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Partial Differential Equations — *An obstacle problem with gradient term and asymptotically linear reaction*, by B. Abdellaoui*, S. M. Bouguima and I. Peral, communicated on 10 December 2010.

In memoria di Giovanni Prodi matematico e gentiluomo.

Abstract. — We will consider the following obstacle problem

$$\int_\Omega \nabla u \nabla T_k(v-u)\,dx + \int_\Omega h(u)|\nabla u|^q T_k(v-u)\,dx \ge \int_\Omega (g(x,u)+f)T_k(v-u)\,dx,$$

with the condition that $u \ge \psi$ a.e in Ω. Under suitable condition relating g, h and q, we show the existence of a solution for all $f \in L^1(\Omega)$.

The main feature is, assuming that $g(x,s)$ is asymptotically linear as $|s| \to \pm\infty$ and independently of the values of

$$\lim_{s\to\pm\infty} \frac{g(x,s)}{s},$$

to obtain a solution for all $\lambda > 0$ and $f \in L^1(\Omega)$. In this sense we could say that the first order term break down any resonant effect.

Key words: Nonlinear obstacle problems, existence and nonexistence, regularization, resonance.

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1. Introduction

In this paper we deal with a nonlinear elliptic obstacle problem of the form

$$\text{(1.1)}\quad \begin{cases} u \ge \psi \text{ a.e in } \Omega, \quad \text{for all } v \in \mathcal{K}(\psi), \text{ we have} \\ \displaystyle\int_\Omega \nabla u \nabla (v-u)\,dx + \int_\Omega h(u)|\nabla u|^q (v-u)\,dx \ge \int_\Omega (g(x,u)+f)(v-u)\,dx, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $1 < q \le 2$, ψ is a bounded function such that $\psi \in W_0^{1,2}(\Omega)$ and

$$\mathcal{K}(\psi) = \{v \in W_0^{1,2} \cap L^\infty(\Omega) : v \ge \psi \text{ in } \Omega\}.$$

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We suppose that $f \in L^1(\Omega)$, g is a Caratheodory function asymptotically linear, that is, verifying

$$|g(x,s)| \leq \lambda g_0(x)|s| + g_1(x) \tag{1.2}$$

where $g_1 \in L^1(\Omega)$ and g_0 satisfies

$$\begin{cases} g_0 \gneqq 0, \\ g_0 \in L^1(\Omega), \\ C(g_0,q) > 0, \ \text{where } C(g_0,q) = \inf\limits_{\phi \in W_0^{1,q}(\Omega)\setminus\{0\}} \dfrac{\left(\int_\Omega |\nabla\phi|^q\,dx\right)^{1/q}}{\int_\Omega g_0|\phi|\,dx}. \end{cases} \tag{1.3}$$

It is easy to check that if g_0 satisfies (1.3), then $g_0 \in W^{-1,q'}(\Omega) \cap L^1(\Omega)$, $q' = \frac{q}{q-1}$.

We say that g_0 is an *admissible weight* if (1.3) holds.

If $\psi \equiv 0$ and we consider the equation

$$-\Delta u = g(u) + f, \quad \text{in } \Omega,\ u \geq 0 \text{ and } u \in W_0^{1,2}(\Omega), \tag{1.4}$$

where g is a lipschitz function such that $g(0) = 0$ and verifying the condition

$$\lim_{s\to\pm\infty} \frac{g(s)}{s} = \lambda_\pm, \tag{1.5}$$

for $\lambda_- < \lambda_1 < \lambda_+ < \lambda_2$, λ_1 and λ_2 are the first and second eigenvalue of the Laplacian. The problem (1.4) was solved in the famous work by Ambrosetti-Prodi [2]. The authors establish a sharp existence, nonexistence and multiplicity result related to the value of the projection of the datum $f \in L^2(\Omega)$ on the first positive normalized eigenfunction of the Laplacian,

$$\int_\Omega f(x)\phi_1(x)\,dx = t.$$

More precisely they prove that there exists a threshold $\bar{t}$ such that, if $t > \bar{t}$ there is no solution, if $t = \bar{t}$ there exist a solution and if $t < \bar{t}$ there exist two solutions.

One of the goals of this paper is to prove that under some hypotheses on q, for all $f \in L^1(\Omega)$, g satisfying (1.2) (1.3) and h with some structural conditions, there exists a solution to the variational inequality (1.1). In particular in the Ambrosetti-Prodi context we prove that the gradient term give a solution without any condition on $\lambda_\pm$ or the projection of f on ϕ_1.

As a precedent we have the case of an equation with gradient term. It was proved in [1], for the case $g(x,u) = \lambda g_0(x)u$, under a suitable condition on q and g_0, $h(u) \equiv 1$, that the absorption term $|\nabla u|^q$ is sufficient to break down any resonant effect of the linear zero order term and then the existence of a solution is obtained for all $\lambda > 0$ and $f \in L^1(\Omega)$. In this sense this paper could be understood, in particular, as the extension of the result in [1] to variational inequalities with g verifying (1.2) (1.3) and h verifying (3.2) below.

Unilateral problems with gradient term has been largely studied in the literature, we refer, for instance, to [4], [8], [14] and the references therein. In [4] it is studied the existence of unbounded solutions for an obstacle problem with natural growth in the gradient.

To prove the existence of solutions for unilateral problems with L^1 datum, it is necessary to consider entropy solution in the sense that $u \geq \psi$ and

$$\int_\Omega \nabla u \nabla (T_k(v-u))\,dx \geq \int_\Omega f(T_k(v-u))\,dx$$

for all $v \in \mathscr{K}(\psi)$. See for instance [7].

We organize the contents as follows.

In Section 2 we consider a simple model where $\psi \geq 0$, $f \geq 0$, $h \equiv 1$ and $g(x,s) \equiv \lambda g_0(x)u$, with $g_0 \geq 0$. Then for all $\lambda > 0$, we prove the existence of a nonnegative solution. More precisely we show that if g_0 is a nonnegative admissible weight in the sense of condition (1.3), then we have a solution for all $\lambda > 0$ and all $f \in L^1(\Omega)$.

To prove the main result we use a convenient approximate problems and uniform estimates in order to pass to the limit. In Subsection 2.1 a partial uniqueness result is given for $q = 2$ and $\psi \equiv 0$.

Section 3, is devoted to obstacle problem (1.1) without any sign condition on f and ψ. The term $|\nabla u|^q$ will be substituted by the more general $h(u)|\nabla u|^q$ and we will consider the general nonlinearity $g(x,u)$ satisfying (1.5). Under suitable conditions on h we will prove the existence of entropy solution for all $f \in L^1$ and without any restriction on $\lambda_\pm$. In this sense the result can be seen as *breaking of resonance* for the Ambrosetti-Prodi obstacle problem.

It is worthy to point out that in the problem without constraint, condition (1.3) is optimal. It is sufficient to consider $g(x) = |x|^{-2}$, the Hardy potential, for which we have the classical inequality

$$\int_\Omega |\nabla u|^2\,dx \geq \Lambda_N \int_\Omega \frac{u^2}{|x|^2}\,dx, \quad \text{for all } u \in \mathscr{C}_0^\infty(\Omega) \text{ where } \Lambda_N = \left(\frac{N-2}{2}\right)^2.$$

In this case condition (1.3) holds if and only if $q > \frac{N}{N-1}$. Then if $q < \frac{N}{N-1}$ and $\lambda > \Lambda_N = \left(\frac{N-2}{2}\right)^2$, there is no solution to the obstacle problem. (See Theorem 3.1 in [1] for details).

