d) \end{pmatrix} \\ \begin{pmatrix} 1 & m & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ \tau \end{pmatrix} &= \begin{pmatrix} z + m\tau + n \\ \tau \end{pmatrix} \\ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ \tau \end{pmatrix} &= \begin{pmatrix} \varepsilon z \\ \tau \end{pmatrix} \end{aligned}$$

(compare [2, Proposition 3.100 and Proposition 3.102]).

We observe that this action covers the usual action of $\Gamma_1(2)$ on \mathcal{S}_1 , the upper half plane, by fractional linear transformations. The quotient $B^\circ = \Gamma_1(2) \backslash \mathcal{S}_1$ is $\mathbf{P}^1 - 3$ points. Then we see that the quotient $D^\circ = P(\pm(1, 0, 0, 0)) \backslash \mathbf{C} \times \mathcal{S}_1$ is a fiber space over B° . Examining the fiber over a point represented by $\tau \in \mathcal{S}_1$, we see that it is a Kummer curve, i.e., it is the quotient of \mathbf{C} by the lattice $\{m\tau + n \mid m, n \in 2\mathbf{Z}\}$, which is an elliptic curve, and then furthermore by the involution $z \mapsto -z$ of this elliptic curve. (A Kummer curve is \mathbf{P}^1 , but has moduli because it has four distinguished points that are the images of the four fixed points of the involution.) D° is an open Kummer modular surface, and there is a compactification D of D° extending the projection $D^\circ \rightarrow B^\circ$:

$$\begin{array}{ccc} D^\circ & \hookrightarrow & D \\ \downarrow & & \downarrow \\ B^\circ & \hookrightarrow & B. \end{array}$$

B is \mathbf{P}^1 , obtained by adding to B° its three cusps. D is a nonsingular surface, but as a fibre space it is singular, with the fibers over the cusps being two \mathbf{P}^1 's intersecting transversely. (This construction is very analogous to the construction of the well-known elliptic modular surfaces [9].) As an abstract complex surface, $D_\ell = D$.

Now for the corank 2 boundary components. There are much more subtle to obtain, but much easier to describe. For any isotropic plane h , C_h is a configuration of three \mathbf{P}^1 's as in the letter Y , i.e., the three \mathbf{P}^1 's are mutually disjoint except for a common triple point (at which their tangent spaces span the tangent space of \mathfrak{M}_2^*). We call this triple point the *deepest point* of C_h .

The relation between these two sorts of boundary components is that $D_\ell \cap C_h$ is nonempty if and only if $\ell \in h$ (i.e., if there is an edge in $\overline{\mathcal{F}}$ joining ℓ to h). In this case, on the one hand $D_\ell \cap C_h$ is one of the exceptional fibers over a cusp in D_h , consisting of two \mathbf{P}^1 's, and on the other hand $D_\ell \cap C_h$ consists of two of the ‘‘arms’’ of the Y in C_h . In particular, we see that if $\ell_1 \in h$ and $\ell_2 \in h$, then $D_{\ell_1} \cap D_{\ell_2}$ is one arm of the Y in C_h , i.e., is a single \mathbf{P}^1 , which we denote s_{ℓ_1, ℓ_2} .

(Thus if ℓ_1, ℓ_2 , and ℓ_3 are the three lines in h , the three \mathbf{P}^1 's in C_h are s_{ℓ_1, ℓ_2} , s_{ℓ_1, ℓ_3} , and s_{ℓ_2, ℓ_3} , and the deepest point in h is the triple intersection $D_{\ell_1} \cap D_{\ell_2} \cap D_{\ell_3}$.) We also see that every D_ℓ contains 3 deepest points, the intersection of the two \mathbf{P}^1 's in each of its three exceptional fibers.

Next we describe the Humbert surfaces $\{H_\Delta\}$ of discriminant 1 in \mathfrak{M}_2^* . (In our previous work we simply referred to them as the Humbert surfaces, as they were the only ones that appeared.) Again we begin with the indexing set. We call $\Delta = \{\delta, \delta^\perp\}$ a nonsingular pair, (in our previous work we called it a pair of anisotropic planes) where δ and δ^\perp are both planes in \bar{V} , with the restriction of the form $\langle \cdot, \cdot \rangle$ to each of these planes nonsingular, and with each of δ and δ^\perp the orthogonal complement of the other, in which case it follows that \bar{V} is the orthogonal direct sum of δ and δ^\perp . Let us take $\Delta = \{(1000) \wedge (0010), (0100) \wedge (0001)\}$. Then Δ is the image of a pair of subspaces in V , obtained by regarding the entries of Δ as integers rather than integers modulo 2, and the corresponding subgroup of $\Gamma(2)$ consists of matrices of the form

$$\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_3 & 0 & b_3 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_3 & 0 & d_3 \end{pmatrix},$$

with $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma_1(2)$ for $i = 1, 3$.

These matrices stabilize the subspace $\mathcal{S}_1 \times \mathcal{S}_1 = \{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \in \mathcal{S}_2 \}$ and the above matrix acts on $\mathcal{S}_1 \times \mathcal{S}_1$ by

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} (a_1\tau_1 + b_1)/(c_1\tau_1 + d_1) & 0 \\ 0 & (a_3\tau_3 + b_3)/(c_3\tau_3 + d_3) \end{pmatrix}$$

and so we see that the quotient $H^\circ = (\Gamma_1(2) \backslash \mathcal{S}_1) \times (\Gamma_1(2) \backslash \mathcal{S}_1) = B^\circ \times B^\circ \subset \mathfrak{M}_2$. Then $H^\circ \subset H \subset \mathfrak{M}_2^*$ where $H = B \times B (= \mathbf{P}^1 \times \mathbf{P}^1$ as an abstract surface). (It is not obvious that this is the compactification of H° in \mathfrak{M}_2^* but this turns out to be true.) Again $\text{Sp}(4, \mathbf{Z})$ acts transitively on $\{\Delta\}$, and there are 10 of these, so the $\{H_\Delta\}$ are all mutually isomorphic, and are isomorphic to H .

Finally, $H_\Delta \cap D_\ell$ is nonempty exactly when $\ell \in \Delta$, by which we mean $\ell \in \delta$ or $\ell \in \delta^\perp$. For fixed Δ this occurs for 6 values of ℓ , and the intersection is $\mathbf{P}^1 \times \text{cusp}$ or $\text{cusp} \times \mathbf{P}^1$ in H_Δ , and for fixed ℓ this occurs for 4 values of Δ , and the intersection is a section of the singular fiber space $D \rightarrow B$, these sections extending over the cusps.

We shall let $\mathcal{D} = \bigcup_\ell D_\ell$ and $\mathcal{H}_1 = \bigcup_\Delta H_\Delta$.

2. INVOLUTIONS IN $\text{PSp}(4, \mathbf{Z})$

We now consider involutions in $\text{PSp}(4, \mathbf{Z})$. By [10], every involution in $\text{PSp}(4, \mathbf{Z})$ is conjugate in $\text{PSp}(4, \mathbf{Z})$ to one of the following two:

$$j_1 = \pm \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad j_2 = \pm \begin{pmatrix} 1 & & & 1 \\ & -1 & -1 & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

(Since we are in $\mathrm{PSp}(4, \mathbf{Z})$, j_1 and j_2 are only defined up to sign. We will choose the positive sign to obtain representatives.) We observe that $j_1 \in \Gamma(2)$ but $j_2 \notin \Gamma(2)$.

Our first step is to obtain a finer classification.

We remind the reader of our conventions:

We let $V = \mathbf{Z}^4$ and $V_{\mathbf{Q}} = \mathbf{Z}^4 \otimes \mathbf{Q} = \mathbf{Q}^4$. We have the nonsingular symplectic form \langle , \rangle on $V_{\mathbf{Q}}$ preserved by $\mathrm{Sp}(4, \mathbf{Q})$. We regard V as a space of row vectors and we let $\mathrm{Sp}(4, \mathbf{Z})$ act on the left on V by $g(v) = vg^{-1}$.

Recall that a sublattice W of V is *pure* if $W = (W \otimes \mathbf{Q}) \cap V$, i.e., if $nw \in W$ for some $n \in \mathbf{Z}$, $n \neq 0$, implies $w \in W$.

DEFINITION 2.1. An ordered pair (W^1, W^2) (resp. a pair $\{W^1, W^2\}$) of sublattices of V is *\mathbf{Q} -nonsingular* if W^1 and W^2 are each pure sublattices of V of rank 2, the restrictions of \langle , \rangle to $W^1_{\mathbf{Q}} = W^1 \otimes \mathbf{Q}$ and to $W^2_{\mathbf{Q}} = W^2 \otimes \mathbf{Q}$ are each nonsingular, $W^1_{\mathbf{Q}}$ and $W^2_{\mathbf{Q}}$ are orthogonal with respect to \langle , \rangle , and hence $V_{\mathbf{Q}} = W^1_{\mathbf{Q}} \oplus W^2_{\mathbf{Q}}$. The *discriminant* of (W^1, W^2) (or $\{W^1, W^2\}$) is the cardinality of the quotient $V/(W^1 \oplus W^2)$.

THEOREM 2.2. *Up to conjugation by $\Gamma(2) \subset \mathrm{PSp}(4, \mathbf{Z})$:*

- (1) *The single conjugacy class of j_1 in $\mathrm{PSp}(4, \mathbf{Z})$ splits into 10 conjugacy classes $j_{1,\Delta}$ naturally indexed by nonsingular pairs Δ .*
- (2) *The single conjugacy class of j_2 in $\mathrm{PSp}(4, \mathbf{Z})$ splits into 15 conjugacy classes $j_{2,h}$ naturally indexed by isotropic planes h .*

PROOF. Our analysis here follows the analysis in [2], except that here we are in the principally polarized case, so that the distinction between “short” and “long” vectors in [2] no longer exists.

