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On $\mathbb{Q}$-Conic Bundles, III

To the memory of the late Professor Masayoshi Nagata

By

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Abstract

A $\mathbb{Q}$-conic bundle germ is a proper morphism from a threefold with only terminal singularities to the germ $(Z \ni o)$ of a normal surface such that fibers are connected and the anti-canonical divisor is relatively ample. Building upon our previous paper [MP08a], we prove the existence of a Du Val anti-canonical member under the assumption that the central fiber is irreducible.

§1. Introduction

The present paper is a continuation of a series of papers [MP08a], [MP08b].

Recall that a $\mathbb{Q}$-*conic bundle* is a projective morphism $f\colon X \to Z$ from an (algebraic or analytic) threefold with terminal singularities to a surface that satisfies the following properties:

(i) $f_*\mathscr{O}_X = \mathscr{O}_Z$ and all fibers are one-dimensional,

(ii) $-K_X$ is f-ample.

For $f\colon X \to Z$ as above and for a point $o \in Z$, we call the *analytic* germ $(X, f^{-1}(o)_{\mathrm{red}})$ a $\mathbb{Q}$-*conic bundle germ.*

In this paper we complete the proof of the following

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Theorem 1.1 (Reid's general elephant conjecture). *Let $f\colon (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ with irreducible central fiber C. Then a general member E_X of $|-K_X|$ has only Du Val singularities.*

We recall that the existence of a Du Val member $E_X \in |-K_X|$ follows from [MP08a, (1.3.7)] if the base (Z, o) is singular. Thus the case of smooth base remains to be studied for the theorem.

Remark 1.2. Let $f : (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ over a smooth base. For each singular point P of X with $P \in C$, consider the germ $(P \in C \subset X)$. All such germs (or all such singular points, for simplicity) are classfied into five types (IA), (IC), (IIA), (IIB), and (III), whose definitions we refer the reader to [KM92] or [MP08a]. By the index of the germ, we simply mean the index of P.

For each given f, one can consider the set $S(f)$ of all such germs $(P \in C \subset X)$ with $P \in C$ a singular point of X. If for instance $S(f)$ consists of two germs of type (IA) and (IA), we write $S(f)$ =(IA)+(IA) as a convenient shorthand. Or if f is smooth, we may write $S(f) = \emptyset$. By abuse of language, $S(f)$ may be called the *type* of (X, C).

It is proved in [MP08a] that $S(f)$ must be one of the following:

(i) $\emptyset$, (III), (III)+(III),

(ii) (IA), (IA)+(III), (IA)+(III)+(III),

(iii) (IIA), (IIA)+(III),

(iv) (IC), (IIB),

(v) (IA)+(IA) of indices 2 and odd m (≥ 3),

(vi) (IA)+(IA)+(III) of indices 2, odd m (≥ 3) and 1.

We recall that in the cases (ii) and (iii) (the case (i) is trivial) a general member $E_X \in |-K_X|$ does not contain C and has only Du Val singularity at $E_X \cap C$ [MP08a, (1.3.7)]. Cases (iv)–(vi) are treated in this paper. More precisely we give a complete classification of $E_X \in |-K_X|$:

Theorem 1.3 (cf. [KM92, (2.2)]). *Let $f\colon (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ with smooth base surface Z. Assume that $S(f) =$ (IC), (IIB), (IA)+(IA), or (IA)+(IA)+(III) (cf. Remark 1.2). Then a general member E_X of $|-K_X|$ and $E_Z := \operatorname{Spec}_Z f_*\mathscr{O}_{E_X}$ have only Du Val singularities. To be more explicit, the minimal resolutions of E_Z and E_X coincide. Furthermore the* *following assertions hold depending on $S(f)$, the type of (X, C) (below, $o' \in E_Z$ is the image of C):*

Case 1.3.1 ((IC), [KM92, (2.2.2)]). (E_Z, o') is of type D_m and $\Delta(E_Z, o')$ is

$$\begin{array}{r} \circ \\ | \\ \underbrace{\circ - \cdots - \circ}_{m-3} - \circ - \bullet, \end{array}$$

where m, the index of the (IC) point of C, is odd and $m \geq 5$.

Case 1.3.2 ((IIB), [KM92, (2.2.2′)]). (E_Z, o') is of type E_6 and $\Delta(E_Z, o')$ is

$$\begin{array}{c} \circ \\ | \\ \circ - \circ - \circ - \circ - \bullet. \end{array}$$

Case 1.3.3 ((IA)+(IA), [KM92, (2.2.3)], [Mor07]). The two (IA) points are an ordinary point of odd index $m \geq 3$ and an index 2 point of type $cA/2$, $cAx/2$ or $cD/2$ with axial multiplicity k such that $k \geq 2$ if $m = 3$. (E_Z, o') is of type D_{2k+m}, $\operatorname{Sing} E_X$ is of type $A_{m-1} + D_{2k}$ ($A_{m-1} + A_1 + A_1$ if $k = 1$) and $\Delta(E_Z, o')$ is

$$\begin{array}{r} \circ \\ | \\ \underbrace{\circ - \cdots - \circ}_{m-1} - \bullet - \circ - \cdots - \circ - \circ \end{array}.$$

Case 1.3.4 ((IA)+(IA)+(III), [KM92, (2.2.3′)]). The two (IA) points are both ordinary and of indices 2 and m (odd, ≥ 3). (E_Z, o') is of type D_{m+2} ($m \geq 3$) or E_6 ($m = 3$). The graph $\Delta(E_Z, o')$ is

$$\begin{array}{r} \circ \\ | \\ \underbrace{\circ - \cdots - \circ}_{m-1} - \bullet - \circ \end{array} \ (m \geq 3), \quad \text{or} \quad \begin{array}{c} \circ \\ | \\ \circ - \circ - \bullet - \circ - \circ \end{array} \ (m = 3).$$

Above, we use the usual notation of graphs $\Delta(E_Z, o')$: $\bullet$ corresponds to the curve C and each $\circ$ corresponds to a (-2)-curve on the minimal resolution of E_X.

Note that we do not assert that all the above cases really occur. The question about existence will be discussed in our subsequent paper (in preparation). Thus at the moment the less investigated case is the case where the base is smooth and the central fiber C is reducible. This class of $\mathbb{Q}$-conic bundles was studied only under additional assumption of "semistability" [Pro04], [Pro05].

The techniques used in this paper are very similar to those in [KM92, §2]. The main difference is that for the conic bundle case we do not have vanishing of $H^1(X, \omega_X)$ which was used in [KM92, (2.5)] to extend sections from $D \in |-2K_X|$. Instead of the vanishing we use Proposition 2.1 and Corollary 2.2.