We will use the following notation. For a measurable function u we define the truncation $T_k(u)$ by

$$T_k(u) = \max\{-k, \min\{k,u\}\}.$$

We set $G_k(u) = u - T_k(u)$.

2. Existence of nonnegative solutions to the simple model

In this section we deal with the simple case where $\psi \geq 0$, $f \geq 0$, $h \equiv 1$ and $g(x,s) \equiv \lambda g_0(x)u$, with $g_0 \geq 0$. Define the convex set

$$\mathscr{K}(\psi) = \{v \in W_0^{1,2} \cap L^\infty(\Omega) : v \geq \psi \text{ in } \Omega\}.$$

We find the following result.

Theorem 2.1. *Assume that g_0 is an admissible weight in the sense of condition* (1.3), *then for all $\lambda > 0$ and for all $f \in L^1(\Omega)$, there exits a positive $u \geq \psi$ such that $|\nabla u|^q \in L^1(\Omega)$, $T_k(u) \in W_0^{1,2}(\Omega)$ for all $k > 0$ and for all $v \in \mathscr{K}(\psi)$ we have*

$$\int_\Omega \nabla u \nabla (T_k(v-u))\,dx + \int_\Omega |\nabla u|^q (T_k(v-u))\,dx \geq \int_\Omega (\lambda g u + f)(T_k(v-u))\,dx. \tag{2.1}$$

We will say that u is an entropy solution to the obstacle problem if (2.1) *holds.*

To prove Theorem 2.1 we start by proving the result in some particular cases and then we proceed by approximation of g and f. Notice that since $1 < q \leq 2$, then $\frac{N}{2} \leq \frac{N}{q}$.

Theorem 2.2. *Assume that $f, g \in L^r(\Omega)$ are positive functions with $r > \frac{N}{q}$, then for all $\lambda \geq 0$, there exists $u \in \mathscr{K}(\psi)$ a weak positive solution to problem*

$$\begin{cases} \displaystyle\int_\Omega \nabla u \nabla (v-u)\,dx + \int_\Omega |\nabla u|^q (v-u)\,dx \\ \qquad \displaystyle\geq \int_\Omega (\lambda g u + f)(v-u)\,dx \quad \textit{for all } v \in \mathscr{K}(\psi), \end{cases} \tag{2.2}$$

Proof. We divide the proof in several steps.

Step 1: Let $k > 0$ be fixed, then for all $n \in \mathbb{N}$, using classical results (see for instance [13] and [12]), there exists $w_n \in \mathscr{K}(\psi)$, a solution to the obstacle problem

$$\int_\Omega \nabla w_n \nabla (v-w_n)\,dx + \int_\Omega \frac{|\nabla w_n|^q}{1+\frac{1}{n}|\nabla w_n|^q}(v-w_n)\,dx \geq \int_\Omega (\lambda g T_k(w_n) + f)(v-w_n)\,dx \tag{2.3}$$

for all $v \in \mathscr{K}(\psi)$.

For k fixed we pass to the limit in n. Let $v = T_m(w_n)$, since $\psi \in L^\infty(\Omega)$, then choosing m very large we conclude that v is an admissible test function in (2.3). Since $v - w_n = -G_m(w_n)$, it follows that

$$\int_\Omega |\nabla G_m(w_n)|^2\,dx + \int_\Omega \frac{|\nabla G_m(w_n)|^q}{1+\frac{1}{n}|G_m(w_n)|^q} G_m(w_n) \le \int_\Omega (\lambda g T_k(w_n) + f) G_m(w_n)\,dx.$$

Thus

$$\int_\Omega |\nabla G_m(w_n)|^2\,dx \le \lambda k^2 \|g\|_1 + \int_\Omega f G_m(w_n)\,dx.$$

Using Poincaré inequality we get that $\int_\Omega |\nabla G_m(w_n)|^2\,dx \le C$ for all m.

Notice that choosing $m \gg k$ it follows that

$$\int_\Omega |\nabla G_m(w_n)|^2\,dx \le \int_\Omega f G_m(w_n)\,dx, \tag{2.4}$$

and then by using the classical Stampacchia estimates, see [15], we obtain that $\|w_n\|_{L^\infty} \le C$ where C is independent of n.

We set now $v = \psi$, then

$$\int_\Omega \nabla w_n \nabla (w_n - \psi)\,dx + \int_\Omega \frac{|\nabla w_n|^q}{1+\frac{1}{n}|\nabla w_n|^q}(w_n - \psi)\,dx$$
$$\le \int_\Omega (\lambda g T_k(w_n) + f)(w_n - \psi)\,dx.$$

Since $w_n \ge \psi$, we get

$$\int_\Omega |\nabla w_n|^2\,dx \le \int_\Omega \nabla w_n \nabla \psi\,dx + \int_\Omega (\lambda g T_k(w_n) + f)(w_n - \psi)\,dx.$$

Thus using Hölder inequality and the previous estimate we obtain that

$$\int_\Omega |\nabla w_n|^2\,dx \le C(f, g, \Omega, k) \quad \text{uniformly in } n,$$

therefore, up to a subsequence, $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$. By weak-*convergence in $L^\infty(\Omega)$ we also have that $u_k \in W_0^{1,2} \cap L^\infty(\Omega)$ and $u_k \ge \psi$. Next we investigate the inequality satisfied by u_k. To do that we prove the following claim.

Convergence claim. $w_n \to u_k$ strongly in $W_0^{1,2}(\Omega)$.

Proof of the convergence claim. It is clear that for all $v \in \mathscr{K}(\psi)$,

$$(\lambda g T_k(w_n) + f)(w_n - v) \to (\lambda g T_k(u_k) + f)(u_k - v) \quad \text{strongly in } L^1(\Omega).$$

Let $v = w_n - (w_n - u_k)^+$, then $v \in \mathscr{K}(\psi)$ and $v - w_n = -(w_n - u_k)^+$, so we have

$$\int_\Omega \nabla w_n \nabla (w_n - u_k)^+ \, dx + \int_\Omega \frac{|\nabla w_n|^q}{1 + \frac{1}{n}|\nabla w_n|^q}(w_n - u_k)^+ \, dx$$
$$\leq \int_\Omega (\lambda g T_k(w_n) + f)(w_n - u_k)\, dx.$$

It is clear that $\int_\Omega (\lambda g T_k(w_n) + f)(w_n - u_k)\, dx \to 0$ as $n \to \infty$. Hence we conclude that

$$\int_\Omega |\nabla (w_n - u_k)^+|^2 \, dx + \int_\Omega \frac{|\nabla w_n|^q}{1 + \frac{1}{n}|\nabla w_n|^q}(w_n - u_k)^+ \, dx$$
$$\leq -\int_\Omega \nabla u_k \nabla (w_n - u_k)^+ \, dx + o(1) = o(1).$$

Thus $\int_\Omega |\nabla (w_n - u_k)^+|^2 \, dx = o(1)$ and then $(w_n - u_k)^+ \to 0$ strongly in $W_0^{1,2}(\Omega)$.