By [2, Lemma 5.2] there is a 1-1 correspondence

$$\{\alpha \in \mathrm{Sp}(4, \mathbf{Q}) \mid \alpha^2 = 1, \alpha \neq \pm 1\} \leftrightarrow \{(V^+, V^-) \text{ a } \mathbf{Q}\text{-nonsingular ordered pair}\},$$

where $V^+ = V_{\mathbf{Q}}^+ \cap V$ and $V^- = V_{\mathbf{Q}}^- \cap V$, given by

$$\alpha \leftrightarrow (V_{\mathbf{Q}}^+ = (+1) \text{ eigenspace of } \alpha, V_{\mathbf{Q}}^- = (-1) \text{ eigenspace of } \alpha),$$

and furthermore by [2, Proposition 5.9] this correspondence is equivariant in the sense that for any α and for any $g \in \mathrm{Sp}(4, \mathbf{Q})$,

$$(V_{\mathbf{Q}}^+(g(\alpha)), V_{\mathbf{Q}}^-(g(\alpha))) = (g(V_{\mathbf{Q}}^+(\alpha)), g(V_{\mathbf{Q}}^-(\alpha)))$$

where $g(\alpha) = g\alpha g^{-1}$ and $g(V_{\mathbf{Q}}^{\pm}(\alpha)) = V_{\mathbf{Q}}^{\pm}(g(\alpha))g^{-1}$.

Then we immediately obtain a 1-1 correspondence

$$\{\alpha \in \text{PSp}(4, \mathbf{Q}) \mid \alpha^2 = 1, \alpha \neq \pm 1\} \leftrightarrow \{\{V^+, V^-\} \text{ a } \mathbf{Q}\text{-nonsingular pair}\}$$

as the action of $-I \in \text{Sp}(4, \mathbf{Q})$ interchanges the $(+1)$ and (-1) eigenspaces.

Specializing to $\alpha \in \text{PSp}(4, \mathbf{Z})$, we see from [2, Proposition 5.11] that there are two possibilities for the quotient $V/(V^+(\alpha) \oplus V^-(\alpha))$: either this quotient is $\{0\}$, in which case $\{V^+(\alpha), V^-(\alpha)\}$ has discriminant 1, or this quotient is isomorphic to $(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z})$, in which case $\{(V^+(\alpha), V^-(\alpha))\}$ has discriminant 4. Clearly the discriminant of $\{(V^+(\alpha), V^-(\alpha))\}$ is an invariant of the conjugacy class of α in $\text{PSp}(4, \mathbf{Z})$.

In particular we observe that for the involution j_1 ,

$$V^+(j_1) = (1, 0, 0, 0) \wedge (0, 0, 1, 0) \quad \text{and} \quad V^-(j_1) = (0, 1, 0, 0) \wedge (0, 0, 0, 1),$$

so $\{V^+(j_1), V^-(j_1)\}$ has discriminant 1, and for the involution j_2 ,

$$V^+(j_2) = (2, 0, 0, 1) \wedge (0, 0, 1, 0) \quad \text{and} \quad V^-(j_2) = (0, 2, 1, 0) \wedge (0, 0, 0, 1),$$

so $\{V^+(j_2), V^-(j_2)\}$ has discriminant 4.

Thus we see that the classification of involutions in $\text{PSp}(4, \mathbf{Z})$ up to conjugacy is the same as the classification of \mathbf{Q} -nonsingular pairs of discriminants 1 and 4 of V up to linear transformation.

Following the argument of [2, Proposition 5.22] we may show that there is an element g of $\text{PSp}(4, \mathbf{Z})$ that takes any \mathbf{Q} -nonsingular pair of discriminant 1 to $\{(V^+(j_1), V^-(j_1))\}$ and any \mathbf{Q} -nonsingular pair of discriminant 4 to $\{V^+(j_2), V^-(j_2)\}$, merely recovering Ueno’s result. But of course we are interested here in the finer classification up to the action of $\Gamma(2)$. Now for any vector $(x_1, x_2, x_3, x_4) \in V$, if $(y_1, y_2, y_3, y_4) = (x_1, x_2, x_3, x_4)g^{-1}$, then $y_i \equiv x_i \pmod{2}$. Thus a necessary condition for two involutions α and α' of $\text{PSp}(4, \mathbf{Z})$ to be equivalent under conjugation by $\Gamma(2)$ is that we must have $V^\pm(\alpha) \equiv V^\pm(\alpha') \pmod{2}$, where by this congruence we mean that we must be able to choose a basis for each of these lattices so that the vectors in the basis are congruent $\pmod{2}$. But the proof of [2, Proposition 5.22] shows that this necessary condition is sufficient as well.

Examining $\{V^+(j_1), V^-(j_1)\}$ we see that these subspaces reduced mod 2 form a nonsingular pair, so the conjugates of $j_1 \pmod{\Gamma(2)}$ are in 1-1 correspondence with nonsingular pairs Δ , i.e., are appropriate elements $j_{1,\Delta}$ for each Δ .

Examining $\{V^+(j_2), V^-(j_2)\}$ we see that these two subspaces reduce to the same subspace mod 2, and that this subspace is an isotropic subspace h , so that conjugates of $j_2 \pmod{\Gamma(2)}$ are in 1-1 correspondence with isotropic subspaces h , i.e., are appropriate elements $j_{2,h}$ for each h .

Finally, we have already given the cardinalities of $\{\Delta\}$ and for $\{h\}$, and indeed they are listed explicitly in [7], but let us indicate how to count the elements of these sets anyway. Let $V_{\mathbf{Z}/2\mathbf{Z}} = (\mathbf{Z}/2\mathbf{Z})^4$ and note that $\text{PSp}(4, \mathbf{Z}/2\mathbf{Z}) =$

$\mathrm{Sp}(4, \mathbf{Z}/2\mathbf{Z})$ acts transitively on $\{\ell\} = V_{\mathbf{Z}/2\mathbf{Z}} - \{(0, 0, 0, 0)\}$, a set of cardinality $2^4 - 1 = 15$.

Consider $\ell_0 = (1000)$. Let $\Delta = \{\delta, \delta^\perp\}$. If $\ell \in \Delta$, then $\ell \in \delta$ or δ^\perp . For the sake of definiteness, let $\ell \in \delta$. Then δ must have a unique vector of the form $(0, x_2, 1, y_2)$ with x_2 and y_2 arbitrary, and there are four choices, so we count a total of $15 \cdot 4 = 60$ lines in all the Δ 's. (Note δ determines δ^\perp so there are no further choices.) But each plane δ and δ^\perp contains 3 lines, so each Δ contains 6 lines and then there are $60/6 = 10$ Δ 's. On the other hand, if $\ell \in h$, then h must have a unique nonzero vector of the form $(0, x_2, 0, y_2)$, and there are three choices, so we count a total of $15 \cdot 3 = 45$ lines in all the h 's. But each plane h contains 3 lines and then there are $45/3 = 15$ h 's. \square

Following the notation of the proof of Theorem 2.2, we make the following definition.

DEFINITION 2.3. The *discriminant* of an involution $\alpha \in \mathrm{PSp}(4, \mathbf{Z})$, $\alpha \neq \pm 1$, is the discriminant of the pair $\{V_+(\alpha), V_-(\alpha)\}$.

COROLLARY 2.4. Each involution $j_{1,\Delta}$ has discriminant 1 and each involution $j_{2,h}$ has discriminant 4.

3. DESCRIPTION OF THE COMPLEX SURFACES K_h

Our goal in this section is to define and describe the surfaces K_h in \mathfrak{M}_2^* that are our main focus of interest in the paper. But we begin by recalling some properties of the surfaces H_Δ that we have already seen.

THEOREM 3.1. Fix a nonsingular pair Δ , and let $j = j_{1,\Delta}$ be the associated involution of discriminant 1. The action of j on \mathcal{S}_2 fixes (pointwise) a Humbert surface of discriminant 1. The compactification of its image in \mathfrak{M}_2^* , denoted H_Δ , has the following properties:

- (1) $H_\Delta \cap D_\ell$ is nonempty if and only if $\ell \in \Delta$ (i.e., if and only if $\ell \in \delta$ or $\ell \in \delta^\perp$, where $\Delta = \{\delta, \delta^\perp\}$), in which case $H_\Delta \cap D_\ell$ is a section of B_ℓ . For fixed Δ , this is the case for 6 values of ℓ , and for fixed ℓ this is the case for 4 values of Δ . If ℓ is fixed, these 4 sections are the images of the points of order 1 or 2 in each fiber.
- (2) $H_\Delta \cap C_h$ is nonempty if and only if there is some ℓ with $\ell \in \Delta$ and $\ell \in h$, in which case $H_\Delta \cap C_h$ is a single point. For fixed Δ , this is the case for 9 values of h , and for fixed h this is the case for 6 values of Δ .

PROOF. This is well known, and we are merely stating it for completeness and ease of reference, and also for comparison with Theorem 3.2. But we shall make a few observations.

We observe that if we choose

$$j = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

then $j \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = \begin{pmatrix} \tau_1 & -\tau_2 \\ -\tau_2 & \tau_3 \end{pmatrix}$, so j fixes $\left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \right\}$, which by [1, Definition 3.1.7], is a Humbert surface of discriminant 1.

We are claiming in the statement of the theorem that, in case $\ell \in \Delta$, not only is $H_\Delta \cap D_\ell^\circ$ a section over B_ℓ° , but that this section extends over the cusps. This is a familiar fact for elliptic modular surfaces, cf. [9], and the situation here is analogous.