§2. Preliminaries

Proposition 2.1 (cf. [KM92, (2.5)]). *Let $f\colon (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ with smooth base surface Z. Let $D \in |-nK_X|$ for some integer $n > 0$ such that the restriction $g = f_D\colon D \to Z$ is finite. The standard exact sequence*

$$0 \to \omega_X \to \omega_X(D) \to \omega_D \to 0$$

induces the exact sequence

$$f_*\omega_X(D) \xrightarrow{\alpha} g_*\omega_D \xrightarrow{\operatorname{Tr}_{D/Z}} \omega_Z,$$

where $\omega_Z \simeq R^1 f_*\omega_X$ by [MP08a, Lemma (4.1)], *and the natural map $g^*\colon \omega_Z \to g_*\omega_D$ has the property $\operatorname{Tr}_{D/Z} \circ g^* = 2n \operatorname{id}_{\omega_Z}$.*

Proof. In view of the relative duality, this follows from the fact that $\mathscr{O}_Z \longrightarrow g_*\mathscr{O}_D \xrightarrow{\operatorname{Tr}_{Z/D}} \mathscr{O}_Z$ is the multiplication by $\deg(D/Z) = 2n$. □

Corollary 2.2. *The homomorphism induced by Proposition* 2.1

$$f_*\omega_X(D) \to (g_*\omega_D)/\omega_Z$$

is surjective.

Notation 2.3. Everywhere below

$$f\colon (X, C) \to (Z, o)$$

denotes a $\mathbb{Q}$-conic bundle germ. We assume that the curve C is irreducible (and so $C \simeq \mathbb{P}^1$) and the base surface (Z, o) is smooth. Notation and techniques of [Mor88] will be used freely. Additionally, $(c + \sum d_i P_i^\sharp)$ denotes the element of $\operatorname{Cl}^{\mathrm{sc}}(X) \simeq \operatorname{Pic}^\ell(X) \simeq \operatorname{QL}(C)$ corresponding to $c + \sum d_i P_i^\sharp \in \operatorname{QL}(C)$ (see [KM92, (2.7)]).

The symbol $\Delta(E_Z, o')$ at the end of Theorem 1.3 is extended as follows. Let E be a normal surface and $C \subset E$ a curve such that the proper transform $\tilde{C}$ of C and the exceptional divisors Γ_i on the minimal resolution of E form a simple normal crossing divisor. The graph $\Delta(E, C)$ is the dual graph of the divisor $\tilde{C}+\sum \Gamma_i$, where each component of $\tilde{C}$ is drawn $\bullet$ and each Γ_i is drawn $\circ$, and if no weight is specified we mean that the corresponding Γ_i is a (-2)-curve. We note that $\Delta(E_Z, o') = \Delta(E_X, C)$.

Remark 2.4. As explained in [Mor88, §1b] and [MP08a, §6], any local deformation near points $P_i \in (X, C)$ on $\mathbb{Q}$-conic bundle germ (X, C) can be extended to a global deformation (X_λ, C_λ). A general element (X_λ, C_λ) of the family can be either an extremal neighborhood or again a $\mathbb{Q}$-conic bundle germ. In some cases this allows us to obtain certain restrictions on the possible configurations of singular points. We will use these arguments several times below.

§3. Case of (IC)

3.1 (Cf. [KM92, (2.10)]**).** Let P be the (IC) point of index m and

$$(y_1, y_2, y_4)/\boldsymbol{\mu}_m(2, m-2, 1)$$

be coordinates for the canonical cover $P^\sharp \in C^\sharp \subset X^\sharp$ given in [Mor88, (A.3)] so that $C^\sharp$ is parametrized by $(t^2, t^{m-2}, 0)$. In this case, P is the only singular point of X on C [MP08a, (8.2)]. Since $y_1^{m-2} - y_2^2$ and y_4 generate the defining ideal of $C^\sharp$, they form an ℓ-free ℓ-basis of $\operatorname{gr}^1_C \mathscr{O}_X$. It is easy to see that $\Omega = dy_1 \wedge dy_2 \wedge dy_4$ is an ℓ-free ℓ-basis of $\operatorname{gr}^0_C \omega_X$. Then $\operatorname{ql}_C(\omega_X) = -P^\sharp$ and $D = \{y_1 = 0\}/\boldsymbol{\mu}_m \in |-2K_X|$ by $(D \cdot C) = 2/m$. By

$$\operatorname{ql}_C(\operatorname{gr}^0_C(\omega^*)) = \operatorname{ql}_C(\omega^*) = -\operatorname{ql}_C(\omega) = P^\sharp,$$

one has

$$\deg(\operatorname{gr}^0_C(\omega^*)) = \operatorname{TL}(P^\sharp) = -U(-1) = -1$$

(see [Mor88, (8.9.1)(iii)]), where

$$U(x) = \min\{z \in \mathbb{Z} \mid mz - x \in 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+\}$$

[Mor88, (2.8)]. Thus

$$\operatorname{gr}^0_C(\omega^*) = \omega^*/F^1_C(\omega^*) \simeq \mathscr{O}_C(-1)$$

and $H^0(\mathscr{O}_X(-K_X))=H^0(F_C^1(\omega^*))$. Hence a general section $s\in H^0(\mathscr{O}_X(-K_X))$ is written as $(\lambda\cdot y_4+\mu\cdot(y_1^{m-2}-y_2^2))/\Omega$ near P, where $\lambda\in\mathscr{O}_X$ and $\mu\in\mathscr{O}_{X^\sharp}$ with $\operatorname{wt}\mu\equiv 5 \mod (m)$. Now apply Corollary 2.2 with $n=2$. We have $\omega_D\simeq\mathscr{O}_D(-K_X)$, and

$$(3.1.1)\quad g^*\omega_Z\subset\wedge^2\mathscr{O}_D\bigl(dy_2^m,\ dy_2y_4^2,\ dy_2^{(m+1)/2}y_4,\ dy_4^m\bigr)\subset$$
$$\mathscr{O}_D\bigl(y_2^{(3m-1)/2},\ y_2^my_4,\ y_2^{(m+1)/2}y_4^2,\ y_2^{m-1}y_4^{m-1},\ y_2^{(m-1)/2}y_4^m,\ y_4^{m+1}\bigr)dy_2\wedge dy_4.$$

Since $y_4dy_2\wedge dy_4$ corresponds to y_4/Ω, we have $\lambda(0)\neq 0$. Hence s induces a section $\overline{s}$ of $\operatorname{gr}_C^1(\omega^*)=F_C^1(\omega^*)/F_C^2(\omega^*)$ and $\overline{s}$ is a part of an ℓ-free ℓ-basis of $\operatorname{gr}_C^1(\omega^*)$ at P. This induces an ℓ-exact sequence

$$(3.1.2)\qquad 0\to(a)\to\operatorname{gr}_C^1(\omega^*)\to(b+5P^\sharp)\to 0,$$

where $a,b\in\mathbb{Z}$, $a\geq 0$. This is because y_4/Ω and $(y_1^{m-2}-y_2^2)/\Omega$ have weights $\equiv 0$ and $m-5 \mod (m)$, respectively. We claim an ℓ-isomorphism

$$(3.1.3)\qquad \operatorname{gr}_C^1\mathscr{O}\simeq(4P^\sharp)\,\tilde{\oplus}\,(-1+(m-1)P^\sharp).$$

First recall that m is odd and $m\geq 5$ since P is an (IC) point. By (3.1.2) $\tilde{\otimes}$ $\operatorname{gr}_C^0\omega$, there is an ℓ-exact sequence

$$(3.1.4)\qquad 0\to((a-1)+(m-1)P^\sharp)\to\operatorname{gr}_C^1\mathscr{O}\to(b+4P^\sharp)\to 0.$$