To complete the proof we follow closely the argument used in [6]. Consider $\phi(s) = s e^{(1/4)s^2}$, which satisfies $\phi'(s) - |\phi(s)| \geq \frac{1}{2}$.

Let $v = w_n + \phi((w_n - u_k)^-)$, then $v \in \mathscr{K}(\psi)$ and $v - w_n = \phi((w_n - u_k)^-)$. It is clear that

$$\nabla (v - w_n) = \begin{cases} 0 & \text{if } w_n \geq u_k, \\ \phi'((w_n - u_k)^-)(\nabla u_k - \nabla w_n) & \text{if } w_n \leq u_k. \end{cases}$$

Using v as a test function in (2.3) we obtain that

$$\int_\Omega \nabla w_n \phi'((w_n - u_k)^-) \nabla (w_n - u_k)^- \, dx + \int_\Omega H_n(\nabla w_n)\phi(w_n - u_k)^- \, dx \tag{2.5}$$
$$\leq \lambda \int_\Omega g(x) T_k w_n \phi((w_n - u_k)^-)\, dx + \int_\Omega f(x)\phi((w_n - u_k)^-)\, dx.$$

where $H_n(s) = \frac{|s|^q}{1 + \frac{1}{n}|s|^q}$. Therefore

$$\int_{w_n \leq u_k} \phi'((w_n - u_k)^-) \nabla w_n \nabla (w_n - u_k)\, dx - \int_\Omega H_n(\nabla w_n)|\phi(w_n - u_k)^-|\, dx \tag{2.6}$$
$$\leq \lambda \int_{w_n \leq u_k} g(x) T_k w_n \phi(u_k - w_n)\, dx + \int_{w_n \leq u_k} f(x)\phi(u_k - w_n)\, dx.$$

Since $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$, a direct computation shows that

$$\int_\Omega \nabla w_n \phi'((w_n - u_k)^-)\nabla(w_n - u_k)^- \, dx$$
$$= \int_\Omega |\nabla((w_n - u_k)^-)|^2 \phi'((w_n - u_k)^-)\, dx + o(1).$$

As $q \le 2$, $\forall \varepsilon > 0$ there exists a non negative constant C_ε such that

$$s^q \le \varepsilon s^2 + C_\varepsilon, \quad s \ge 0. \tag{2.7}$$

Hence the second term in the left-hand side can be estimated in the following way,

$$\int_\Omega H_n(\nabla w_n)\phi((w_n - u_k)^-)\, dx$$
$$\le \varepsilon \int_\Omega |\nabla w_n|^2 |\phi((w_n - u_k)^-)|\, dx + C(\varepsilon)\int_\Omega |\phi((w_n - u_k)^-)|\, dx$$
$$= \varepsilon \int_\Omega |\nabla((w_n - u_k)^-)|^2 |\phi((w_n - u_k)^-)|\, dx - \varepsilon \int_\Omega |\nabla u_k|^2 |\phi((w_n - u_k)^-))|\, dx$$
$$+ 2\varepsilon \int_\Omega \nabla w_n \nabla u_k |\phi((w_n - u_k)^-)|\, dx + C(\varepsilon)\int_\Omega |\phi((w_n - u_k)^-)|\, dx.$$

Since $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$ and $|\phi((w_n - u_k)^-)| \to 0$ almost everywhere and in $L^2(\Omega)$, it follows that,

(i) $\int_\Omega |\nabla u_k|^2 |\phi((w_n - u_k)^-)|\, dx \to 0$ as $n \to \infty$,

(ii) $\int_\Omega \nabla w_n \nabla u_k \phi((w_n - u_k)^-)\, dx \to 0$ as $n \to \infty$.

Therefore, passing to the limit as n tends to ∞, we have

$$\int_\Omega H_n(\nabla w_n)\phi((w_n - u_k)^-)\, dx \le \varepsilon \int_\Omega |\nabla w_n - \nabla u_k|^2 |\phi((w_n - u_k)^-)|\, dx + o(1).$$

Moreover, it is clear that the right-hand side in (2) goes to zero as $n \to \infty$. Since $\phi'(s) - |\phi(s)| > \frac{1}{2}$, choosing $\varepsilon \le 1$ we conclude that

$$\frac{1}{2}\int_\Omega |\nabla((w_n - u_k)^-)|^2\, dx$$
$$\le \int_\Omega (\phi'((w_n - u_k)^-) - \varepsilon|\phi((w_n - u_k)^-)|)|\nabla((w_n - u_k)^-)|^2\, dx$$
$$\le o(1),$$

whence $w_n \to u_k$ in $W_0^{1,2}(\Omega)$ and the claim is proved. Moreover, from (2.7) it follows that

$$H_n(\nabla w_n) \leq c_1|\nabla w_n|^2 + c_2.$$

By the claim, we have in particular the almost everywhere convergence of the gradients and therefore we conclude that

$$H_n(\nabla w_n) \to |\nabla u_k|^q \quad \text{in } L^1(\Omega).$$

Hence we find that $u_k \in \mathscr{K}(\psi)$ solves

$$\begin{aligned} \int_\Omega \nabla u_k \nabla(v - u_k)\,dx + \int_\Omega |\nabla u_k|^q (v - u_k)\,dx \\ \geq \int_\Omega (\lambda g T_k(u_k) + f)(v - u_k)\,dx \end{aligned} \tag{2.8}$$

for all $v \in \mathscr{K}(\psi)$.

Step 2: We claim the existence of a universal $M > 0$ that does not depend on k such that $\|u_k\|_{L^\infty(\Omega)} \leq M$. To prove the claim we use the fact that $f, g_0 \in L^r(\Omega)$ where $r > \frac{N}{2}$. Let $v = T_m(u_k)$, using v as a test function in (2.8) it follows that

$$\int_\Omega |\nabla G_m(u_k)|^2\,dx + \int_\Omega |\nabla G_m(u_k)|^q G_m(u_k) \leq \lambda \int_\Omega g_0 G_m^2(u_k)\,dx + \int_\Omega f G_m(u_k)\,dx.$$

Notice that, using Poincaré inequality we get

$$\int_\Omega |\nabla G_m(u_k)|^q G_m(u_k) = \frac{1}{(1 + 1/q)^q} \int_\Omega |\nabla G_m^{1+1/q}(u_k)|^q\,dx \geq C \int_\Omega g_0 G_m^{1+q}(u_k)\,dx,$$

and

$$\begin{aligned} \lambda \int_\Omega g_0 G_m^2(u_k)\,dx + \int_\Omega f G_m(u_k)\,dx \\ \leq \varepsilon \int_\Omega g_0 G_m^{1+q}(u_k)\,dx + C(\varepsilon) \int_{u \geq m} g_0\,dx + C\Big(\int_\Omega G_m^{2^*}(u_k)\,dx\Big)^{1/2^*}. \end{aligned}$$

Therefore we conclude that

$$\int_\Omega |\nabla G_m(u_k)|^2\,dx + c\int_\Omega |\nabla G_m(u_k)|^q G_m(u_k) \leq \int_{u \geq m} g_0\,dx + C\Big(\int_\Omega G_m^{2^*}(u_k)\,dx\Big)^{1/2^*}$$

where $C > 0$ is a positive constant that depends only on the data and is independent of m and k.