We refer the reader to [6] or [8] for a picture of this situation. □

THEOREM 3.2. *Fix an isotropic plane h and let $j = j_{2,h}$ be the associated involution of discriminant 4. The action of j on the three-fold \mathfrak{M}_2^* fixes (pointwise) a complex surface K_h , a Humbert surface of discriminant 4. This surface K_h has the following properties:*

- (1) *If $\ell \in h$ (which occurs for three values of ℓ), $K_h \cap D_\ell$ is a double section of B_ℓ , i.e., a branched double cover of the base curve B_ℓ .*
 - (a) *The intersection of K_h with each general fiber is two points.*
 - (b) *Let $h = \{\ell_1, \ell_2, \ell_3\}$ with $\ell_1 = \ell$. In the exceptional fiber $D_\ell \cap C_h = s_{\ell_1, \ell_2} \cup s_{\ell_1, \ell_3}$, K_h intersects each of the \mathbf{P}^1 's s_{ℓ_1, ℓ_2} and s_{ℓ_1, ℓ_3} in a single point.*
 - (c) *Let $\ell \in \{h_1, h_2, h_3\}$ with $h_1 = h$. In the exceptional fibers $D_\ell \cap C_{h_2}$ and $D_\ell \cap C_{h_3}$, K_h passes through the deepest point that is the intersection of the two \mathbf{P}^1 's in that exceptional fiber. Thus these two points are the branch points of the double cover $K_h \cap D_\ell \rightarrow B_\ell$.*
 - (d) *$K_h \cap (D_\ell \cap \mathcal{H}_1)$ is empty.*
- (2) *If $h = \{\ell_1, \ell_2, \ell_3\}$, then the intersection $K_h \cap C_h$ is three points, one in each of the \mathbf{P}^1 's s_{ℓ_1, ℓ_2} , s_{ℓ_1, ℓ_3} , and s_{ℓ_2, ℓ_3} in C_h . Also, $K_h \cap (C_h \cap \mathcal{H}_1)$ is empty.*
- (3) *Let h' be nearby h . Then if $\ell \in h$, $h' = \{\ell, \ell', \ell''\}$ for some lines ℓ', ℓ'' . The intersection $K_h \cap C_{h'}$ is the entire \mathbf{P}^1 $s_{\ell', \ell''}$. (Note that this \mathbf{P}^1 contains a deepest point as in 1(c).) Also, $K_h \cap (C_{h'} \cap \mathcal{H}_1)$ is two points in this \mathbf{P}^1 . There are six such values of h' .*
- (4) *Let ℓ' be nearby ℓ . Then $\ell' \in h'$ for some h' nearby h . The intersection $K_h \cap D_{\ell'}$ is a \mathbf{P}^1 $s_{\ell', \ell''}$ in an exceptional fiber over $B_{\ell'}$. Also, $K_h \cap (D_{\ell'} \cap \mathcal{H}_1)$ is two points in this \mathbf{P}^1 . (These intersections are the same intersections as in (3).) There are six such values of ℓ' .*
- (5) *If $\ell'' \notin h$ and $\ell'' \notin h'$ for any h' nearby h , then $K_h \cap D_{\ell''}$ is empty. If $h'' \neq h$ and h'' is not nearby h , then $K_h \cap C_{h''}$ is empty. There are eight such values of ℓ'' and eight such values of h'' .*

PROOF. Since all the involutions $j_{2,h}$ are congruent under the action of $\Gamma(1) = \text{Sp}(4, \mathbf{Z})$, there is an automorphism of \mathfrak{M}_2^* taking any surface K_h to any other,

so it suffices to prove this for a single value of h . We first take $h = h_1 = (0010) \wedge (0001)$ and then

$$j = j_{2,h} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Explicit computation then shows that

$$j \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = \begin{pmatrix} \tau_1 & 1 - \tau_2 \\ 1 - \tau_2 & \tau_3 \end{pmatrix} \quad \text{for } \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathcal{S}_2$$

so j fixes the surface

$$\left\{ \begin{pmatrix} \tau_1 & 1/2 \\ 1/2 & \tau_3 \end{pmatrix} \right\} \subset \mathcal{S}_2,$$

obviously a complex surface, and from [1, Definition 3.1.7] we see that this is a Humbert surface of discriminant 4. Then j descends to an involution on \mathfrak{M}_2 fixing the image of this surface, which we denote by $K = K_h$, under the projection $\mathcal{S}_2 \rightarrow \mathfrak{M}_2 = \Gamma(2) \backslash \mathcal{S}_2$. Furthermore, the compactification process is equivariant so the action of j extends to an action on \mathfrak{M}_2^* .

A key point to note is that the action of j is equivariant with respect to the indexing: For any corank 1 boundary component ℓ , the image of D_ℓ under j is $D_{j(\ell)}$, and similarly for corank 2 boundary components. In particular, we first choose $\ell = \ell_1 = (0010)$, and then we have that $j(\ell) = \ell$, so j leaves D_ℓ invariant (though certainly not pointwise fixed), and also $j(h) = h$, so j leaves C_h invariant (though again not pointwise fixed).

Now \mathfrak{M}_2^* is a nonsingular complex variety, and j acts smoothly, indeed analytically, on \mathfrak{M}_2^* , and hence any component of the fixed point set of j , or of the restriction of j to D_ℓ , will be a smooth, and indeed analytic subvariety.

The analysis of j on D_ℓ° is not difficult, as the toroidal compactification process for a corank 1 open boundary component is relatively straightforward. Recall our discussion in Section 1. Very roughly speaking, if P is the subgroup of $\Gamma(2)$ stabilizing a neighborhood N of the inverse image in \mathcal{S}_2 of this boundary component, which we can think of as $\left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mid \text{Im}(\tau_1) \gg 0 \right\}$, then we can think of this boundary component as $\left\{ \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} i\infty & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \right\}$ and $\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in P' \backslash N$ approaches $\begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} \in \mathbf{C} \times \mathcal{S}_1$ by letting τ_1 approach $i\infty$, where P' is an appropriate subgroup of P . We then take a further quotient to obtain a neighborhood of D_ℓ° in \mathfrak{M}_2^* . In this description, the action of j on \mathfrak{M}_2 extends to the action on D_ℓ° given by $\begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 - \tau_2 \\ \tau_3 \end{pmatrix}$, where here we are looking at representatives, since D_ℓ° is a quotient of $\mathbf{C} \times \mathcal{S}_1$, and so we see that points represented by $\begin{pmatrix} 1/2 \\ \tau_3 \end{pmatrix}$ are fixed, as are the points represented by $\begin{pmatrix} 1/2 + \tau_3 \\ \tau_3 \end{pmatrix}$, since the image of this point, $\begin{pmatrix} 1/2 - \tau_3 \\ \tau_3 \end{pmatrix}$, is equivalent to it modulo $2\mathbf{Z} + 2\tau_3\mathbf{Z}$. This can all be made absolutely precise, but we prefer

not to do so here, for reasons of brevity, and because this description, while enlightening, is mostly superfluous to our needs. But see [2, Proposition 3.102].

However, the toroidal compactification process for corank 2 boundary components is far from straightforward, and an analysis of j there would involve not only a careful development of that compactification but also a careful local analysis of j as well. (The compactification process is thoroughly described in [2] and the reader can see what is involved.)

Thus instead we choose an approach that avoids almost any sort of local analysis, whether for corank 1 or corank 2 boundary components.

Our approach is based on the following very simple topological fact, which we will use repeatedly: An orientation preserving involution on the two-sphere S^2 fixes either exactly two points or all of S^2 . We call this Fact I.

Let us now get to work. We could proceed with our choice of h (and hence $j = j_{2,h}$) and ℓ as above, but we shall instead change to $h = h_1 = (1000) \wedge (0100)$, giving

$$j = j_{2,h} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

and $\ell = (1000)$. This is purely for convenience (for ourselves and for the reader). These were the choices of ℓ and h we made in [6], and this enables us to use the indexing in the figures in that paper. As the reader will see, our arguments heavily use the indexing, and so if our indexing here were different than our indexing there, everyone would become hopelessly confused. (We did not begin with this choice of ℓ and h as we wished to begin with an element j of $\text{Sp}(4, \mathbf{Z})$ whose action on \mathcal{S}_2 was transparent, as was our original action, while this new value of j has the lower left hand 2-by-2 block nonzero, a situation everyone who works on Siegel space tries to avoid whenever possible, for good reason.) Again we set $K = K_h$ for this value of h .

To accompany the following argument we refer the reader to [6, Figure 1].

We begin by noting that j leaves each fiber of D_ℓ° invariant. (This is the *only* fact from the above description of the corank 1 compactification process that we need.) We have observed that j is equivariant with respect to the indexing, and direct computation shows that, in the notation of [6], j interchanges Δ_1 and Δ_2 , and also Δ_3 and Δ_4 . Thus $j(H_{\Delta_1} \cap D_\ell) = H_{\Delta_2} \cap D_\ell$, etc. In particular, since each $H_{\Delta_i} \cap D_\ell$ intersects every general fiber, and the curves $H_{\Delta_i} \cap D_\ell$ are pairwise disjoint, no general fiber is entirely fixed. Hence by Fact I, the intersection of K with every general fiber of D_ℓ° is two points, i.e., $K \cap D_\ell^\circ$ is a double section, unbranched over B_ℓ° .

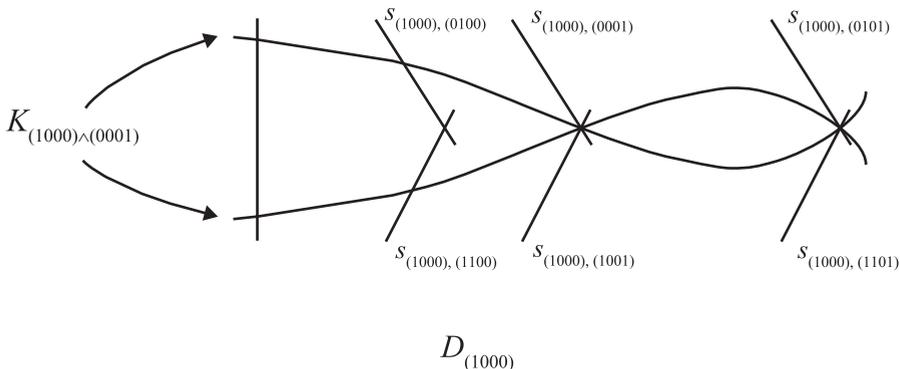
Now to analyze the exceptional fibers. We note that in the exceptional fiber indexed by h , the equivariance of the indexing shows that j leaves each of $s_{1000,0100}$ and $s_{1000,1100}$, the two \mathbf{P}^1 's in $C_h \cap D_\ell$, invariant, and hence their intersection, which is a single point, invariant and hence fixed.

Now neither of these two \mathbf{P}^1 's is pointwise fixed, again because each of them is intersected by a pair of H_Δ 's that are interchanged, so by Fact I the action of j must fix exactly one other point in each, and so $K_h \cap s_{1000,0100}$ and $K_h \cap s_{1000,1100}$ are each a single point. Hence $K_h \cap D_\ell \rightarrow B_\ell$ is not branched here.