It follows from $i_P(1)=2$ [Mor88, (6.5)] that $\deg\operatorname{gr}_C^1\mathscr{O}=-1$. By

$$\deg((a-1)+(m-1)P^\sharp)=a-1$$

and $\deg(b+4P^\sharp)=b$ [Mor88, (8.9.1)(iii)], we have $a+b=0$. Hence from

$$\begin{aligned}\operatorname{ql}_C((a-1)+(m-1)P^\sharp-(b+4P^\sharp))&=\\ &=\operatorname{ql}_C(2a-1+(m-5)P^\sharp)=2a-1\geq -1,\end{aligned}$$

we see that (3.1.4) is ℓ-split by [KM92, (2.6)]. Since $H^1(C,\operatorname{gr}_C^1\mathscr{O})=0$ by [MP08a, Corollary (2.3.1)], we have $b\geq -1$ and hence $(a,b)=(0,0)$ or $(1,-1)$. Whence (3.1.3) follows if $(a,b)=(0,0)$ or $m=5$. Assuming $(a,b)=(1,-1)$ and $m\geq 7$, we will derive a contradiction. Now (3.1.2) $\tilde{\otimes}\,\omega_X^{\tilde{\otimes}2}$ gives us an ℓ-exact sequence

$$0\to(-1+(m-2)P^\sharp)\to\operatorname{gr}_C^1\omega\to(-1+3P^\sharp)\to 0.$$

Since $\operatorname{Spec}\mathscr{O}_X/I_C^{(2)}\not\supset f^{-1}(o)$, by [MP08a, Theorem (4.4)] we have $H^1(C,\operatorname{gr}_C^1\omega)=0$. Whence,

$$-1\leq\deg(-1+3P^\sharp)=\operatorname{TL}(-1+3P^\sharp)=-2.$$

This is a contradiction and (3.1.3) is proved. Thus,

$$\mathrm{gr}^1_C(\omega^*) = (5P^\sharp)\,\tilde{\oplus}\,(0). \tag{3.1.5}$$

We claim that $\overline{s}$ is a nowhere vanishing section of the locally free sheaf $\mathrm{gr}^1_C(\omega^*) \simeq \omega^* \,\tilde{\otimes}\, \mathrm{gr}^1_C \mathscr{O}$. In case $m \geq 7$, there is a splitting $\mathrm{gr}^1_C(\omega^*) \simeq \mathscr{O}_C \oplus \mathscr{O}_C$ or $\mathscr{O}_C \oplus \mathscr{O}_C(-1)$ by (3.1.5) and $\overline{s}(P) \neq 0 \in \mathrm{gr}^1_C(\omega^*) \otimes \mathbb{C}(P)$ whence $\overline{s}$ is nowhere vanishing. In case $m = 5$, there is a splitting $\mathrm{gr}^1_C(\omega^*) \simeq \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and

$$\overline{s}(P) = \left(\lambda(0) \cdot y_4 + \mu(0) \cdot (y_1^{m-2} - y_2^2)\right)/\Omega \in \mathrm{gr}^1_C(\omega^*) \otimes \mathbb{C}(P)$$

is a general element because $\lambda(0)$ and $\mu(0)$ can be chosen arbitrary by Corollary 2.2 and by (3.1.1). Indeed, note that $y_4 dy_2 \wedge dy_4$ and $y_2^2 dy_2 \wedge dy_4$ are linearly independent modulo $\wedge^2 \mathscr{O}_D(dy_2^m,\, dy_2 y_4^2,\, dy_2^{\frac{m+1}{2}} y_4,\, dy_4^m)$. Thus $\overline{s}$ is nowhere vanishing and the claim is proved. We study $E_X = \{s = 0\} \in |-K_X|$. Since $\overline{s}$ is a nowhere vanishing section of $\mathrm{gr}^1_C(\omega^*) \simeq \omega^* \,\tilde{\otimes}\, \mathrm{gr}^1_C \mathscr{O}$, E_X is smooth on $C \setminus \{P\}$. The canonical cover $E_{X^\sharp}$ at P is defined by $y_4 + y_2(\cdots) + y_1(\cdots) = 0$. Therefore $(E_X, P) = (y_1, y_2)/\boldsymbol{\mu}_m(2, m-2)$ has only Du Val singularities, whence so is E_Z by $(K_X \cdot C) = 0$.

For the precise result, we express $(E_X, P) = (x_1, x_2, x_3; x_1 x_2 = x_3^m)$, where $x_1 = y_1^m$, $x_2 = y_2^m$ and $x_3 = y_1 y_2$. The curve C is the image of $C^\sharp$, the locus of (t^2, t^{m-2}), where C is the locus of (s^2, s^{m-2}, s) in this embedding of (E_X, P), where $s = t^m$. Then it is easy to check

Computation 3.2 (see [KM92, (2.10.5)]). *Let (E, P) be an A_{m-1} -singularity*:

$$(E, P) = (x_1, x_2, x_3; x_1 x_2 = x_3^m),$$

and C the locus of (s^2, s^{m-2}, s). Then $\Delta(E, C)$ is as in 1.3.1.

Thus the proof of Theorem 1.3 is completed in the case (IC). □

§4. Case of (IIB)

4.1 (**Cf.** [KM92, (2.11)]). Let $P \in (X, C)$ be of type (IIB). Then

$$(X, P) \simeq (y_1, y_2, y_3, y_4; \phi)/\boldsymbol{\mu}_4(3, 2, 1, 1; 2)$$

with $C^\sharp$ the locus of $(t^3, t^2, 0, 0)$ [Mor88, (A.3)], where

$$\phi = y_1^2 - y_2^3 + \psi$$

and $\psi \in (y_3, y_4)$ satisfies $\operatorname{wt}\psi \equiv 2 \mod (4)$ and $\psi(0,0,y_3,y_4) \notin (y_3,y_4)^3$. The last condition comes from the classification of terminal singularities [Rei87, (6.1)(2)]. In this case, P is the only singular point of X on C [Mor88, (B.1)]. Since y_3 and y_4 generate the defining ideal of $C^\sharp$, they form an ℓ-free ℓ-basis of $\operatorname{gr}^1_C \mathscr{O}_X$. By residue,

$$\Omega = \operatorname{Res}\frac{dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4}{\phi} = \frac{dy_2 \wedge dy_3 \wedge dy_4}{\partial\phi/\partial y_1}$$

is an ℓ-free ℓ-basis of $\operatorname{gr}^0_C \omega_X$ with $\operatorname{wt}\Omega \equiv 1 \mod (4)$.

Lemma 4.2 (cf. [KM92, (2.11), p. 549])**.** $i_P(1) = 2$.