Recall that $f, g_0 \in L^r$ with $r > N/2$, then using Sobolev inequality,

$$C\Big(\int_\Omega G_m^{2^*}(u_k)\,dx\Big)^{2/2^*} + \Big(\int_\Omega G_m^{(1+1/q)q^*}(u_k)\,dx\Big)^{q/q^*}$$
$$\le C\Big(|u_k \ge m|^{1/r'} + C\int_\Omega G_m^{2^*}(u_k)\,dx\Big)^{1/2^*}|u_k \ge m|^{1-1/r-1/2^*}.$$

From Young's inequality there results that

$$C\Big(\int_\Omega G_m^{2^*}(u_k)\,dx\Big)^{2/2^*} + \Big(\int_\Omega G_m^{(1+1/q)q^*}(u_k)\,dx\Big)^{q/q^*}$$
$$\le C(|u_k \ge m|^{1/r'} + C|u_k \ge m|^{2(1-1/r-1/2^*)} \le C|u_k \ge m|^{\gamma},$$

where $\gamma = \min\{2(1 - 1/r - 1/2^*), 1/r'\}$. By a direct computation we get easily that $2^*\gamma/2 > 1$.

We set $\beta(m) = |u_k \ge m|$, then for $m_1 < m_2$ we have

$$\begin{aligned}\beta^{1/2^*}(m_2)(m_2 - m_1) &\le \Big(\int_{u_k \ge m_2} |u_k - m_1|^{2^*}\,dx\Big)^{1/2^*}\\ &\le \Big(\int_{u_k \ge m_1} |u_k - m_1|^{2^*}\,dx\Big)^{1/2^*}\\ &\le \beta^{\gamma/2}(m_1).\end{aligned}$$

Thus

$$\beta(m_2) \le \frac{\beta^{2^*\gamma/2}(m_1)}{(m_2 - m_1)^{2^*}}.$$

Since $2^*\gamma/2 > 1$, using the Stampacchia classical result, (see [15]), there exists a universal constant $\bar{m} > 0$ such that $\beta(m) = 0$ if $m \ge \bar{m}$. Thus $u_k \le \bar{m}$, and then choosing $k \gg \bar{m}$, we obtain that $u = u_k$ solves

$$\text{(2.9)}\qquad \begin{cases} u \in \mathcal{K}(\psi) \quad \text{for all } v \in \mathcal{K}(\psi), \text{ we have} \\ \displaystyle\int_\Omega \nabla u \nabla(v-u)\,dx + \int_\Omega |\nabla u|^q (v-u)\,dx \ge \int_\Omega (\lambda g_0 u + f)(v - u_n)\,dx. \end{cases}$$

Hence we conclude the proof. □

In the following result, we still consider a weight g_0 with the same summability condition as in Theorem 2.2, but now we assume $f \in L^1(\Omega)$.

THEOREM 2.3. *Assume that f, g_0 are positive functions, $f \in L^1(\Omega)$ and $g_0 \in L^r(\Omega)$ with $r > \frac{N}{q}$, then for all $\lambda \geq 0$ there exists $u \in W_0^{1,q}(\Omega)$ such that $u \geq \psi$ and for all $v \in \mathcal{K}(\psi)$ we have*

$$\int_\Omega \nabla u \nabla (T_k(v-u))\,dx + \int_\Omega |\nabla u|^q (T_k(v-u))\,dx \geq \int_\Omega (\lambda g_0 u + f)(T_k(v-u))\,dx.$$

PROOF. Consider a sequence $f_n \in L^\infty(\Omega)$ such that $f_n \uparrow f$ in $L^1(\Omega)$. By Theorem 2.2, there exists a sequence of positive bounded functions $\{u_n\}$, solutions to problems,

$$\begin{cases} u_n \in \mathcal{K}(\psi), \quad \text{for all } v \in \mathcal{K}(\psi), \text{ we have} \\ \int_\Omega \nabla u_n \nabla (v-u_n)\,dx + \int_\Omega |\nabla u_n|^q (v-u_n)\,dx \geq \int_\Omega (\lambda g_0 u_n + f)(v-u_n)\,dx. \end{cases} \tag{2.10}$$

Consider the function

$$\Psi_k(s) = \begin{cases} 0 & \text{if } s \leq k \\ s-k & \text{if } k \leq s \leq k+1 \\ 1 & \text{if } s \geq k+1. \end{cases} \tag{2.11}$$

Define $v = u_n - \Psi_k(u_n)$, then $v \geq \psi$. Using v as a test function in (2.10) it follows that,

$$\int_{k \leq u_n \leq k+1} |\nabla u_n|^2\,dx + \int_\Omega |\nabla u_n|^q \Psi_k(u_n)\,dx$$
$$\leq \lambda \int_\Omega g(x) u_n \Psi_k(u_n)\,dx + \int_\Omega f_n(x) \Psi_k(u_n)\,dx.$$

Notice that,

$$\int_\Omega |\nabla u_n|^q \Psi_k(u_n)\,dx = \int_\Omega |\nabla \Theta(u_n)|^q\,dx,$$

where $\Theta(s) = \int_0^s \Psi_k^{1/q}(s)\,ds$. Using the hypothesis (1.3) on g we obtain,

$$\int_\Omega |\nabla \Theta(u_n)|^q\,dx \geq C(g_0,q)\Big(\int_\Omega g\Theta(u_n)\,dx\Big)^q.$$

Therefore, using the fact that $s\Psi_k(s) \leq \Theta(s) + C$, it follows that,

$$\int_{k \leq u_n \leq k+1} |\nabla u_n|^2\,dx + C(g_0,q)\Big(\int_\Omega g_0\Theta(u_n)\,dx\Big)^q$$
$$\leq \lambda \int_\Omega g_0\Theta(u_n)\,dx + C\lambda \int_\Omega g_0(x)\,dx + \int_\Omega f_n(x)\,dx.$$

Thus using Young's inequality there results

$$\int_{k\le u_n\le k+1} |\nabla u_n|^2\,dx + C(g_0,q)\Big(\int_\Omega g_0\Theta(u_n)\,dx\Big)^q \le C,$$

and then

$$\int_\Omega |\nabla u_n|^q\Psi_k(u_n)\,dx \le C, \quad \lambda\int_\Omega g_0(x)u_n\Psi_k(u_n)\,dx \le C,$$

where C is a positive constant depending only on the data.