Now for the other two exceptional fibers. In each of these, j interchanges the two \mathbf{P}^1 's: $s_{1000,0001}$ and $s_{1000,1001}$, and also $s_{1000,0101}$ and $s_{1000,1101}$. So the only fixed point of j on each of these fibers is the single intersection point $s_{1000,0001} \cap s_{1000,1001}$ and $s_{1000,0101} \cap s_{1000,1101}$ on each. K_h must pass through each of these points (and no other points in these exceptional fibers). Hence $K_h \cap D_\ell \rightarrow B_\ell$ is branched at both of these points.

Finally, since no $H_{\Delta_i} \cap D_\ell$ contains any fixed point of j , it cannot intersect $K_h \cap D_\ell$.

Thus we have proved all parts of (1). For the further edification of the reader, we draw the schematic diagram of $K_h \cap D_\ell$ that we have just shown to be true.



Next let us deal with the corank 2 boundary component C_h . This consists of three \mathbf{P}^1 's meeting at a single point. They are s_{ℓ_1, ℓ_2} , s_{ℓ_1, ℓ_3} , and s_{ℓ_2, ℓ_3} , where ℓ_1, ℓ_2 , and ℓ_3 are the three lines in h , $\ell_1 = (1000)$, $\ell_2 = (0100)$, $\ell_3 = (1100)$. In our analysis of D_{ℓ_1} , we have just found $K_h \cap s_{\ell_1, \ell_2}$ and $K_h \cap s_{\ell_1, \ell_3}$. They are each a single point. Performing the same analysis for D_{ℓ_2} would show that $K_h \cap s_{\ell_1, \ell_2}$ is a single point (again), and that $K_h \cap s_{\ell_2, \ell_3}$ is a single point (and then the analysis for D_{ℓ_3} would again recover the two intersection points $K_h \cap s_{\ell_1, \ell_3}$ and $K_h \cap s_{\ell_2, \ell_3}$). Furthermore, we have already seen that none of these points lie on any H_Δ . Thus we have proven (2).

Now let us consider a nearby cusp component $C_{h'}$. We take $h' = (1000) \wedge (0001)$ so that the three lines in h' are $\ell_1 = (1000)$, $\ell_4 = (0001)$, and $\ell_5 = (1001)$. Then $j(\ell_1) = \ell_1$, $j(\ell_4) = \ell_5$, and $j(\ell_5) = \ell_4$. Hence j interchanges the \mathbf{P}^1 's s_{ℓ_1, ℓ_4} and s_{ℓ_1, ℓ_5} , and since they are disjoint except for the deepest point, no point other than possibly the deepest point can be in the fixed set K_h . Now by the same argument, $j(s_{\ell_4, \ell_5}) = s_{\ell_4, \ell_5}$ so this \mathbf{P}^1 is invariant. We must analyze the action on this \mathbf{P}^1 . We know already that the deepest point on this \mathbf{P}^1 is fixed as this is one of the branch points of $K_h \cap D_\ell \rightarrow B_\ell$ that we have seen already. Now s_{ℓ_4, ℓ_5} is inter-

sected by H_Δ for two values of Δ , namely $\Delta = \Delta' = \{(1110) \wedge (0001), (1001) \wedge (0011)\}$ and $\Delta = \Delta'' = \{(0110) \wedge (0001), (1001) \wedge (0010)\}$. But $j(\Delta') = \Delta'$ and $j(\Delta'') = \Delta''$, so j also fixes the two points $s_{\ell_4, \ell_5} \cap H_{\Delta'}$ and $s_{\ell_4, \ell_5} \cap H_{\Delta''}$ of s_{ℓ_4, ℓ_5} . Thus j fixes at least three points of s_{ℓ_4, ℓ_5} so by Fact I j must leave s_{ℓ_4, ℓ_5} pointwise fixed, proving (3).

As for (4), we take $\ell' = \ell_4$, so $j(\ell_4) = \ell_5 = \ell''$. Then any point in $D_{\ell'}$ fixed by j must be in $D_{\ell'} \cap j(D_{\ell'}) = D_{\ell'} \cap D_{\ell''} = s_{\ell', \ell''}$. But we have just seen in (3) that in fact this entire \mathbf{P}^1 is fixed.

As for (5), for any value of ℓ'' other than those considered above, $D_{\ell''} \cap j(D_{\ell''}) = D_{\ell''} \cap D_{j(\ell'')}$ is empty, so $D_{\ell''}$ cannot contain any point of the fixed set K_h . For any value of h'' other than those considered above, $j(h'') \neq h''$, and since the corank 2 boundary components are pairwise disjoint, we certainly have that $C_{h''} \cap j(C_{h''}) = C_{h''} \cap C_{j(h'')}$ is empty, so $C_{h''}$ cannot contain any point of the fixed set K_h , completing the proof. □

As we have observed, D_ℓ is a Kummer modular surface. The fiber over a general point $\tau \in B_\ell^\circ$ (more precisely, in the equivalence class of $\tau \in \mathcal{S}_1$ under the action of $\Gamma_1(2)$) is the quotient of the elliptic curve $\mathbf{C}/(2\mathbf{Z} \oplus 2\tau\mathbf{Z})$ by the involution $z \mapsto -z$. This involution has four fixed points, the points of order 1 or 2. There are $12 = 4^2 - 4$ points of order 4 on this elliptic curve, and they are interchanged pairwise by this involution, so their images in the quotient are six distinct points. An analysis of the degeneration in the exceptional fibers over the cusps in B_ℓ shows that, in each cusp, four of these points remain distinct while the other two “collapse” into the deepest point.

COROLLARY 3.3. *Fix a line ℓ and let h_1, h_2, h_3 be the three isotropic planes with $\ell \in h_i, i = 1, 2, 3$. Then $K_{h_1} \cap D_\ell^\circ, K_{h_2} \cap D_\ell^\circ$, and $K_{h_3} \cap D_\ell^\circ$ together form a six-section over B_ℓ° consisting of the six points in each fiber over a general point τ that are the images of the points of order 4 in $\mathbf{C}/(2\mathbf{Z} \oplus 2\tau\mathbf{Z})$ under the involution $z \mapsto -z$. Consequently, the intersection of the curves $K_{h_1} \cap D_\ell$ and $K_{h_2} \cap D_\ell$ is a single point, a deepest point in an exceptional fiber that is the branch point of the double cover $K_{h_i} \cap D_\ell \rightarrow B_\ell$ that they have in common.*

PROOF. We see immediately from the proof of Theorem 3.2 that, in the notation of that proof, setting $h_1 = h, K_{h_1} \cap D_\ell^\circ = K_h \cap D_\ell^\circ$ contains the point that is the image of $1/2 \in \mathbf{C}/(2\mathbf{Z} \oplus 2\tau\mathbf{Z})$ in each general fiber, and that is a point of order 4. It is easy to see that it contains the image of $1/2 + \tau$ as well. Then, with proper ordering, $K_{h_2} \cap D_\ell^\circ$ contains the image of $\tau/2$ and $1 + \tau/2$, and $K_{h_3} \cap D_\ell^\circ$ contains the image of $1/2 + \tau/2$ and $3/2 + \tau/2$, and these are all six such points. Consequently, $\{K_{h_i} \cap D_\ell^\circ\}$ are pairwise distinct, so the only possible intersections of $K_{h_i} \cap D_\ell$ with $K_{h_j} \cap D_\ell, i \neq j$, occur in exceptional fibers.

But similarly, this can only occur if $K_{h_i} \cap D_\ell$ and $K_{h_j} \cap D_\ell$ both pass through a deepest point in an exceptional fiber, and that only occurs at the unique branch point they both have in common. □

REMARK 3.4. This corollary provides another proof that $K_h \cap D_\ell^\circ$ and $H_\Delta \cap D_\ell^\circ$ are disjoint for $\ell \in h$ and $\ell \in \Delta$, as $H_\Delta \cap D_\ell^\circ$ is a point of order 1 or 2 in every

fiber $\mathbf{C}/(2\mathbf{Z} + 2\tau\mathbf{Z})$ and that is distinct from a point of order 4. Then a local analysis around the cusps shows that $K_h \cap D_\ell$ and $H_\Delta \cap D_\ell$ remain disjoint.

REMARK 3.5. K_h is not the entire fixed point set of the involution $j_{2,h}$. As we see from the proof of Theorem 3.2, $j_{2,h}$ also has the deepest point of C_h as an isolated fixed point.

The relationship between Δ and h in Theorem 3.2(2) will turn out to be important to us, and so we make an explicit definition.

DEFINITION 3.6. Let Δ be a nonsingular pair and let h be an isotropic plane. We write $\Delta \sim h$ if there is a line ℓ with $\ell \in \Delta$ and $\ell \in h$, and $\Delta \not\sim h$ otherwise.

Having explicitly found the curve $K_h \cap D_\ell$ in D_ℓ for $\ell \in h$, it is natural to ask for its homology class.

The analog of the following lemma is true for any line ℓ and for any three distinct isotropic planes h_1, h_2, h_3 with $\ell \in h_i, i = 1, 2, 3$. We state it in this one case for convenience.

LEMMA 3.7. Let $h_1 = (1000) \wedge (0100), h_2 = (1000) \wedge (0001),$ and $h_3 = (1000) \wedge (0101)$. Then the intersections $K_{h_1} \cap D_{(1000)}, K_{h_2} \cap D_{(1000)},$ and $K_{h_3} \cap D_{(1000)}$ are all homologous in $D_{(1000)}$. The homology class $u_{(1000)}$ they represent is given by

$$u_{(1000)} = -t_{(1000)} + s_{(1000),(0100)} + s_{(1000),(0001)} + s_{(1000),(0101)} + 2m_{(1000)\wedge(0010)}.$$

Also, this class has self-intersection number $u_{(1000)}^2 = 1$ in $D_{(1000)}$. Here $t_{(1000)}$ denotes the class of the general fiber of $D_{(1000)} \rightarrow B_{(1000)}$.

PROOF. First we remind the reader that $m_{(1000)\wedge(0010)} = H_{\Delta_1} \cap D_{(1000)}$ where $\Delta_1 = \{(1000) \wedge (0010), (0100) \wedge (0001)\}$.