Proof. Using the parametrization $(t^3, t^2, 0, 0)$ of $C^\sharp$ and ℓ-free ℓ-basis (y_3, y_4) of $\operatorname{gr}^1_C \mathscr{O}_X$, we see the following on $C^\sharp \subset X^\sharp$:

$$\begin{aligned} \operatorname{gr}^0_C \omega \mid_{\tilde{C}} &= \mathscr{O}_{\tilde{C}} t^3 \Omega \mid_{\tilde{C}} \\ &= \mathscr{O}_{\tilde{C}} t dt \wedge dy_3 \wedge dy_4, \\ \wedge^2(\operatorname{gr}^1_C \mathscr{O}) \otimes \Omega^1_C \mid_{\tilde{C}} &= \mathscr{O}_{\tilde{C}} (t^3 y_3) \wedge (t^3 y_4) \otimes d(t^4) \\ &= \mathscr{O}_{\tilde{C}} t^9 y_3 \wedge y_4 \otimes dt. \end{aligned}$$

Thus (cf. [Mor88, (2.2)])

$$\wedge^2(\operatorname{gr}^1_C \mathscr{O}) \otimes \Omega^1_C = t^8 \operatorname{gr}^0_C \omega.$$

Hence $i_P(1) = 2$ as claimed because t^4 is a coordinate of C at P. □

By [MP08a, (4.4.3)] we have $\deg \operatorname{gr}^0_C \omega = -1$. Then using [MP08a, (3.1.2)] we obtain $\deg \operatorname{gr}^1_C \mathscr{O} = -1$. Thus we see $\operatorname{gr}^0_C \omega \simeq (-1 + 3P^\sharp)$ and $\operatorname{gr}^1_C \mathscr{O} \simeq (3P^\sharp) \tilde{\oplus} (-1 + 3P^\sharp)$ with ℓ-structures using their ℓ-free ℓ-bases at P above. Let $D = \{y_2 = 0\}/\boldsymbol{\mu}_4$. Then $D \in |-2K_X|$ by $(D \cdot C) = 1/2$. By

$$\operatorname{ql}_C(\operatorname{gr}^0_C(\omega^*)) = \operatorname{ql}_C(\omega^*) = -\operatorname{ql}_C(\omega) = P^\sharp,$$

one has

$$\deg(\operatorname{gr}^0_C(\omega^*)) = \operatorname{TL}(P^\sharp) = -U(-1) = -1$$

because

$$U(x) = \min\{z \in \mathbb{Z} \mid 4z - x \in 2\mathbb{Z}_+ + 3\mathbb{Z}_+\}$$

[Mor88, (8.9.1)(iii)]. Thus $\operatorname{gr}^0_C(\omega^*) \simeq \mathscr{O}_C(-1)$ and a general section $s \in H^0(\mathscr{O}_X(-K_X))$ vanishes along C, i.e. $s \in H^0(F^1_C(\omega^*))$. Hence $s = (\lambda \cdot y_3 + \mu \cdot$

$y_4)/\Omega$ for some λ and $\mu \in \mathscr{O}_X$. We see that $\lambda(0)$ and $\mu(0) \in \mathbb{C}$ are general by Corollary 2.2 (cf. [KM92, (2.5)]). Indeed, in Corollary 2.2 with $n = 2$, we have $\omega_D \simeq \mathscr{O}_D(-K_X)$ with

$$\left.\frac{y_j dy_3 \wedge dy_4}{y_1 + (\cdots)}\right|_{D^\sharp} \overset{\text{Res}}{\longleftrightarrow} \frac{y_j}{\Omega}, \qquad j = 3,\ 4.$$

In view of

$$g^*\omega_Z \subset \wedge^2 \mathscr{O}_D\big(dy_1^4,\ dy_3^4,\ dy_4^4,\ dy_1 y_3,\ dy_1 y_4,\ dy_3^3 y_4,\ dy_3^2 y_4^2,\ dy_3 y_4^3\big)$$

one easily sees that $g^*\omega_Z \subset \mathscr{O}_{D^\sharp}(y_1, y_3, y_4)^2/\Omega$, and y_3/Ω and y_4/Ω are independent $\mod g^*\omega_Z$.

We study $E_X = \{s = 0\} \in |-K_X|$. We see that s induces a section $\overline{s}$ of

$$\mathrm{gr}^1_C(\omega^*) \simeq (\mathrm{gr}^0_C\,\omega)^{\tilde{\otimes}(-1)} \,\tilde{\otimes}\, \mathrm{gr}^1_C \mathscr{O} \simeq (0) \,\tilde{\oplus}\, (1)$$

such that $\overline{s}(P)$ is general in $\mathrm{gr}^1_C(\omega^*) \otimes \mathbb{C}(P)$. Thus $\overline{s}$ is nowhere vanishing, whence $E_X \supset C$ and E_X is smooth on $C \setminus \{P\}$. Eliminating y_4, we see $(E_X, P) \simeq (y_1, y_2, y_3; \overline{\phi})/\boldsymbol{\mu}_4(3, 2, 1)$ with C the locus of $(t^3, t^2, 0)$, where

$$\overline{\phi} = (y_1^2 - y_2^3) + y_3(cy_3 + \cdots) \in \mathbb{C}\{y_1, y_2, y_3\}$$

for some $c \in \mathbb{C}^*$ by independence of $\lambda(0)$ and $\mu(0)$. We claim that we may take

$$\overline{\phi} = y_1^2 - y_2^3 + y_3^2 \tag{4.2.1}$$

modulo multiplication by units and $\boldsymbol{\mu}_m$-automorphisms fixing C. First by Weierstrass preparation Theorem, we may assume $\overline{\phi} = y_1^2 + \alpha(y_2, y_3)y_1 + \beta(y_2, y_3)$ with $\mathrm{wt}\,\alpha \equiv 3$ and $\mathrm{wt}\,\beta \equiv 2 \mod (4)$. Since $\overline{\phi}(t^3, t^2, 0) = 0$, we see $\alpha \equiv 0$ and $\beta \equiv y_2^3 \mod (y_3)$. Hence we may assume $\alpha = 0$, after replacing y_1 by $y_1 - \alpha/2$. Since $\mathrm{wt}((\beta - y_2^3)/y_3) \equiv 1$ and $\mathrm{wt}\,y_2 \equiv 2 \mod (4)$, we see $\beta \equiv y_2^3 \mod (y_3^2)$. Thus (4.2.1) holds by $c \in \mathbb{C}^*$. Then it is easy to check (cf. [Rei87, (4.10)])

Computation 4.3 (see [KM92, (2.11.2)]). *Let*

$$(E, P) = (y_1, y_2, y_3;\ y_1^2 - y_2^3 + y_3^2)/\boldsymbol{\mu}_4(3, 2, 1; 2)$$

and $C \subset E$ the locus of $(t^3, t^2, 0)$. Then (E, P) is of type D_5 and $\Delta(E, C)$ is as in 1.3.2.

Thus the proof of Theorem 1.3 is completed in case (IIB). ◻

§5. Case of (IA)+(IA)+(III)

5.1 (**Cf.** [KM92, (2.12)]). The configuration of singular points on (X, C) is the following: a (IA) point P of odd index $m \geq 3$ and a (IA) point Q of index 2 and a (III) point R [MP08a, (9.1)]. We know that $\operatorname{siz}_P = 1$ [MP08a, (8.5)], $i_P(1) = i_Q(1) = i_R(1) = 1$ [MP08a, (9.2.1)], and hence $\operatorname{gr}^1_C \mathscr{O} \simeq \mathscr{O}(-1) \oplus \mathscr{O}(-1)$ from the formula [Mor88, (2.3.4)]. It follows from [MP08a, (3.1.1), (9.2.1), (2.8)] that

$$w_P(0) = 1 + (K_X \cdot C) - w_Q(0) = (m-1)/2m$$

(because $w_Q(0) \in \frac{1}{2}\mathbb{Z}$). We start with the set-up.