We set now $v = u_n - T_k(u_n-\psi)$. It is clear that $v\ge\psi$, using v as a test function in (2.10) we get

$$\int_\Omega \nabla u_n\nabla T_k(u_n-\psi)\,dx + \int_\Omega |\nabla u_n|^q T_k(u_n-\psi)\,dx$$

$$\le \lambda\int_\Omega g_0u_nT_k(u_n-\psi)\,dx + \int_\Omega fT_k(u_n-\psi)\,dx.$$

Using the fact that $u_n\ge\psi$ and that $\lambda\int_\Omega g_0u_nT_k(u_n-\psi)\,dx + \int_\Omega fT_k(u_n-\psi)\,dx \le C$ for all n, it follows that

$$\int_{|u_n-\psi|\le k} \nabla u_n\nabla(u_n-\psi)\,dx \le C.$$

Then using Hölder's and Young's inequalities we get

$$\int_{|u_n-\psi|\le k} |\nabla u_n|^2\,dx \le C.$$

Let $k>0$, then

$$\int_{|u_n|\le k} |\nabla u_n|^2\,dx \le \int_{|u_n-\psi|\le k+\|\psi\|_{L^\infty}} |\nabla u_n|^2\,dx \le C.$$

Hence $\{T_k(u_n)\}$ is bounded in $W_0^{1,2}(\Omega)$ and $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$. Thus we get the existence of u such that $u_n\rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$ and $T_ku_n\rightharpoonup T_ku$ weakly in $W_0^{1,2}(\Omega)$. It is clear by the assumption on g_0 that $g_0u_n\to g_0u$ strongly in $L^1(\Omega)$.

Define $\Phi_{k-1}(s) = T_1(G_{k-1}(s))$, then $\Phi_{k-1}(u_n)|\nabla u_n|^q \ge |\nabla u_n|^q\chi_{\{u_n\ge k\}}$.

Let $v = u_n - \Phi_{k-1}(u_n)$, then $v\ge\psi$. Using v as a test function in (2.10) there results

$$\int_\Omega |\nabla\Phi_{k-1}(u_n)|^2\,dx + \int_\Omega \Phi_{k-1}(u_n)|\nabla u_n|^q\,dx \le \int_\Omega (\lambda g_0(x)u_n + f_n(x))\Phi_{k-1}(u_n)\,dx.$$

Since $\{u_n\}$ is uniformly bounded in $L^p(\Omega)$, $\forall p \le q^*$, it follows that

$$|\{x \in \Omega,\ \text{such that } k-1 < u_n(x) < k\}| \to 0,$$
$$|\{x \in \Omega,\ \text{such that } u_n(x) > k\}| \to 0 \quad \text{as } k \to \infty,$$

uniformly in n. Thus we conclude

$$\lim_{k\to\infty} \int_{\{u_n \ge k\}} |\nabla u_n|^q \, dx = 0, \quad \text{uniformly in } n. \tag{2.12}$$

We claim that $\nabla u_n \to \nabla u$, a.e. in Ω.

To prove the claim we follow the same arguments as in the proof of Theorem 2.2.

Let $v = u_n - (T_k(u_n) - T_k(u))^+$, then $v \in \mathscr{K}(\psi)$ and $v - u_n = -(T_k(u_n) - T_k(u))^+$, hence there result

$$\int_\Omega \nabla u_n \nabla (T_k(u_n) - T_k(u))^+ \, dx + \int_\Omega |\nabla u_n|^q (T_k(u_n) - T_k(u))^+ \, dx$$
$$\le \int_\Omega (\lambda g_0 u_n + f_n)(T_k(u_n) - T_k(u))^+ \, dx.$$

A direct computation shows that

$$\int_\Omega \nabla u_n \nabla (T_k(u_n) - T_k(u))^+ \, dx$$
$$= \int_\Omega |\nabla (T_k(u_n) - T_k(u))^+|^2 \, dx + \int_\Omega \nabla G_k(u_n) \nabla T_k(u) \, dx$$
$$+ \int_\Omega \nabla T_k(u) \nabla (T_k(u_n) - T_k(u))^+ \, dx$$
$$= \int_\Omega |\nabla (T_k(u_n) - T_k(u))^+|^2 \, dx + o(1).$$

It is clear that $\int_\Omega (\lambda g_0 u_n + f_n)(T_k(u_n) - T_k(u))^+ \, dx \to 0$ as $n \to \infty$, therefore we conclude that

$$\int_\Omega |\nabla (T_k(u_n) - T_k(u))^+|^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

Thus $(T_k(u_n) - T_k(u))^+ \to 0$ strongly in $W_0^{1,2}$. Take now $v = u_n + \phi((T_k(u_n) - T_k(u))^-)$, then $v \in \mathscr{K}(\psi)$. Using v as a test function in (2.10) we obtain that

$$\int_\Omega \nabla\phi'((w_n-u_k)^-)\nabla u_n \nabla(T_k(u_n)-T_k(u))^-\,dx$$
$$+\int_\Omega |\nabla u_n|^q \phi((T_k(u_n)-T_k(u))^-)\,dx$$
$$\le \lambda \int_\Omega g_0(x)u_n\phi((T_k(u_n)-T_k(u))^-)\,dx + \int_\Omega f_n(x)\phi((T_k(u_n)-T_k(u))^-)\,dx.$$

Thus

$$\int_{T_k(u_n)\le T_k(u)} \phi'((T_k(u_n)-T_k(u))^-)\nabla u_n \nabla(u_n-u)\,dx$$
$$-\int_\Omega |\nabla u_n)|\phi((T_k(u_n)-T_k(u))^-|\,dx$$
$$\le \lambda \int_{T_k(u_n)\le T_k(u)} g_0(x)u_n\phi((T_k(u_n)-T_k(u))^-\,dx$$
$$+\int_{T_k u_n \le T_k u} f_n(x)\phi((T_k(u_n)-T_k(u))^-)\,dx.$$

Since $T_k u_n \rightharpoonup T_k u$ weakly in $W_0^{1,2}(\Omega)$, then

$$\int_{T_k(u_n)\le T_k(u)} \phi'((T_k(u_n)-T_k(u))^-)\nabla u_n \nabla(u_n-u)\,dx$$
$$=\int_\Omega |\nabla((T_k(u_n)-T_k(u))^-)|^2\phi'((T_k(u_n)-T_k(u))^-)\,dx + o(1).$$

Since $q \le 2$, as in the computation in the proof of Theorem 2.2 it follows that

$$\frac{1}{2}\int_\Omega |\nabla((T_k(u_n)-T_k(u))^-)|^2\,dx$$
$$\le \int_\Omega (\phi'((T_k(u_n)-T_k(u))^-)$$
$$-\varepsilon|\phi((T_k(u_n)-T_k(u))^-)|\nabla((T_k(u_n)-T_k(u))^-|^2\,dx$$
$$\le o(1),$$

whence $(T_k(u_n)-T_k(u))^-)\to 0$ in $W_0^{1,2}(\Omega)$ and then $T_k(u_n)\to T_k(u)$ strongly in $W_0^{1,2}$. Hence the claim follows.