We begin by recalling [6, Proposition 2.3.1]: $H_2(D_{(1000)}; \mathbf{Z})$ has basis $\{m_{(1000)\wedge(0010)}, s_{(1000),(0100)}, s_{(1000),(0001)}, s_{(1000),(0101)}, s_{(1000)\wedge(1100)}\}$ and each of these classes has self-intersection number -1 . We have that $t_{(1000)} = s_{(1000),(0100)} + s_{(1000),(1100)}$ and so we also have the basis $\{m_{(1000)\wedge(0010)}, s_{(1000),(0100)}, s_{(1000),(0001)}, s_{(1000),(0101)}, t_{(1000)}\}$ and it will be convenient for us to use this latter basis. We note $t_{(1000)}^2 = 0$.

Now consider the involution

$$j = j_{2,h_1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

This involution fixes K_{h_1} and leaves $D_{(1000)}$ invariant. Thus the restriction of j to $D_{(1000)}$, which we denote by i , is an involution of $D_{(1000)}$ fixing $u = K_{h_1} \cap D_{(1000)}$.

We have the induced map on homology. Hence $i_* : H_2(D_{(1000)}; \mathbf{Q}) \rightarrow H_2(D_{(1000)}; \mathbf{Q})$, with $i_*(u) = u$ in homology. Now we can easily compute i_* . The action of i_* is given by its action on the subscripts:

$$\begin{aligned} i_*(m_{(1000)\wedge(0010)}) &= m_{(1000)\wedge(0110)} \\ i_*(s_{(1000),(0100)}) &= s_{(1000),(0100)} \\ i_*(s_{(1000),(0001)}) &= s_{(1000),(1001)} = t_{(1000)} - s_{(1000),(0001)}. \\ i_*(s_{(1000),(0101)}) &= s_{(1000),(1101)} = t_{(1000)} - s_{(1000),(0101)} \\ i_*(t_{(1000)}) &= t_{(1000)}. \end{aligned}$$

We need to compute the homology class $m_{(1000)\wedge(0110)}$ and this was done in the proof of [6, Proposition 2.3.1]:

$$m_{(1000)\wedge(0110)} = m_{(1000)\wedge(0010)} - t_{(1000)} + s_{(1000),(0001)} + s_{(1000),(0101)}.$$

(The classes $m_{(1000)\wedge(0010)}$ and $m_{(1000)\wedge(0110)}$ were denoted in [6] by H_{Δ_1} and H_{Δ_2} respectively. As we remarked in [8, Remark 2.7], the statement and the proof of [6, Proposition 2.3.1] are correct but the proof contained a misprint in that the term $s_{(1000),(0001)}$ was inadvertently omitted.)

Since $i_*(u) = u$, u must be in the $+1$ eigenspace of i_* , and computation with the above basis shows that the $+1$ eigenspace of i_* is 3 dimensional with basis $\{t_{(1000)}, s_{(1000),(0100)}, s_{(1000),(0001)} + s_{(1000),(0101)} + 2m_{(1000)\wedge(0010)}\}$.

Thus

$$u = \alpha t_{(1000)} + \beta s_{(1000),(0100)} + \gamma (s_{(1000),(0001)} + s_{(1000),(0101)} + 2m_{(1000),(0010)}),$$

with the coefficients yet to be determined. We determine these coefficients by taking intersection numbers:

$$\begin{aligned} 2 &= u \cdot t_{(1000)} = \alpha(0) + \beta(0) + \gamma(0 + 0 + 2(+1)) \\ 1 &= u \cdot s_{(1000),(1100)} = \alpha(0) + \beta(1) + \gamma(0 + 0 + 2(0)) \\ 0 &= u \cdot m_{(1000)\wedge(0010)} = \alpha(1) + \beta(1) + \gamma(1 + 1 + 2(-1)) \end{aligned}$$

with solution $\alpha = -1$, $\beta = 1$, $\gamma = 1$, yielding the expression for u given in the statement of the lemma.

A priori we should call this class $u_{h_1,\ell}$ (for $\ell = (1000)$) as it depends on h_1 . But we see that the expression we have obtained is symmetric in the cusps, so the classes $u_{h_1,\ell}$, $u_{h_2,\ell}$ and $u_{h_3,\ell}$ are all equal in homology, and so we are justified in just denoting this class by u_ℓ .

Finally, given our expression for u_ℓ and our knowledge of intersection and self-intersection numbers of the homology classes in this expression, it is routine to compute u_ℓ^2 . □

We have a project underway with J. W. Hoffman to analyze Humbert surfaces in Siegel modular thresholds. Humbert surfaces of square discriminant D^2 are

fixed points of involutions in $\mathrm{Sp}(4, \mathbf{Q})$, and their behavior in Siegel modular varieties $(\Gamma(N) \backslash \mathcal{S}_2)^*$ ($\Gamma(N)$ being the principal congruence subgroup of level N) is dependent on what common factors D and N have. Our work in [2] was not in the principally polarized case with full level structure, but rather in the case of $(1, p)$ polarization with level structure of canonical type *with p odd*, but the fact that $N = D = 2$ here in type II, while $D = 2$ is prime to $1 \cdot p = p$ there, is responsible for much of the difference in behavior. (The type I cases, with $D = 1$, are very analogous here and there.)

4. THE HOMOLOGY CLASS REPRESENTED BY K_h

Up until now, we have used the natural indexing from the symplectic group to index the various subvarieties of \mathfrak{M}_2^* , with the indexes transforming in the natural way under the action of $\mathrm{PSp}(4, \mathbf{Z}/2\mathbf{Z})$. But $\mathrm{PSp}(4, \mathbf{Z}/2\mathbf{Z})$ is isomorphic to the symmetric group S_6 , and indeed isomorphic by an (almost canonical) isomorphism. In the remainder of this paper we shall use the indexing coming from S_6 , as doing so makes it much easier to follow the various combinatorial arguments. The isomorphism between $\mathrm{PSp}(4, \mathbf{Z}/2\mathbf{Z})$ and S_6 was given in [7], and we now recall the results of that paper. We consider that S_6 acts on the set $\{1, 2, 3, 4, 5, 6\}$ by permutations in the natural way. We have [7, Theorem 4.5]:

THEOREM 4.1. *Let S_6 act on the set $\{1, 2, 3, 4, 5, 6\}$ in the natural way, and let $\mathrm{PSp}(4, \mathbf{Z}/2\mathbf{Z})$ act on $(\mathbf{Z}/2\mathbf{Z})^4$ in the natural way. Then there is an isomorphism $\eta : S_6 \rightarrow \mathrm{PSp}(4, \mathbf{Z}/2\mathbf{Z})$ inducing permutation isomorphisms of the following sets:*

- (a) $\{\text{monads } \{a\}\} \leftrightarrow \{\text{spreads of lines } \sigma\}$
- (b) $\{\text{duads } \{a, b\}\} \leftrightarrow \{\text{lines } \ell\}$
- (c) $\{\text{duadic synthemes } \{\{a, b\}, \{c, d\}, \{e, f\}\}\} \leftrightarrow \{\text{isotropic planes } h\}$
- (d) $\{\text{triadic synthemes } \{\{a, b, c\}, \{d, e, f\}\}\} \leftrightarrow \{\text{nonsingular pairs } \Delta\}$.

(The names of the objects on the left were introduced by Sylvester in 1844.) Here we understand distinct letters to refer to distinct elements of $\{1, \dots, 6\}$. Also, this permutation isomorphism reflects inclusions:

- (a) If $\ell \leftrightarrow \{i, j\}$ and $h \leftrightarrow \{\{a, b\}, \{c, d\}, \{e, f\}\}$, then $\{i, j\} = \{a, b\}, \{c, d\}$, or $\{e, f\}$ if and only if $\ell \in h$.
- (b) If $\ell \leftrightarrow \{i, j\}$ and $\Delta = \{\{a, b, c\}, \{d, e, f\}\}$, in which case, recalling that $\Delta = \{\delta, \delta^\perp\}$, so that $\delta = \{a, b, c\}$ and $\delta^\perp = \{d, e, f\}$, or vice-versa, then $\ell \in \Delta$ if and only if $\ell \in \delta$, i.e., $\{i, j\} \subset \{a, b, c\}$, or $\ell \in \delta^\perp$, i.e., $\{i, j\} \subset \{d, e, f\}$.
- (c) If $\Delta = \{\{a, b, c\}, \{d, e, f\}\}$ and $h = \{\{p, q\}, \{r, s\}, \{t, u\}\}$, then $\Delta \sim h$, i.e., there exists a line ℓ with $\ell \in \Delta$ and $\ell \in h$, if and only if there is an $\{i, j\}$ with $\{i, j\} = \{p, q\}, \{r, s\}$, or $\{t, u\}$, and $\{i, j\} \subset \{a, b, c\}$ or $\{i, j\} \subset \{d, e, f\}$.

This isomorphism is canonical up to renumbering of $\{1, \dots, 6\}$ (the obvious indeterminacy) and up to the outer automorphism of S_6 , but with respect to the

latter we have made the “right” choice. Otherwise, all the identifications given above would change.

Henceforth, for clarity, we will drop the commas and braces in our symmetric group indexing of lines, isotropic planes, and nonsingular pairs.

With this language in hand, we proceed.

We have the following theorem from [6]:

THEOREM 4.2. (1) *The homology groups $H_i(\mathfrak{M}_2^*; \mathbf{Z})$ are free Abelian of rank 1, 0, 16, 0, 16, 0, 1 for $i = 0, \dots, 6$.*

(2) *The classes $\{D_\ell\}$ (15 classes) and $\{H_\Delta\}$ (10 classes) span $H_4(\mathfrak{M}_2^*; \mathbf{Z})$.*

(3) *The classes $\{D_{\ell_1} \cap D_{\ell_2}\}$ and $\{D_\ell \cap H_\Delta\}$ span $H_2(\mathfrak{M}_2^*; \mathbf{Z})$.*

In that paper we further determined the homology groups $H_i(\mathfrak{M}_2^*; \mathbf{Q})$ as representation spaces of S_6 . Using the usual Young diagram notation, $H_4(\mathfrak{M}_2^*; \mathbf{Q}) = 2[6] + [51] + [42]$. (Here [6] is the trivial 1 dimensional representation, [51] has dimension 5, and [42] has dimension 9.) Also, the (permutation) representation of S_6 on the vector space with basis $\{D_\ell\}$ is $[6] + [51] + [42]$ and on $\{H_\Delta\}$ is $[6] + [42]$. Thus we see that the space of relations between $\{D_\ell\}$ and $\{H_\Delta\}$ in \mathfrak{M}_2^* , i.e., the kernel of the map

$$\bigoplus_{\ell} H_4(D_\ell; \mathbf{Q}) \oplus \bigoplus_{\Delta} H_4(H_\Delta; \mathbf{Q}) \rightarrow H_4(\mathfrak{M}_2^*; \mathbf{Q})$$

is a 9 dimensional vector space that is isomorphic to the irreducible representation [42] of S_6 . We will explicitly determine this kernel. In order to do so we need an explicit generating set for $H_2(\mathfrak{M}_2^*; \mathbf{Q})$. In fact, we can extract a generating set for $H_2(\mathfrak{M}_2^*; \mathbf{Z})$ from the classes given by the above theorem, but since we will only be using this set to compute intersection numbers with 4 dimensional classes, it is easiest to use a different generating set for $H_2(\mathfrak{M}_2^*; \mathbf{Q})$.