Lemma 5.2 ([KM92, (2.12.1)]). *We can express*

$$(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\boldsymbol{\mu}_m(1, \tfrac{m+1}{2}, -1, 0; 0) \supset (C, P) = y_1\text{-axis}/\boldsymbol{\mu}_m,$$

$$(X, Q) = (z_1, z_2, z_3, z_4; \beta)/\boldsymbol{\mu}_2(1, 1, 1, 0; 0) \supset (C, Q) = z_1\text{-axis}/\boldsymbol{\mu}_2,$$

$$(X, R) = (w_1, w_2, w_3, w_4; \gamma) \supset (C, R) = w_1\text{-axis},$$

using equations α, β and γ such that $\alpha \equiv y_1 y_3 \mod (y_2, y_3)^2 + (y_4)$, $\beta \equiv z_1 z_3 \mod (z_2, z_3)^2 + (z_4)$ and $\gamma \equiv w_1 w_3 \mod (w_2, w_3, w_4)^2$.

Proof. Express $(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\boldsymbol{\mu}_m(a_1, a_2, -a_1, 0; 0)$ so that $C^\sharp$ is the locus of $(t^{a_1}, t^{a_2}, 0, 0)$, where a_1 and a_2 are positive integers such that $\gcd(a_1 a_2, m) = 1$. Since $w_P(0) = (m-1)/2m$, it holds that $a_2 = (m+1)/2$ [Mor88, (4.9)(i)]. By $\operatorname{siz}_P = 1 = U(a_1 a_2)$, we have $a_1 a_2 \leq m$ and $a_1 = 1$. We need only to replace y_2 by $y_2 - y_1^{(m+1)/2}$ to get the assertion for (X, P). We can choose α so that $\alpha \equiv y_1 y_3$ because P is a cA point [Mor88, (B.1)(g)]. The rest is similar except for $\beta \equiv z_1 z_3$ and $\gamma \equiv w_1 w_3$ which follow from $i_Q(1) = 1$ and $i_R(1) = 1$ and [Mor88, (2.16)]. $\square$

We will improve the set-up in two steps.

Lemma 5.3 ([KM92, (2.12.2)]). *The point P is ordinary, that is,*

$$(X, P) = (y_1, y_2, y_3)/\boldsymbol{\mu}_m(1, (m+1)/2, -1) \supset (C, P) = y_1\text{-axis}/\boldsymbol{\mu}_m.$$

Proof. Suppose that P is not ordinary. We will derive a contradiction. By our hypothesis, we may assume $\alpha \equiv y_1 y_3 \mod (y_2, y_3, y_4)^2$. Apply L-deformation at Q [KM92, (2.9.1)], see also Remark 2.4. If a general member of the corresponding family is an extremal neighborhood, the assertion follows from [KM92, (2.12.2)]. Thus we may assume that Q is ordinary and hence $\beta = z_4 + z_1 z_3$ in our $\mathbb{Q}$-conic bundle case. Hence $\{y_2, y_4\}$ and $\{z_2, z_3\}$ are the ℓ-free ℓ-bases of $\mathrm{gr}^1_C \mathscr{O}$ at P and Q, respectively. By [KM92, (2.12.1)], we see

$$\mathrm{gr}^0_C \omega \simeq \left(-1 + \tfrac{m-1}{2} P^\sharp + Q^\sharp\right)$$

and

$$\mathrm{gr}^0_C(\omega^*) \simeq \left(-1 + \tfrac{m+1}{2} P^\sharp + Q^\sharp\right).$$

Thus $H^0(\omega^*) = H^0(F^1_C(\omega^*))$. Let $D = \{y_1 + h(y_2, y_3, y_4) = 0\}/\boldsymbol{\mu}_m$ with general h such that $\mathrm{wt}\ h = \mathrm{wt}\ y_1$. Then $D \in |-2K_X|$ by $(D \cdot C) = 1/m$.

5.3.1. Now apply Corollary 2.2 with $n = 2$. We obtain $\omega_D \simeq \mathscr{O}_D(-K_X)$ which gives the correspondence of ℓ-bases

$$\left.\frac{dy_2 \wedge dy_3}{\partial\alpha/\partial y_4}\right|_{D^\sharp} = \left.\frac{dy_4 \wedge dy_2}{\partial\alpha/\partial y_3}\right|_{D^\sharp} = \left.\frac{dy_3 \wedge dy_4}{\partial\alpha/\partial y_2}\right|_{D^\sharp} \overset{\mathrm{Res}}{\longleftrightarrow} \frac{(\mathrm{unit})}{\Omega},$$

and

$$g^*\omega_Z \subset \wedge^2 \mathscr{O}_D\big(dy_2^m,\ dy_3^m,\ dy_4,\ dy_2 y_3^{(m+1)/2},\ dy_2^2 y_3\big).$$

Hence,

$$\begin{array}{rc} g^*\omega_Z \subset & \sum_{i,j=2,3,4} \mathscr{O}_{D^\sharp}(y_2, y_3)^2 dy_i \wedge dy_j \\ & \updownarrow \mathrm{Res} \\ & \sum_{k=2,3,4} \mathscr{O}_{D^\sharp}(y_2, y_3)^2 (\partial\alpha/\partial y_k)\Omega^{-1}. \end{array}$$

So we have the lifting modulo $\mathscr{O}_{D^\sharp}(y_2, y_3)^2(\partial\alpha/\partial y_2, \partial\alpha/\partial y_3, \partial\alpha/\partial y_4)$. Therefore there exists $s \in H^0(F^1_C(\omega^*))$ inducing $(y_2 + (y_1 + h)\mathscr{O}_X)/\Omega \in \mathscr{O}_D(-K_X)$, where

$$\Omega = \left.\frac{dy_1 \wedge dy_2 \wedge dy_3}{\partial\alpha/\partial y_4}\right|_{D^\sharp}.$$

5.3.2. Thus s induces a global section $\overline{s}$ of $\mathrm{gr}^1_C(\omega^*) \simeq \mathrm{gr}^1_C \mathscr{O} \mathbin{\tilde{\otimes}} \mathrm{gr}^0_C(\omega^*)$ which is a part of ℓ-free ℓ-basis at P. Hence there is an exact sequence

$$0 \to \mathrm{gr}^0_C \omega \to \mathrm{gr}^1_C \mathscr{O} \to \mathrm{gr}^1_C \mathscr{O}/\mathrm{gr}^0_C \omega \to 0.$$