To finish the proof, we have to show that $|\nabla u_n|^q \to |\nabla u|^q$ strongly in $L^1(\Omega)$. Since the sequence of gradients converges a.e. in Ω, we have just to prove the equi-integrability of the sequence $\{|\nabla u_n|^q\}$ and then apply Vitali's Theorem. Let $E \subset \Omega$ be a measurable set. Then,

$$\int_E |\nabla u_n|^q\,dx \le \int_E |\nabla T_k u_n|^q\,dx + \int_{\{u_n \ge k\}\cap E} |\nabla u_n|^q\,dx.$$

Since $q \le 2$, then for all $k > 0$, $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,q}(\Omega)$. Hence the integral $\int_E |\nabla T_k(u_n)|^q\,dx$ is uniformly small if $|E|$ is small enough. On the other hand, by (2.12) we obtain that

$$\int_{\{u_n \ge k\}\cap E} |\nabla u_n|^q\,dx \le \int_{\{u_n \ge k\}} |\nabla u_n|^q\,dx \to 0 \quad \text{as } k \to \infty \text{ uniformly in } n.$$

The equi-integrability of $|\nabla u_n|^q$ follows immediately, and the proof is complete. $\square$

Proof of Theorem 2.1. Consider $g_n(x) = \min\{g_0(x), n\} \in L^\infty(\Omega)$. It is clear that $g_n \to g_0$ strongly in $W^{-1,q'}$.

Using Theorem 2.3, we get the existence of a sequence of nonnegative functions $\{u_n\}$ such that u_n solves

$$(2.13)\qquad \begin{cases} u_n \ge \psi \quad \text{for all } v \in \mathcal{K}(\psi), \text{ we have} \\ \displaystyle\int_\Omega \nabla u_n \nabla(T_k(v-u_n))\,dx + \int_\Omega |\nabla u_n|^q (T_k(v-u_n))\,dx \\ \qquad \displaystyle\ge \int_\Omega (\lambda g_n u_n + f)(T_k(v-u_n))\,dx. \end{cases}$$

By setting $v = u_n - \Psi_k(u_n)$, where Ψ_k is defined in (2.11), and using the same computations as in the proof of Theorem 2.3, it follows that

$$\int_{k \le u_n \le k+1} |\nabla u_n|^2\,dx + C(g_n,q)\Big(\int_\Omega g_n \Theta(u_n)\,dx\Big)^q$$
$$\le \lambda \int_\Omega g_n \Theta(u_n)\,dx + C\lambda \int_\Omega g_0(x)\,dx + \int_\Omega f_n(x)\,dx,$$

where

$$C(g_n,q) = \inf_{\phi \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{(\int_\Omega |\nabla\phi|^q\,dx)^{1/q}}{\int_\Omega g_n|\phi|\,dx}.$$

It is clear that $C(g_n,q) \uparrow C(g_0,q) > 0$. Then

$$\int_{k \le u_n \le k+1} |\nabla u_n|^2\,dx + \Big(\int_\Omega g_n \Theta(u_n)\,dx\Big)^q \le C,$$

and

$$\int_\Omega |\nabla u_n|^q \Psi_k(u_n)\,dx \le C, \quad \lambda \int_\Omega g_n(x) u_n \Psi(u_n)\,dx \le C,$$

where C is a positive constant depending only on the data. As in the proof of Theorem 2.3 we can prove that $\int_\Omega |\nabla u_n|^q\,dx \le C$ and then $u_n \to u$ weakly in $W_0^{1,q}(\Omega)$. Since $g_n \to g$ strongly in $W^{-1,q'}$, then $\int_\Omega g_n u_n\,dx \to \int_\Omega g_0 u\,dx$ strongly in $L^1(\Omega)$. Now to complete the proof we follow closely the argument used in the proof of Theorem 2.3. □

2.1. Partial uniqueness result. In this subsection we consider the case $q = 2$. We will prove a uniqueness result for positive solutions. We will use the next Comparison Principle that is a variation of the uniqueness result obtained in [1]. For the reader's convenience we include a short proof.

LEMMA 2.4. *Let $f \in L^1(\Omega)$ is a non negative function and suppose that g_0 is an admissible function in the sense of condition* (1.3). *Let $u_1, u_2 \in W_0^{1,2}(\Omega)$ be functions such that $u_1 > 0$* (*resp. $u_2 > 0$*) *is a subsolution* (*resp. supesolution*) *to problem*

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda g_0(x) u + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

then $u_1 \le u_2$ in Ω.

PROOF. For $i = 1, 2$ we set $v_i \equiv 1 - \mathrm{e}^{-u_i}$, then $0 < v_i \le 1$ in Ω and v_1 (resp. v_2) is a subsolution (resp. supersolution) to problem

$$\begin{cases} -\Delta v = \lambda g_0(x)(1-v)\log\left(\frac{1}{1-v}\right) + (1-v) f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Define

$$H(x,v) = \begin{cases} \lambda g(x)(1-v)\log\left(\frac{1}{1-v}\right) + (1-v) f(x), & \text{if } 0 < v < 1, \\ 0, & \text{if } v \ge 1. \end{cases}$$

By a direct computation we find that $\frac{H(v,x)}{v}$ is a non-increasing function in v for $v \ge 0$, then by similar arguments as in [1], we conclude that $v_1 \le v_2$, therefore the result follows.

Then we can prove the following result about uniqueness.

THEOREM 2.5. *Assume $q = 2$, $\psi \equiv 0$ and that the hypotheses of Theorem* 2.1 *hold. Then problem* (2.1) *has a unique positive solution.*

PROOF. The existence of a nonnegative solution is a consequence of the Theorem 2.1.

Assume that u_1 and u_2 are two nonnegative solution to the obstacle problem (2.1). We claim that u_1 is strictly positive in Ω (the same conclusion holds for u_2). To prove the claim we consider $\phi \in \mathscr{C}_0^\infty(\Omega)$ a nonnegative function. Let $v = T_h(u) + \phi$, it is clear that for h large we have $v \geq \psi$ and $v \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$. Using v as a test function in (2.1), it follows that

$$\int_\Omega \nabla u_1 \nabla(T_k(T_h(u_1) - u_1 + \phi))\,dx + \int_\Omega |\nabla u_1|^2 (T_k(T_h(u_1) - u_1 + \phi))\,dx \geq \int_\Omega (\lambda g_0 u_1 + f)(T_k(T_h(u_1) - u_1 + \phi))\,dx.$$

Since $u_1 \in W_0^{1,2}(\Omega)$, then for fixed k, using the Dominated Convergence Theorem and a duality argument we can pass to the limit in h, hence

$$\int_\Omega \nabla u_1 \nabla(T_k(\phi))\,dx + \int_\Omega |\nabla u_1|^2 (T_k(\phi))\,dx \geq \int_\Omega (\lambda g_0 u_1 + f)(T_k(\phi))\,dx. \tag{2.16}$$

Since $\phi \in \mathscr{C}_0^\infty(\Omega)$, then choosing k large enough we conclude that

$$\int_\Omega \nabla u_1 \nabla \phi\,dx + \int_\Omega |\nabla u_1|^2 \phi\,dx \geq \int_\Omega (\lambda g_0 u_1 + f)\phi\,dx.$$

Thus u_1 is a nonnegative supersolution to problem

$$\begin{cases} -\Delta w + |\nabla w|^2 = \lambda g_0(x) w + f & \text{in } \Omega, \\ w \geq 0 \text{ in } \Omega \quad \text{and} \quad w \in W_0^{1,2}(\Omega). \end{cases} \tag{2.17}$$

From [1] we know that the above problem has a unique positive solution and that if w_1 is a supersolution to problem (2.17), then $w_1 \geq w$. Since u_1 is a supersolution, then the claim follows.