To that end, we introduce two new sorts of 2-dimensional homology classes.

DEFINITION 4.3. For a line ℓ , the homology class t_ℓ is the class of a general fiber of D_ℓ .

Note that any general fiber is homologous to the sum of the two \mathbf{P}^1 's in any exceptional fiber of D_ℓ .

Thus if $\ell = (ab)$ and $h = (ab, cd, ef)$, then $t_{(ab)} = s_{(ab), (cd)} + s_{(ab), (ef)}$.

DEFINITION 4.4. For a nonsingular pair $\Delta = \{\delta, \delta^\perp\}$, the homology class n_Δ is the sum $n_\Delta = m_\delta + m_{\delta^\perp}$ of generators of $H_2(H_\Delta; \mathbf{Z})$.

Thus if $\Delta = (abc, def)$, then $n_{(abc, def)} = m_{(abc)} + m_{(def)}$.

We have the following intersection numbers:

LEMMA 4.5. *The following intersection numbers are correct:*

- (a) $s_{(ab), (cd)} \cdot D_{(ef)} = 1$
- (b) $s_{(ab), (cd)} \cdot D_{(ab)} = -1$

- (c) $s_{(ab),(de)} \cdot H_{(abc,def)} = 1$
- (d) $m_{(abc)} \cdot D_{(de)} = 1$
- (e) $m_{(abc)} \cdot D_{(ab)} = 0$
- (f) $m_{(abc)} \cdot H_{(abc,def)} = -1$
- (g) all other intersection numbers of classes $s_{(-),(-)}$ or $m_{(-)}$ with classes $D_{(-)}$ or $H_{(-),(-)}$ are 0.

PROOF. This follows directly from the computations in [6] and is explicitly stated as [8, Lemma 2.10]. □

COROLLARY 4.6. *The following intersection numbers are correct:*

- (a) $t_{(ab)} \cdot D_{(ab)} = -2$
- (b) $t_{(ab)} \cdot H_{(abc,def)} = 1$
- (c) $n_{(abc,def)} \cdot D_{(ab)} = 1$
- (d) $n_{(abc,def)} \cdot H_{(abc,def)} = -2$
- (e) all other intersection numbers of classes $t_{(-)}$ or $n_{(-),(-)}$ with classes $D_{(-)}$ or $H_{(-),(-)}$ are 0.

PROOF. Some of these simply follow from the fact that the relevant intersections are a transverse point or empty, but in any case all of these follow from Lemma 4.5 and the equations (in homology) $t_{(ab)} = s_{(ab),(cd)} + s_{(ab),(ef)}$ and $n_{(abc,def)} = m_{(abc)} + m_{(def)}$. □

LEMMA 4.7. *The homology group $H_2(\mathfrak{M}_2^*; \mathbf{Q})$ has basis given by the 15 classes $\{t_\ell\}$ and the 1 class $\sum_\Delta n_\Delta$.*

PROOF. From Corollary 4.6 we have that $t_\ell \cdot D_\ell = -2$ and $t_\ell \cdot D_{\ell'} = 0$ for $\ell' \neq \ell$. Thus we immediately see that the classes $\{t_\ell\}$ are linearly independent. (The matrix with entries $t_\ell \cdot D_{\ell'}$ is a 15-by-15 matrix that is -2 times the identity matrix, which is nonsingular.) The class $\sum_\Delta n_\Delta$ is acted on trivially by S_6 , so we need only check its independence from $\sum_\ell t_\ell$, which is a generator of the 1 dimensional subspace of the vector space generated by $\{t_\ell\}$ that is acted on trivially by S_6 .

Again from Corollary 4.6 we have that $\sum_\ell t_\ell \cdot \sum_{\ell'} D_{\ell'} = 15(-2) = -30$ and $\sum_\Delta n_\Delta \cdot \sum_\Delta H_\Delta = 10(-2) = -20$. Furthermore, $\sum_\ell t_\ell \cdot \sum_\Delta H_\Delta = 15(4) = 60$, as each t_ℓ has intersection number 1 with 4 H_Δ 's, and $\sum_\Delta n_\Delta \cdot \sum_{\ell'} D_{\ell'} = 10(6) = 60$, as $n_\Delta = m_\delta + m_{\delta^\perp}$ and each m_δ has intersection number 1 with 3 $D_{\ell'}$'s. But then the matrix

$$\begin{bmatrix} -30 & 60 \\ 60 & -20 \end{bmatrix}$$

is nonsingular, completing the proof. □

Actually, in our computations of intersection numbers, we will always (except in the following lemma) be using the individual classes n_Δ , rather than just their

sum $\sum_{\Delta} n_{\Delta}$, as in almost all cases the only way to find intersection numbers with $\sum_{\Delta} n_{\Delta}$ is to find the individual intersection numbers with each n_{Δ} and then add them.

LEMMA 4.8. *The kernel of the map*

$$\bigoplus_{\ell} H_4(D_{\ell}; \mathbf{Q}) \oplus \bigoplus_{\Delta} H_4(H_{\Delta}; \mathbf{Q}) \rightarrow H_4(\mathfrak{M}_2^*; \mathbf{Q})$$

is spanned by

$$2(H_{(abc, def)} - H_{(abd, cef)}) - [(D_{(ad)} + D_{(bd)} + D_{(ce)} + D_{(cf)}) - (D_{(ac)} + D_{(bc)} + D_{(de)} + D_{(df)})]$$

as (a, b, c, d, e, f) runs over the permutations of $(1, 2, 3, 4, 5, 6)$.

PROOF. We know that this space is 9 dimensional and in fact as a representation space of S_6 is [42]. In this case it is automatically orthogonal to any other irreducible representation of S_6 , and in particular to the fixed class $\sum_{\Delta} n_{\Delta}$. Thus we need only to check orthogonality with all t_{ℓ} 's.

But it is easy to compute from Corollary 4.6 that $2(H_{(abc, def)} - H_{(abd, cef)})$ has intersection number -2 with each of the classes $t_{(ad)}$, $t_{(bd)}$, $t_{(ce)}$, $t_{(cf)}$, intersection number $+2$ with each of the classes $t_{(ac)}$, $t_{(bc)}$, $t_{(de)}$, $t_{(df)}$, and intersection number 0 with the remaining 7 t_{ℓ} 's. However, we immediately see from Corollary 4.6 that the class $(D_{(ad)} + D_{(bd)} + D_{(ce)} + D_{(cf)}) - (D_{(ac)} + D_{(bc)} + D_{(de)} + D_{(df)})$ has exactly the same intersection numbers with each of the t_{ℓ} 's. \square

REMARK 4.9. Since the representation of S_6 on $\{H_{\Delta}\}$ is $[6] + [42]$, any relation must lie in the subspace orthogonal to $[6]$, i.e., in the subspace where the sum of the coefficients is 0 , and we see that is the case. Since the representation of S_6 on $\{D_{\ell}\}$ is $[6] + [51] + [42]$, any relation must lie in the subspace orthogonal to $[6] + [51]$. This subspace is the permutation representation on spreads, i.e., the permutation representation on the individual letters $\{a, b, c, d, e, f\}$, and so the sum of the coefficients of the terms involving each of these letters must be 0 , and we see that this is the case as well.

Now we come to the consideration of the classes K_h .

DEFINITION 4.10. Let $K_h = \sum_{\ell} \alpha_{h, \ell} D_{\ell} + \sum_{\Delta} \beta_{h, \Delta} H_{\Delta}$. Then this expression for K_h has D -weight $A = \sum_{\ell} \alpha_{h, \ell}$ and H -weight $B = \sum_{\Delta} \beta_{h, \Delta}$.

A priori, the D -weight and H -weight depend on the particular expression for K_h , but in fact they do not. This follows from Lemma 4.8, where we see that every relation in $H_4(\mathfrak{M}_2^*)$ has both D -weight 0 and H -weight 0 . But independently of that lemma we have the following more precise result.

LEMMA 4.11. *For any fixed h , every expression $K_h = \sum_{\ell} \alpha_{h, \ell} D_{\ell} + \sum_{\Delta} \beta_{h, \Delta} H_{\Delta}$ has D -weight $A = 3$ and H -weight $B = 2$.*

PROOF. Consider a single expression as in the statement of the lemma. Note that the action of any element g of the symmetric group S_6 takes the coefficient $\alpha_{h,\ell}$ to $\alpha_{g(h),g(\ell)}$ and takes the coefficient $\beta_{h,\Delta}$ to $\beta_{g(h),g(\Delta)}$. In particular, g takes this expression for K_h to an expression for $K_{h'}$, $h' = g(h)$, of the same weight.