It is split because $\mathrm{gr}^0_C \omega \simeq \mathscr{O}(-1)$ and $\mathrm{gr}^1_C \mathscr{O} \simeq \mathscr{O}(-1) \oplus \mathscr{O}(-1)$. Then it is ℓ-split at Q because ℓ-bases of $\mathrm{gr}^0_C \omega$ and $\mathrm{gr}^1_C \mathscr{O}$ have all weights $\equiv 1 \mod (2)$. Hence $\mathrm{gr}^1_C \mathscr{O}/\mathrm{gr}^0_C \omega$ is an ℓ-invertible sheaf such that

$$\mathrm{ql}_C(\mathrm{gr}^1_C \mathscr{O}/\mathrm{gr}^0_C \omega) = \mathrm{ql}_C(\mathrm{gr}^1_C \mathscr{O}) - \mathrm{ql}_C(\mathrm{gr}^0_C \omega) = -1 + Q^\sharp.$$

Applying [KM92, (2.6)] to $\mathrm{ql}_C(\mathrm{gr}^0_C\,\omega)-\mathrm{ql}_C(\mathrm{gr}^1_C\,\mathscr{O}/\,\mathrm{gr}^0_C\,\omega)=\frac{m-1}{2}P^\sharp$, we obtain an ℓ-splitting

$$\mathrm{gr}^1_C\,\mathscr{O}\simeq\left(-1+\tfrac{m-1}{2}P^\sharp+Q^\sharp\right)\tilde{\oplus}\,(-1+Q^\sharp). \tag{5.3.3}$$

We may further assume that y_2, z_2 and w_2 (resp. y_4, z_3 and w_4) are the ℓ-free ℓ-bases of $\left(-1+\frac{m-1}{2}P^\sharp+Q^\sharp\right)$ (resp. $(-1+Q^\sharp)$) at P, Q and R, by making coordinates changes to the ones in [KM92, (2.12.1)].

Let J be the C-laminal ideal of width 2 such that $J/F^2_C\mathscr{O}=\left(-1+\frac{m-1}{2}P^\sharp\right.$ $\left.+Q^\sharp\right)$. Then $\{y_2,y_3,y_4^2\}$ form an ℓ-basis of J at P. By replacing y_3 by an element of the form $y_3+y_4^2(\cdots)$ if necessary, we may assume $\alpha\equiv y_1y_3+cy_4^2$ mod $J^\sharp I_{C^\sharp}$ for some $c\in\mathbb{C}$. If $c\neq 0$ then $I\supset J$ is $(1,2,2)$-monomializable at P (see [Mor88, (8.9-8.10)]). If $c=0$ we may still assume that $I\supset J$ is $(1,2,2)$-monomializable at P by deformation arguments [KM92, (2.9.2)] (see also Remark 2.4) because Lemma 5.3 would follow from [KM92, (2.12.2)] if our (X,C) deformed to an extremal neighborhood. In the same way, we may assume that $I\supset J$ is $(1,2,2)$-monomializable at R. At the ordinary point Q, $I\supset J$ is $(1,2)$-monomializable. Thus there are ℓ-isomorphisms

$$\begin{aligned}
\mathrm{gr}^1(\mathscr{O},J)&\simeq(-1+Q^\sharp),\\
\mathrm{gr}^{2,0}(\mathscr{O},J)&\simeq\left(-1+\tfrac{m-1}{2}P^\sharp+Q^\sharp\right),\\
\mathrm{gr}^{2,1}(\mathscr{O},J)&\simeq\mathrm{gr}^1(\mathscr{O},J)^{\tilde{\otimes}2}\,\tilde{\otimes}\,(1+P^\sharp)\simeq(P^\sharp),\\
\mathrm{gr}^{3,0}(\mathscr{O},J)&\simeq\mathrm{gr}^{2,0}(\mathscr{O},J)\,\tilde{\otimes}\,\mathrm{gr}^1(\mathscr{O},J)\simeq\left(-1+\tfrac{m-1}{2}P^\sharp\right),\\
\mathrm{gr}^{3,1}(\mathscr{O},J)&\simeq\mathrm{gr}^{2,1}(\mathscr{O},J)\,\tilde{\otimes}\,\mathrm{gr}^1(\mathscr{O},J)\simeq(-1+P^\sharp+Q^\sharp)
\end{aligned}$$

(cf. [Mor88, (8.6), (8.12)]) and the following:

$$\begin{aligned}
\mathrm{gr}^1(\omega,J)&\simeq\mathrm{gr}^1(\mathscr{O},J)\,\tilde{\otimes}\,gr^0_C\omega\simeq(-1+\tfrac{m-1}{2}P^\sharp),\\
\mathrm{gr}^{2,0}(\omega,J)&\simeq\mathrm{gr}^{2,0}(\mathscr{O},J)\,\tilde{\otimes}\,gr^0_C\omega\simeq\left(-1+(m-1)P^\sharp\right),\\
\mathrm{gr}^{2,1}(\omega,J)&\simeq\mathrm{gr}^{2,1}(\mathscr{O},J)\,\tilde{\otimes}\,gr^0_C\omega\simeq\left(-1+\tfrac{m+1}{2}P^\sharp+Q^\sharp\right),\\
\mathrm{gr}^{3,0}(\omega,J)&\simeq\mathrm{gr}^{3,0}(\mathscr{O},J)\,\tilde{\otimes}\,\mathrm{gr}^0_C\,\omega\simeq\left(-2+(m-1)P^\sharp+Q^\sharp\right),\\
\mathrm{gr}^{3,1}(\omega,J)&\simeq\mathrm{gr}^{3,1}(\mathscr{O},J)\,\tilde{\otimes}\,\mathrm{gr}^0_C\,\omega\simeq\left(-1+\tfrac{m+1}{2}P^\sharp\right),
\end{aligned}$$

by $gr^0_C\omega\simeq(-1+\frac{m-1}{2}P^\sharp+Q^\sharp)$.

Hence there are an ℓ-isomorphism and ℓ-exact sequences

$$\mathrm{gr}^1(\omega, J) \simeq \left(-1 + \tfrac{m-1}{2} P^\sharp\right),$$

$$0 \to \left(-1 + \tfrac{m+1}{2} P^\sharp + Q^\sharp\right) \to \mathrm{gr}^2(\omega, J) \to (-1 + (m-1)P^\sharp) \to 0,$$

$$0 \to \left(-1 + \tfrac{m+1}{2} P^\sharp\right) \to \mathrm{gr}^3(\omega, J) \to (-2 + (m-1)P^\sharp + Q^\sharp) \to 0,$$

by [Mor88, (8.6)]. Now from the exact sequences

$$0 \to \mathrm{gr}^n(\omega, J) \to \omega_X / F^{n+1}(\omega, J) \to \omega_X / F^n(\omega, J) \to 0 \tag{5.3.4}$$

we obtain $H^1(\omega/F^4(\omega, J)) \neq 0$ which is a contradiction. Then it follows from [MP08a, (4.4)] that $V := \mathrm{Spec}_X\, \mathscr{O}_X / F^4(\mathscr{O}, J) \supset f^{-1}(o)$. Hence, $2 = (-K_X \cdot f^{-1}(o)) \le (-K_X \cdot V)$. On the other hand, near a general point $S \in C$, for a suitable choice of coordinates (x, y, z) in (X, S), we may assume that $F^1(\mathscr{O}, J) = I_C = (x, y)$, $F^2(\mathscr{O}, J) = J = (x, y^2)$, $F^3(\mathscr{O}, J) = I_C J + J = (x^2, xy, y^3)$, $F^4(\mathscr{O}, J) = J^2 = (x^2, xy^2, y^4)$. Hence,

$$2 \le (-K_X \cdot V) = \tfrac{1}{2m} \,\mathrm{length}_S\, \mathbb{C}\{x, y\}/(x^2, xy^2, y^4) = \tfrac{6}{2m},$$

which is a contradiction. Lemma 5.3 is proved. □

Lemma 5.4 ([KM92, (2.12.3)]). *The point Q is ordinary, that is,*

$$(X, Q) = (z_1, z_2, z_3)/\boldsymbol{\mu}_2(1, 1, 1) \supset (C, P) = z_1\text{-axis}/\boldsymbol{\mu}_2.$$