We follow now closely the argument used in [3]. Define

$$v_1 = u_1 - \delta\phi, \quad \text{and} \quad v_2 = u_2 + \delta\phi.$$

It is clear that $v_2 \geq 0$. We show that for δ small enough, depending on ϕ, then $v_1 \geq 0$. It is clear that $v_1 \geq 0$ in $\Omega \backslash Supp(\phi)$. Since $Supp(\phi) \subset\subset \Omega$, then by using the strict positivity of u_1 there exists a positive constant c such that $u_1 \geq c$ in $Supp(\phi)$. Hence for $x \in Supp(\phi)$, we have

$$v_1(x) \geq C - \delta\|\phi\|_{L^\infty}.$$

Choosing $\delta \geq \frac{C}{2\|\phi\|_{L^\infty}}$, we conclude that $v_1(x) \geq C/2$ for all $x \in Supp(\phi)$. Hence $v_1 \geq 0$ in Ω. Notice that the same conclusion holds if we substitute v_1 by $v_1^h \equiv T_h(u_1) - \delta\phi$ where h is large enough.

It is clear that we cannot use v_1 and v_2 directly as a test function in the corresponding obstacle problem of u_1 and u_2. Thus we use an approximation argument. Set

$$v_1^h = T_h(u_1) - \delta\phi, \quad v_2^h = T_h(u_2) + \delta\phi.$$

As above, for k fixed and passing to the limit in h, it follows that

$$\int_\Omega \nabla u_1 \nabla(T_k(\delta\phi))\,dx + \int_\Omega |\nabla u_1|^2 (T_k(\delta\phi))\,dx \le \int_\Omega (\lambda g_0 u_1 + f)(T_k(\delta\phi))\,dx$$

and

$$\int_\Omega \nabla u_2 \nabla(T_k(\delta\phi))\,dx + \int_\Omega |\nabla u_2|^2 (T_k(\delta\phi))\,dx \ge \int_\Omega (\lambda g_0 u_2 + f)(T_k(\delta\phi))\,dx.$$

Letting $k \to \infty$ and using the fact that $\phi \in \mathscr{C}_0^\infty(\Omega)$, there results that

$$\int_\Omega \nabla u_1 \nabla\phi\,dx + \int_\Omega |\nabla u_1|^2 \phi\,dx \le \int_\Omega (\lambda g_0 u_1 + f)\phi\,dx$$

and

$$\int_\Omega \nabla u_2 \nabla\phi\,dx + \int_\Omega |\nabla u_2|^2 \phi\,dx \ge \int_\Omega (\lambda g_0 u_2 + f)\phi\,dx.$$

Thus u_1 (resp. u_2) is a nonnegative subsolution (resp. supersolution) to (2.17), then by Lemma 2.4 we conclude that $u_1 \le u_2$. Following the same argument as above we get easily that $u_2 \le u_1$. Hence $u_1 = u_2$ and then the uniqueness result follows. □

3. The general result

In this section we will consider the general case, that is, f and ψ can change sign. More precisely we will consider the following obstacle problem

$$(3.1)\qquad \begin{cases} u \ge \psi \text{ a.e in } \Omega, \quad \text{for all } v \in \mathscr{K}(\psi), \text{ we have} \\ \displaystyle\int_\Omega \nabla u \nabla(v-u)\,dx + \int_\Omega h(u)|\nabla u|^q (v-u)\,dx \ge \int_\Omega (g(x,u)+f)(v-u)\,dx, \end{cases}$$

where $\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $f \in L^1(\Omega)$. We will assume that h satisfies the next sign condition,

$$(3.2)\qquad h(s) \text{ is an continous increasing function such that, } h(s)s \ge 0, \ \forall s \in \mathbb{R}$$

and

$$|g(x,s)| \le \lambda g_0(x)|s| + g_1(x) \tag{3.3}$$

where g_0 satisfies the condition (1.3) and $g_1 \in L^1(\Omega)$.

We can formulate the general existence result as follows.

THEOREM 3.1. *Assume that conditions* (3.2) *and* (1.2) *hold. Then for all* $f \in L^1(\Omega)$, *there exists a function* u *such that* $h(u)|\nabla u|^q \in L^1(\Omega)$, $T_k(u) \in W_0^{1,2}(\Omega)$ *for all* $k > 0$, *and for all* $v \in \mathcal{K}(\psi)$ *we have*

$$\int_\Omega \nabla u \nabla (T_k(v-u))\,dx + \int_\Omega h(u)|\nabla u|^q (T_k(v-u))\,dx \ge \int_\Omega (g(x,u)+f)(T_k(v-u))\,dx. \tag{3.4}$$

We will say that u *is an entropy solution to the obstacle problem* (3.1).

As in section 2 we will prove the existence result for regular data and then we will pass to the limit.

For $n, m > 0$, we define,

$$h_n(s) = \frac{h(s)}{1+\frac{1}{n}|h(s)|} \quad \text{and} \quad g_m(x,s) = \frac{g(x,s)}{1+\frac{1}{m}|g(x,s)|}.$$

Then we have

THEOREM 3.2. *Assume that the above conditions* (3.2) *and* (1.2) *hold, then for all* $f \in L^\infty(\Omega)$ *and for all* m *fixed, there exits a* $u_m \in W_0^{1,2}(\Omega)$ *such that* $h(x,u_m)|\nabla u_m|^q \in L^1(\Omega)$ *and for all* $v \in \mathcal{K}(\psi)$ *we have*

$$\int_\Omega \nabla u_m \nabla (v-u_m)\,dx + \int_\Omega h(u_m)|\nabla u_m|^q (v-u_m)\,dx \ge \int_\Omega (g_m(x,u_m)+f)(v-u_m)\,dx. \tag{3.5}$$

PROOF. Fixed $m > 0$, since $f \in L^\infty(\Omega)$, using the classical results in [13] or [12], there exists $u_n \in \mathcal{K}(\psi)$, a solution to the obstacle problem

$$\int_\Omega \nabla u_n \nabla (v-u_n)\,dx + \int_\Omega h_n(u_n)\frac{|\nabla u_n|^q}{1+\frac{1}{n}|\nabla u_n|^q}(v-u_n)\,dx \ge \int_\Omega (\lambda g_m(x,u_n)+f)(v-u_m)\,dx \tag{3.6}$$

for all $v \in \mathcal{K}(\psi)$. For fixed m, we want to pass to the limit in n. Let $v = T_k(w_n)$, since $\psi \in L^\infty(\Omega)$, then choosing k such that $k \gg \|\psi\|_\infty$, we conclude that v is an admissible test function. Notice that $v - u_n = -G_k(u_n)$, hence

$$\int_\Omega |\nabla G_k(u_n)|^2\, dx + \int_\Omega h_n(u_n) \frac{|\nabla G_k(u_n)|^q}{1 + \frac{1}{n}|G_k(u_n)|^q} G_k(u_n) \leq \int_\Omega (g_m(x, u_n) + f) G_k(u_n)\, dx.$$

Since $h_n(u_n)G_k(u_n) \geq 0$, then

$$\int_\Omega |\nabla G_k(u_n)|^2\, dx \leq \int_\Omega (m + |f|)|G_k(u_n)|\, dx.$$