Now let us consider the images of this expression under all elements of the symmetric group, and add them. Since there are 15 different values of h , the stabilizer of any single value of h is a subgroup of index 15 and hence order 48 of S_6 . Similarly, the stabilizer of any single value of ℓ is a different subgroup of index 15 and hence order 48 of S_6 , and the stabilizer of any single value of Δ is a subgroup of index 10 and hence order 72 of S_6 . We obtain:

$$\begin{aligned} \sum_{g \in S_6} K_{g(h)} &= \sum_{g \in S_6} \alpha_{g(h),g(\ell)} D_{g(\ell)} + \sum_{g \in S_6} \beta_{g(h),g(\Delta)} H_{g(\Delta)} \\ 48 \left(\sum_{h'} K_{h'} \right) &= 48 \left(\sum_{\ell} \alpha_{h,\ell} \right) \left(\sum_{\ell} D_{\ell} \right) + 72 \left(\sum_{\Delta} \beta_{h,\Delta} \right) \left(\sum_{\Delta} H_{\Delta} \right) \\ &= 48A \left(\sum_{\ell} D_{\ell} \right) + 72B \left(\sum_{\Delta} H_{\Delta} \right). \end{aligned}$$

Let us first intersect this relation with a single general fiber t_{ℓ_0} . There are three values of h' for which $K_{h'}$ intersects $D_{\ell_0}^\circ$, and in each of these cases the intersection is a double section, i.e., intersects the general fiber in two points. Hence the left hand side is $48 \cdot 3 \cdot 2$. Now $D_{\ell_0}^\circ \cap D_{\ell}^\circ$ is empty for $\ell \neq \ell_0$, and we have shown in Corollary 4.6 that $D_{\ell_0} \cdot t_{\ell_0} = -2$, so that the first term on the right hand side is $48 \cdot 1 \cdot (-2)A$. There are four values of Δ for which H_{Δ} intersects $D_{\ell_0}^\circ$, and in each of these cases the intersection is a section, i.e., intersects the general fiber in a single point. Hence the second term on the right hand side is $72 \cdot 4 \cdot 1 \cdot B$.

Thus we obtain the equation

$$288 = -96A + 288B.$$

Let us now intersect this relation with a single class n_{Δ_0} in H_{Δ_0} , for some fixed Δ_0 . Recall that $n_{\Delta_0} = m_{\delta_0} + m_{\delta_0^\perp}$ where $\Delta_0 = \{\delta_0, \delta_0^\perp\}$, and the classes m_{δ_0} and $m_{\delta_0^\perp}$ are represented by curves that are entirely contained in $D = \bigcup_{\ell} D_{\ell}$.

We have seen in Theorem 3.1 that for fixed H_{Δ_0} , there are six values of h' for which $H_{\Delta_0} \cap K_{h'} \cap D$ is nonempty (in fact, for which $H_{\Delta_0} \cap K_{h'}$ is nonempty, but we do not need this), and this intersection is a single point. Moreover, since, for proper choice of ℓ , $m_{\delta_0} = D_{\ell} \cap H_{\Delta_0}$, and this intersection point is a transverse triple point of D_{ℓ} , H_{Δ_0} , and $K_{h'}$, we have an intersection number $1 = m_{\delta_0} \cap K_{h'}$. Similarly we have $1 = m_{\delta_0^\perp} \cap K_{h'}$, so $n_{\Delta_0} \cap K_{h'} = 2$ for these six values of h' , and $n_{\Delta_0} \cap K_{h'} = 0$ for the other nine values of h' . Thus the left hand side is $48 \cdot 6 \cdot 2$.

There are six values of ℓ for which $H_{\Delta_0} \cap D_{\ell}$ is nonempty. For three of these values we have $m_{\delta_0} \cap D_{\ell} = 1$ and $m_{\delta_0^\perp} \cap D_{\ell} = 0$ and for the other three values we have $m_{\delta_0} \cap D_{\ell} = 0$ and $m_{\delta_0^\perp} \cap D_{\ell} = 1$, so in any case $n_{\delta_0} \cap D_{\ell} = 1$ for these values

of ℓ and 0 otherwise (Lemma 4.5), so the first term on the right hand side is $48 \cdot 6 \cdot 1 \cdot A$.

Since the surfaces H_Δ are pairwise disjoint, the only possible contributions to $n_{\Delta_0} \cap H_\Delta$ occurs when $\Delta = \Delta_0$. But then $m_{\delta_0} \cap H_{\Delta_0} = m_{\delta_0^\perp} \cap H_{\Delta_0} = -1$ so $n_{\Delta_0} \cap H_{\Delta_0} = -2$ (Corollary 4.6). Thus the second term on the right hand side is $72 \cdot 1 \cdot 1(-2)B$.

Thus we obtain the equation

$$576 = 288A - 144B.$$

Solving this pair of linear equations yields $A = 3, B = 2$. □

Guided by Lemma 4.11, we look for an expression for K_h which reflects the geometry of \mathfrak{M}_2^* . We note that $\ell \in h$ for three values of ℓ , while $\Delta \sim h$ for six values of Δ and $\Delta \not\sim h$ for four values of Δ . This leads us to conjecture the formula in the following theorem, which we can then verify.

THEOREM 4.12. *For any fixed h ,*

$$K_h = \sum_{\ell \in h} D_\ell + \sum_{\Delta \sim h} H_\Delta - \sum_{\Delta \not\sim h} H_\Delta.$$

PROOF. We verify that this formula is correct by showing that the left hand side and the right hand side have the same intersection numbers with each t_{ℓ_0} and each n_{Δ_0} .

We begin with the classes t_{ℓ_0} . There are two cases: $\ell_0 \in h$ and $\ell_0 \notin h$. First suppose $\ell_0 \in h$, e.g., $\ell_0 = (ab)$. Then for the left hand side, $K_{(ab)(cd)(ef)} \cdot t_{(ab)} = 2$ as $K_{(ab)(cd)(ef)}$ is a double section in D_{ab} . For the first term on the right hand side, $t_{(ab)} \cdot D_{ab} = -2$ (by Corollary 4.6) and $t_{(ab)} \cdot D_{(pq)} = 0$ for $(pq) \neq (ab)$, in particular for $(pq) = (cd)$ and $(pq) = (ef)$. For the last two terms, $t_{(ab)} \cdot n_{(pqr,stu)} = 1$ for $(pqr, stu) \in \{(abc, def), (abd, cef), (abe, cdf), (abf, cde)\}$ and 0 otherwise, and for each of these four values of Δ we have $\Delta \sim h$. Then we have the equality $2 = -2 + (4 - 0)$. Next suppose $\ell_0 \notin h$, e.g., $\ell_0 = (ac)$. Then for the left hand side, $K_{(ab)(cd)(ef)} \cdot t_{(ac)} = 0$. For the first term on the right hand side, $t_{(ac)} \cdot D_{(ab)} = t_{(ac)} \cdot D_{(cd)} = t_{(ac)} \cdot D_{(ef)} = 0$. For the last two terms, $t_{(ac)} \cdot n_{(pqr,stu)} = 1$ for $(pqr, stu) \in \{(abc, def), (acd, bef), (ace, bdf), (acf, bde)\}$ and 0 otherwise, and of these four values of Δ , we have $\Delta \sim h$ for two of them and $\Delta \not\sim h$ for the other two. Then we have the equality $0 = 0 + (2 - 2)$.

Now for the classes $n_{(pqr,stu)}$. Again there are two cases to consider. First we consider $n_{(abc,def)} = m_{(abc)} + m_{(def)}$. For the left hand side, we have $K_{(ab)(cd)(ef)} \cdot n_{(abc,def)} = 0$ as $K_{(ab)(cd)(ef)}$ is disjoint from $H_{(abc,def)}$. For the first term on the right hand side, $m_{(abc)} \cdot D_{(ef)} = m_{(def)} \cdot D_{(ab)} = 1$ and all other intersections are 0. For the last two terms on the right hand side, $m_{(abc)} \cdot H_{(abc,def)} = m_{(def)} \cdot H_{(abc,def)} = -1$, and all other intersection numbers are 0, and $(abc, def) \sim (ab)(cd)(ef)$. Thus we obtain $0 = 2 + (-2 - 0)$. Next we consider $n_{(ace,bdf)} = m_{(ace)} + m_{(bdf)}$. For the left hand side, $m_{(ace)} \cdot K_{(ab)(cd)(ef)} = m_{(bdf)} \cdot K_{(ab)(cd)(ef)} = 1$

as this is a single transverse intersection point in the $\mathbf{P}^1_{s(ce),(df)}$ in the corank 2 boundary component $C_{(ab)(ce)(df)}$. For the first term on the right hand side, $m_{(ace)} \cdot D_{(ab)} = m_{(bdf)} \cdot D_{(ab)} = 0$ and similarly for $D_{(cd)}$ and $D_{(ef)}$ as these curves are disjoint from these surfaces. For the last two terms on the right hand side, $m_{(ace)} \cdot H_{(ace,bdf)} = m_{(bdf)} \cdot H_{(ace,bdf)} = -1$, and all other intersection numbers are 0, but now $(ace, bdf) \not\sim (ab)(cd)(ef)$. Thus we obtain $2 = 0 + (0 - (-2))$, completing the verification. \square

We can also obtain other particularly symmetric expressions, though if we wish to have integral coefficients we must pass to multiples of K_h .

COROLLARY 4.13. *For any fixed h ,*

$$\begin{aligned}
 2K_h &= -2 \sum_{\ell \in h} D_\ell + \sum_{\ell \notin h} D_\ell + \sum_{\Delta \not\sim h} H_\Delta, \\
 3K_h &= - \sum_{\ell \in h} D_\ell + \sum_{\ell \notin h} D_\ell + \sum_{\Delta \sim h} H_\Delta, \\
 4K_h &= \sum_{\ell \notin h} D_\ell + 2 \sum_{\Delta \sim h} H_\Delta - \sum_{\Delta \not\sim h} H_\Delta, \\
 5K_h &= \sum_{\ell} D_\ell + 3 \sum_{\Delta \sim h} H_\Delta - 2 \sum_{\Delta \not\sim h} H_\Delta, \\
 &= -3 \sum_{\ell \in h} D_\ell + 2 \sum_{\ell \notin h} D_\ell + \sum_{\Delta} H_\Delta.
 \end{aligned}$$

PROOF. The expression for $2K_h$ is obtained from doubling the expression for K_h in Theorem 4.12 and then adding suitable relations from Lemma 4.8, and then the remaining expressions are obtained by taking suitable linear combinations of the expressions for K_h and for $2K_h$. \square

COROLLARY 4.14. *There is no expression*

$$K_h = \sum_{\ell} \alpha_{h,\ell} D_\ell + \sum_{\Delta} \beta_{h,\Delta} H_\Delta$$

with all $\alpha_{h,\ell}$ and all $\beta_{h,\Delta}$ nonnegative integers.