Proof. Assuming that Q is not ordinary whence $\beta \equiv z_1 z_3 \mod (z_2, z_3, z_4)^2$, we will derive a contradiction. As in the proof of Lemma 5.3, there is a split exact sequence

$$0 \to \mathrm{gr}_C^0\, \omega \to \mathrm{gr}_C^1\, \mathscr{O} \to (\mathrm{gr}_C^1\, \mathscr{O} / \mathrm{gr}_C^0\, \omega) \to 0$$

which is ℓ-split at P. Since ℓ-free ℓ-bases of $\mathrm{gr}_C^0\, \omega$ (resp. $\mathrm{gr}_C^1\, \mathscr{O}$) at Q have weights 1 (resp. 0, 1) $\mod (2)$, the above sequence is also ℓ-split at Q. Thus there are ℓ-exact sequences

$$0 \to \left(-1 + \tfrac{m-1}{2} P^\sharp + Q^\sharp\right) \to \mathrm{gr}_C^1\, \mathscr{O} \to (-1 + P^\sharp) \to 0,$$

$$0 \to (-1 + (m-1)P^\sharp) \to \mathrm{gr}_C^1\, \omega \to \left(-2 + \tfrac{m+1}{2} P^\sharp + Q^\sharp\right) \to 0.$$

Similarly to the argument at the end of Lemma 5.3 $H^1(\omega/F_C^2 \omega) \neq 0$ and one has a contradiction by $2 \le 3/(2m)$. Lemma 5.4 is proved. □

5.5 ([KM92, (2.12.4)]). As in the argument for Lemma 5.3, there is an ℓ-isomorphism

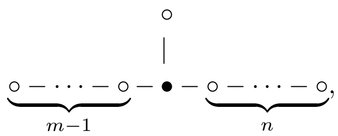
$$\operatorname{gr}^1_C \mathscr{O} \simeq \left(-1 + \tfrac{m-1}{2}P^\sharp + Q^\sharp\right) \tilde{\oplus} \left(-1 + P^\sharp + Q^\sharp\right).$$

Let J be the C-laminal ideal such that $J/F^2_C\mathscr{O} = \left(-1 + \frac{m-1}{2}P^\sharp + Q^\sharp\right)$. After an (equivariant) change of coordinates if necessary, we may assume that (y_2, z_2, w_2) (resp. (y_3, z_3, w_4)) are ℓ-free ℓ-bases of $\left(-1 + \frac{m-1}{2}P^\sharp + Q^\sharp\right)$ (resp. $(-1 + P^\sharp + Q^\sharp)$), whence $J = (w_2, w_3, w_4^2)$ at R. Replacing w_3 by an element $\equiv w_3 \mod (w_2, w_4)^2$ if necessary, we may further assume

$$\gamma \equiv w_1w_3 + c_1w_4^2 + c_2w_4w_2 + c_3w_2^2 \mod (w_3, w_2^2, w_2w_4, w_4^2) \cdot I_C$$

for some $c_1, c_2, c_3 \in \mathbb{C}$. We note $\gamma \equiv w_1w_3 + c_1w_4^2 \mod J \cdot I_C$.

Lemma 5.6 ([KM92, 2.12.5]). *A general member E_X of $|-K_X|$ has singularities A_{m-1}, A_1 and A_n at P, Q and R, respectively and is smooth elsewhere, and $\Delta(E_X, C)$ is*



where n is some integer ≥ 1. We have $n = 1$ if either $m \geq 5$ and $c_1 \neq 0$ or $m = 3$ and $(c_1, c_2, c_3) \neq (0, 0, 0)$.

Proof. There is an ℓ-isomorphism $\operatorname{gr}^1_C(\omega^*) \simeq (0) \tilde{\oplus} (-1 + \frac{m+3}{2}P^\sharp)$. Let $D = \{y_1 + h = 0\}/\boldsymbol{\mu}_m \in |-2K_X|$ as before. We treat the case $m \geq 5$. Then $\operatorname{gr}^1_C \omega^* \simeq \mathscr{O}_C \oplus \mathscr{O}_C(-1)$ and $H^0(\mathscr{O}(-K_X)) = H^0(\operatorname{gr}^2_C(\omega^*, J))$. As in 5.3.1 one has $H^0(\mathscr{O}_X(-K_X)) \twoheadrightarrow \omega_D \mod (y_2, y_3)^2\omega_{D^\sharp}$. So a general section $s \in H^0(\mathscr{O}(-K_X))$ induces $(y_2 + \cdots)/\Omega$, up to some units whence induces a nonzero global section $\overline{s}$ of $\operatorname{gr}^1_C \omega^*$. Hence $\overline{s}$ is nowhere vanishing and the defining equations of $E_X = \{s = 0\}$ are y_2, z_2 and $w_2 \mod F^2_C\mathscr{O}$ up to units at P, Q and R, respectively. Then E_X is smooth outside of P, Q and R, $(E_X, P) \simeq (y_1, y_3)/\boldsymbol{\mu}_m(1, -1)$, $(E_X, Q) \simeq (z_1, z_3)/\boldsymbol{\mu}_2(1, 1)$ and $(E_X, R) \simeq (w_1, w_3, w_4; \overline{\gamma})$, where $\overline{\gamma}(w_1, w_3, w_4) \equiv w_1w_3 + c_1w_4^2 \mod (w_3, w_4^2)(w_3, w_4)$. We are done in case $m \geq 5$. In case $m = 3$, we can see that $\operatorname{gr}^1_C \omega^* \simeq (0) \tilde{\oplus} (0)$ and $H^0(\mathscr{O}(-K_X)) \twoheadrightarrow H^0(\operatorname{gr}^1_C \omega^*)$ and we get a similar assertion on E_X except that $\gamma \equiv w_1w_3 + (c_3t^2 + c_2t + c_1)w_4^2$ for some general $t \in \mathbb{C}$. Thus we are done in case $m = 3$. □

Lemma 5.7 ([KM92, (2.12.6)]). *If $m \geq 5$, then $c_1 \neq 0$ and $n = 1$ in Lemma 5.6. Thus the assertions of the case 1.3.4 hold when $m \geq 5$.*