Since $f \in L^\infty(\Omega)$, then by the classical Stampacchia result, see [15], the following L^∞-estimate holds,

$$\|u_n\|_{L^\infty} \leq C(m, f, \Omega).$$

Take $v = \psi$ as test function in (3.6), then

$$\int_\Omega \nabla u_n \nabla(u_n - \psi)\, dx + \int_\Omega h_n(u_n) \frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} (u_n - \psi)\, dx \leq \int_\Omega (g_m(x, u_n) + f)(u_n - \psi)\, dx.$$

Since $u_n \geq \psi$, we get

$$\int_\Omega |\nabla u_n|^2\, dx \leq \int_\Omega \nabla u_n \nabla \psi\, dx + \int_\Omega g_m(x, u_n) + f)(u_n - \psi)\, dx.$$

Thus using Hölder inequality and the previous estimate we obtain that

$$\int_\Omega |\nabla u_n|^2\, dx \leq C(f, m\Omega, k) \quad \text{uniformly in } n.$$

Therefore, up to a subsequence, $u_n \rightharpoonup u_m$ weakly in $W_0^{1,2}(\Omega)$ as $n \to \infty$. By weak-*convergence in $L^\infty(\Omega)$ we also have that $w_m \in W_0^{1,2} \cap L^\infty(\Omega)$ and $u_m \geq \psi$. Since $\{u_n\}$ is bounded in $L^\infty(\Omega)$, then as in the first section, following closely the argument used in [6], we get the strong convergence of u_n in $W_0^{1,2}(\Omega)$. Thus $u_m \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and it solves the obstacle problem (3.5). Hence the result follows. □

Now we can prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let $\{f_n\}$ be a sequence of bounded functions such that $f_m \to f$ strongly in $L^1(\Omega)$ and consider u_m the solution to the obstacle problem (3.5) obtained above and let $k > 0$, then using $v = u_m - \Psi_k(u_m)$, where Ψ_k is defined in (2.11) with $\Psi(s) = -\Psi(-s)$ for $s < 0$, as a test function in (3.5) it follows that

$$\int_{k\le|u_m|\le k+1} |\nabla u_m|^2\,dx + \int_\Omega h(u_m)|\nabla u_m|^q \Psi_k(u_m)$$
$$\le \lambda \int_\Omega g_0(x)|u_m|\,|\Psi_k(u_m)| + \int_\Omega g_1(x)\,dx + \int_\Omega |f_n(x)|\,dx.$$

Using the properties of h it follows that, for k large,

$$h(u_m)|\nabla u_m|^q\Psi_k(u_m) \ge C(k)|\nabla u_m|^q|\Psi_k(u_m)| \ge C|\nabla\Theta(u_m)|^q$$

where $\Theta(s) = \displaystyle\int_0^s |\Psi(s)|^{1/q}\,ds$. Thus

$$\int_{k\le|u_m|\le k+1} |\nabla u_m|^2\,dx + \int_\Omega |\nabla\Theta(u_m)|^q \le \lambda \int_\Omega g_0(x)|u_m|\,|\Psi_k(u_m)| + C.$$

Notice that $|s|\,|\Psi_k(s)| \le \Theta(s) + C$, then

$$\int_{k\le|u_m|\le k+1} |\nabla u_m|^2\,dx + \int_\Omega |\nabla\Theta(u_m)|^q \le \lambda \int_\Omega g_0(x)\Theta(u_m)\,dx + C.$$

Since g_0 satisfies the condition (1.3), thus

$$\int_{k\le|u_m|\le k+1} |\nabla u_m|^2\,dx + \int_\Omega |\nabla\Theta(u_m)|^q \le C(k).$$

Therefore

$$\int_\Omega |\nabla u_n|^q|\Psi_k(u_m)|\,dx \le C, \quad \lambda \int_\Omega g_0(x)|u_n|\,|\Psi(u_n)|\,dx \le C$$

where C is a positive constant depending only on the data. As in the proof of Theorem 2.3 we can prove that $\displaystyle\int_\Omega |\nabla u_n|^q\,dx \le C$ and then $u_n \to u$ weakly in

$W_0^{1,q}(\Omega)$. It is clear that $\int_\Omega g_0 u_n\,dx \to \int_\Omega g_0 u\,dx$ strongly in $L^1(\Omega)$, then using the Dominated Convergence Theorem we obtain that

$$g_m(x, u_m) \to g(x, u) \quad \text{strongly in } L^1(\Omega).$$

The rest of the proof follows exactly as in the proof of Theorem 2.3. □

References

[1] B. Abdellaoui - I. Peral - A. Primo, *Breaking of resonance and regularizing effect of a first order quasi-linear term in some elliptic equations*, Ann. I. H. Poincaré 25 (2008), 969–985.

[2] A. Ambrosetti - G. Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces*. Ann. Mat. Pura Appl. (4) 93 (1972), 231–246.

[3] G. Barles - F. Murat, *Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions*. Arch. Rational Mech. Anal. 133, no. 1 (1995), 77–101.

[4] A. Bensoussan - L. Boccardo - F. Murat, *On a non linear partial differential equation having natural growth terms and unbounded solutions*, Ann. I. H. Poincaré 5 (1988), 347–364.

[5] L. Boccardo - T. Gallouet, Problème Unilatéraux avec données dans L^1 C.R. Acad. Sci. Paris (Serie I) 311 (1990), 617–619.

[6] L. Boccardo - T. Gallouët - L. Orsina, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, J. Anal. Math. 73 (1997), 203–223.

[7] L. Boccardo - G. R. Cirmi, *Existence and uniqueness of solution of Unilateral Problems with L^1 data*, J. Convex Analysis 6, no. 1 (1999), 195–206.

[8] L. Boccardo - F. Murat - J. P. Puel, *Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique*, Nonlinear Partial Differential Equations and Their Applications, Collège de France, Seminar, Vol. IV, in Research Notes in Math, Vol. 84, Pitman, London, 1983, 19–73.

[9] L. Boccardo - S. Segura - C. Trombetti, *Bounded and unbounded solutions for quasilinear elliptic problems with quadratic term,* J. Math. Pures Appl. 80, no. 9 (2001), 919–940.

[10] H. Brezis - S. Kamin, *Sublinear elliptic equations in $\mathbb{R}^N$*, Manuscripta Math. 74 (1992), 87–106.

[11] H. Brézis - A. Ponce, *Reduced measures for obstacle problems,* Adv. Differential Equations 10 (2005), 1201–1234.

[12] D. Kinderlehrer - G. Stampacchia, *An introduction to vatiational inequalities and their applications*, Academic Press, New York, 1980.

[13] J. L. Lions, *Quelques méthodes de résolution des problemes aux limites non lineaires*, Dunod, Paris, 1969.

[14] A. Mokrane - F. Murat, *The Lewy-Stampacchia inequality for the obstacle problem with quadratic growth in the gradient*. Ann. Mat. Pura Appl. 184, no. 3 (2005), 347–360.

[15] G. Stampacchia, *Equations elliptiques du second ordre a coefficients discontinus*, Les presse de l' univeersité de Montreal, 1966.

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