PROOF. By Lemma 4.11, any such expression would have to have either $\beta_{h,\Delta_1} = \beta_{h,\Delta_2} = 1$ and $\beta_{h,\Delta} = 0$ for $\Delta \neq \Delta_1, \Delta_2$ or $\beta_{h,\Delta_1} = 2$ and $\beta_{h,\Delta} = 0$ for $\Delta \neq \Delta_1$. Now the symmetric group S_6 operates doubly transitively on $\{\Delta\}$, so it suffices to check one expression of each type. Beginning with the formula in Theorem 4.12 and adding suitable relations from Lemma 4.8, we obtain the following two expressions:

$$\begin{aligned}
 K_{(ab)(cd)(ef)} &= -D_{(ab)} + \frac{1}{2}D_{(ac)} + \frac{1}{2}D_{(ad)} + \frac{1}{2}D_{(ae)} + \frac{1}{2}D_{(af)} \\
 &\quad + \frac{1}{2}D_{(bc)} + \frac{1}{2}D_{(bd)} + \frac{1}{2}D_{(be)} + \frac{1}{2}D_{(bf)} \\
 &\quad + H_{(acd,bef)} + H_{(aef,bcd)} \\
 K_{(ab)(cd)(ef)} &= -D_{(ab)} + D_{(ac)} + D_{(ad)} + D_{(be)} + D_{(bf)} + 2H_{(acd,bef)}
 \end{aligned}$$

and in each case the coefficient of $D_{(ab)}$ is negative. □

In fact a stronger result is true.

THEOREM 4.15. *There is no expression*

$$K_h = \sum_{\ell} \alpha_{h,\ell} D_{\ell} + \sum_{\Delta} \beta_{h,\Delta} H_{\Delta}$$

with all $\alpha_{h,\ell}$ and all $\beta_{h,\Delta}$ nonnegative rational numbers.

PROOF. The space of relations is 9-dimensional with basis the relations $R_i : H_{\Delta} - H_{\Delta_0} - \dots = 0$ for any fixed Δ_0 and all $\Delta \neq \Delta_0$, and then it is routine to check that no expression of the form

$$\sum_{\ell \in h} D_{\ell} + \sum_{\Delta \sim h} H_{\Delta} - \sum_{\Delta \sim h} H_{\Delta} + \sum_i c_i R_i,$$

with c_i arbitrary rational numbers, can have all D_{ℓ} coefficients and all H_{Δ} coefficients nonnegative. □

REMARK 4.16. Of course this theorem is equivalent to the result that there does not exist an expression for NK_h , for any integer $N \neq 0$, as above with all coefficients nonnegative integers.

In this paper we have heavily exploited the ‘‘dictionary’’ of [7] showing us how to translate between the finite symplectic group $\text{PSp}(4, \mathbf{Z}/2\mathbf{Z})$ and the symmetric group S_6 . Let us take this opportunity to record an addendum to [7]. Although we have not needed to use theta functions in this paper, [7] also relates this action to the action of $\text{PSp}(4, \mathbf{Z}/2\mathbf{Z})$ on (sets of) theta characteristics in genus 2. We would like to add one more example of this relationship (which we observed shortly after that paper appeared).

We consider the classical theta function with characteristic m given by

$$\theta_m(\tau, z) = \sum_{\xi \in \mathbf{Z}^n} \exp(\pi i(\xi + m'/2)\tau^t(\xi + m''/2) + 2\pi i(\xi + m'/2)^t(z + m''/2))$$

where $\tau \in \mathcal{S}_d$, a point in Siegel space of degree d , $z = (z_1, \dots, z_d)$ is a vector in \mathbf{C}^d , and $m = (m', m'')$ where each of m' and m'' is a vector in \mathbf{Z}^d . We let m be even or odd as the product $\sigma(m) = m'{}^t m''$ is even or odd. These functions satisfy the identity

$$\theta_m(\tau, -z) = (-1)^{\sigma(m)} \theta_m(\tau, z).$$

The theta constant $\theta_m(\tau)$ is defined to be the function $\theta_m(\tau, 0)$ so we see that if m is odd $\theta_m(\tau)$ is identically 0, but for m even it certainly is not. It suffices to consider $m \pmod{2}$, so henceforth we suppose $m \in \{0, 1\}^{2d}$. If $d = 1$ there are three even characteristics, $(0, 0)$, $(0, 1)$, and $(1, 0)$, and one odd characteristic, $(1, 1)$. In that case we have Jacobi's derivative formula

$$\theta'_{11}(\tau) = -\pi \theta_{00}(\tau) \theta_{01}(\tau) \theta_{10}(\tau)$$

where the left hand side denotes $\partial/\partial z(\theta_{11}(\tau, z))$ evaluated at $z = 0$. In degree two there are 10 even characteristics and 6 odd characteristics, there is an analogous formula, discovered by Rosenhain and proved by Thomae and Weber [4]:

$$J[\theta_{m_1}(\tau), \theta_{m_2}(\tau)] = \pm \pi^2 \theta_{m_3}(\tau) \theta_{m_4}(\tau) \theta_{m_5}(\tau) \theta_{m_6}(\tau)$$

where $J[\]$ denotes the Jacobian determinant of the functions $\theta_{m_1}(\tau, z)$ and $\theta_{m_2}(\tau, z)$ with respect to the variables z_1 and z_2 (where $z = (z_1, z_2)$) evaluated at $z = (z_1, z_2) = (0, 0)$, and the six characteristics satisfy:

- m_1 and m_2 are distinct odd characteristics,
- m_3, m_4, m_5 , and m_6 are distinct even characteristics,
- $m_1 + m_2 + m_i$ is odd for $i = 3, 4, 5$, and 6.

By [7, Lemma 4.3] there is a canonical correspondence between odd characteristics and spreads of lines, and a canonical correspondence between even characteristics and nonsingular pairs. (Nonsingular pairs were called anisotropic pairs there.)

THEOREM 4.17. *Let m_1, \dots, m_6 be characteristics as in Rosenhain's formula. Let m_1 and m_2 correspond to spreads of lines σ_1 and σ_2 , and let m_3, \dots, m_6 correspond to nonsingular pairs $\Delta_3, \dots, \Delta_6$ as above. Let ℓ be the unique line in $\sigma_1 \cap \sigma_2$. Then $\Delta_3, \dots, \Delta_6$ are the four nonsingular pairs given by $\ell \in \Delta_i, i = 3, \dots, 6$.*

PROOF. By the equivariance of the correspondence, it suffices to verify this for a single 6-tuple $\{m_1, \dots, m_6\}$. Then direct computation shows that this is true for $\{m_1, \dots, m_6\} = \{0101, 1101, 0010, 0011, 0110, 1111\}$, where the line $\ell = (1000)$. (See [7, Tables 1 and 2].) □

REMARK 4.18. In the indexing given by the symmetric group, if m_1 corresponds to the monad a and m_2 corresponds to the monad b , so that ℓ is the duad $\ell = (ab)$, then $\{\Delta_3, \dots, \Delta_6\}$ are the triadic synthemes $\{(abc, def), (abd, cef), (abe, cdf), (abf, cde)\}$.

5. RELATION WITH $\overline{M}_{0,6}$

The space \mathfrak{M}_2^* has a completely different description, which we now give. (See especially [5, Section 8], and also [6, 7].)

Consider a nonsingular curve, i.e., a Riemann surface, of genus 2. Any such Riemann surface is hyperelliptic, i.e., is a 2-fold cover of \mathbf{P}^1 , branched at 6 points, the Weierstrass points of the surface. We let $M_{0,6}$ be the space of 6 ordered points in \mathbf{P}^1 (a surface of genus 0) modulo the action of $\text{PGL}(2, \mathbf{C})$ on \mathbf{P}^1 by a fractional linear transformations. Given such a six-tuple of points, we associate to it the hyperelliptic curve with these Weierstrass points (this association factoring through the action of $\text{PGL}(2, \mathbf{C})$, which gives automorphisms of the associated surface), and hence its Jacobian, which is a two-dimensional Abelian variety. The ordering of points on the curve corresponds to a level 2 structure, and so we obtain an isomorphism from $M_{0,6}$ into \mathfrak{M}_2^* . The image of this isomorphism is $\mathfrak{M}_2^* - (\mathcal{H}_1 \cup \mathcal{D}_1)$. This isomorphism then extends to an isomorphism $\overline{M}_{0,6} \rightarrow \mathfrak{M}_2^*$, where $\overline{M}_{0,6}$ is the moduli space of stable curves of genus 2. The generic points of \mathcal{H}_1 and \mathcal{D}_1 (i.e., of the divisors H_Δ and D_ℓ) each parameterize a type of singular curve with a single singularity, see [5, Figure 8.4.1] for a description. Fulton asked whether every divisor on $\overline{M}_{0,6}$ is a nonnegative integral linear combination of divisors parameterizing curves with a single singularity, that is, of divisors in $\overline{M}_{0,6} - M_{0,6}$, and this question was answered in the negative by Keel and by Vermeire, who constructed what are now known as Keel-Vermeire divisors. These divisors are exactly our divisors K_h (and Keel came across them by considering involutions: if $h = (ab)(cd)(ef)$ is an isotropic plane in our notation, then there is the associated element of order 2 in S_6 also denoted $(ab)(cd)(ef)$), and so Corollary 4.14 is originally due to them, though our Theorem 4.15 is a strengthening of that result.

(A remark on terminology: Although these spaces have no boundary in the topological sense, it is common among mathematicians who work on Siegel modular varieties to call $\mathcal{D} = \mathfrak{M}_n^* - \mathfrak{M}_n$, the union of the compactification divisors, the boundary. It is common among mathematicians who work on configuration spaces to call $\overline{M}_{0,n} - M_{0,n}$ (the two uses of n are unrelated) the boundary. Translated into \mathfrak{M}_2^* , $\overline{M}_{0,6} - M_{0,6}$ is the union $\mathcal{H}_1 \cup \mathcal{D}$. Thus these two (ab)uses of the word boundary are inconsistent with each other, which is why we have avoided using this term in this paper.)

We would like to close by contrasting the approaches of [11] and this paper. The description of $\overline{M}_{0,6}$ in [11] falls into a family of descriptions of $\overline{M}_{0,n}$ for any n , and so that approach is most useful for generalizations. On the other hand, that description involves a choice which destroys the symmetry of the situation, and so we feel our description here is most useful for understanding \mathfrak{M}_2^* itself.

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