Proof. Assume that $m \geq 5$ and $c_1 = 0$. By $w_1 w_3 \in J \cdot I_C$, we have $w_3 \in F^3(\mathscr{O}, J)$ and $\mathrm{gr}^2(\mathscr{O}, J) = \mathscr{O}_C w_2 \oplus \mathscr{O}_C w_4^2$ at R. Thus there are ℓ-isomorphisms (cf. the proof of Lemma 5.3)

$$\begin{aligned}
\mathrm{gr}^1(\mathscr{O}, J) &= (-1 + P^\sharp + Q^\sharp),\\
\mathrm{gr}^{2,0}(\mathscr{O}, J) &= \left(-1 + \tfrac{m-1}{2} P^\sharp + Q^\sharp\right),\\
\mathrm{gr}^{2,1}(\mathscr{O}, J) &= \mathrm{gr}^1(\mathscr{O}, J)^{\tilde{\otimes} 2} \simeq (-1 + 2P^\sharp).
\end{aligned}$$

Therefore,

$$\mathrm{gr}^1(\omega, J) \simeq \left(-1 + \tfrac{m+1}{2} P^\sharp\right),$$

$$0 \to \left(-2 + \tfrac{m+3}{2} P^\sharp + Q^\sharp\right) \to \mathrm{gr}^2(\omega, J) \to (-1 + (m-1)P^\sharp) \to 0.$$

From the exact sequence

$$0 \to \mathrm{gr}^2(\omega, J) \to \omega_X / F^3(\omega, J) \to \omega_X / F^2(\omega, J) \to 0$$

we obtain $H^1(\omega / F^3(\omega, J)) \neq 0$. Hence $2 \leq 4/(2m)$ [MP08a, (4.4)], a contradiction. $\square$

Lemma 5.8 (cf. [KM92, (2.12.7)]). *Assume* $m = 3$. *If* $(c_1, c_2, c_3) \neq 0$ (*resp.* $= 0$)*, then* $n = 1$ (*resp.* $= 2$) *in Lemma* 5.6*. Thus the assertions of the case* 1.3.4 *hold when* $m = 3$.

Proof. Assume that $(c_1, c_2, c_3) = 0$. Then $w_3 \in F_C^3 \mathscr{O}$. Changing w_1 and w_3, we may further assume $\gamma = w_1 w_3 + \delta(w_2, w_4)$, where δ is a power series in w_2 and w_4 of order $d \geq 3$. Then $d = 3$ because $2 \cdot \mathrm{ldeg}_C(-1 + P^\sharp + Q^\sharp) + 1/d \geq 0$ [KM92, (2.12.8)]. In the proof of Lemma 5.6, it is easy to see $n = d - 1$ from $\mathrm{gr}_C^1(\mathscr{O}) = (-1 + P^\sharp + Q^\sharp) \mathbin{\tilde{\oplus}} (-1 + P^\sharp + Q^\sharp)$. $\square$

Thus we end up with the case 1.3.4 for (IA)+(IA)+(III), and the proof of Theorem 1.3 is completed for (IA)+(IA)+(III).

§6. Case of (IA)+(IA)

6.1 (**Cf.** [KM92, (2.13)]). In this section, we consider the case (IA)+(IA). Since the base (Z, o) is smooth, [MP08a, Theorem 11.1] implies that the singular locus of (X, C) consists of a (IA) point P of odd index $m \geq 3$ and a (IA) point Q of index 2. We know that $\mathrm{siz}_P = 1$, by [MP08a, (8.5)].

We start with the set-up. The following is very similar to Lemma 5.2.

Lemma 6.2 (cf. [KM92, (2.13.1)]). *We can write*

$$(X,P)=(y_1,y_2,y_3,y_4;\alpha)/\boldsymbol{\mu}_m(1,\tfrac{m+1}{2},-1,0;0)\supset(C,P)=y_1\text{-axis}/\boldsymbol{\mu}_m,$$

$$(X,Q)=(z_1,z_2,z_3,z_4;\beta)/\boldsymbol{\mu}_2(1,1,1,0;0)\supset(C,Q)=z_1\text{-axis}/\boldsymbol{\mu}_2,$$

using equations α and β such that $\alpha\equiv y_1y_3 \mod (y_2,y_3)^2+(y_4)$.

We recall $\ell(P)=\operatorname{length}_{P^\sharp}\bigl(I^{\sharp(2)}/I^{\sharp 2}\bigr)$, where $I^\sharp$ is the defining ideal of $C^\sharp$ in $(X^\sharp,P^\sharp)$ and $\ell(Q)$ is defined similarly.

Lemma 6.3 (cf. [KM92, (2.13.2)]). *$i_P(1)=1$ and $\ell(P)\le 1$.*

Proof. This follows from $\alpha\equiv y_1y_3$ and [Mor88, (2.16)]. □

Lemma 6.4 (cf. [KM92, (2.13.3)]). *Either*

Case 6.4.1 ([KM92, (2.13.3.1]). $\ell(Q)\le 1$ (in particular, the point (X,Q) is of type $cA/2$), $i_Q(1)=1$, and $\operatorname{gr}^1_C\mathscr{O}\simeq\mathscr{O}\oplus\mathscr{O}(-1)$, or

Case 6.4.2 ([KM92, (2.13.3.2]). $\ell(Q)=2$, $i_Q(1)=2$, $\operatorname{gr}^1_C\mathscr{O}\simeq\mathscr{O}(-1)\oplus\mathscr{O}(-1)$, and P is ordinary:

$$(X,P)=(y_1,y_2,y_3)/\boldsymbol{\mu}_m\left(1,\tfrac{m+1}{2},-1\right)\supset(C,P)=y_1\text{-axis}/\boldsymbol{\mu}_m.$$

Proof. The assertion on $i_Q(1)$ follows from the one on $\ell(Q)$ by $i_Q(1)=[\ell(Q)/2]+1$ [Mor88, (2.16)]. We assume $\ell(Q)\ge 2$ and denote it by r. Thus we may choose $\beta\equiv z_1^r z_i \mod (z_2,z_3,z_4)^2$, where $i=3$ (resp. 4) if $r\equiv 1$ (resp. 0) $\mod (2)$. If we extend (see Remark 2.4) the deformation $\beta+tz_1^{r-2}z_i=0$ of (X,Q) to a deformation $(X_t,C_t)\ni Q_t$ of $(X,C)\ni Q$ which is trivial outside of a small neighborhood of Q, then X_t has two (IA) points and one (III) point on C_t and $\beta+tz_1^{r-2}z_i=0$ is the equation for $(X_t^\sharp,Q_t^\sharp)$ (cf. [Mor88, (4.12.2)]). Hence Q_t is ordinary, that is, $r=2$ by Lemma 5.4 or [KM92, (2.12)]. □

First we treat the special case 6.4.2.

Lemma 6.5 (cf. [KM92, (2.13.4)]). *Assume that we are in the case* 6.4.2. *Then f is of type* (IA)+(IA) *and the singular point Q is of type $cA/2$, $cAx/2$ or $cD/2$ (see* 1.3.3*).*

Proof. The argument is quite similar to the case (IA)+(IA)+(III) (Section 5). As in the paragraph 5.5, there is an ℓ-isomorphism

$$\operatorname{gr}^1_C\mathscr{O}\simeq\left(-1+\tfrac{m-1}{2}P^\sharp+Q^\sharp\right)\tilde{\oplus}\,(-1+P^\sharp+Q^\sharp),$$

and let J be the C-laminal ideal such that

$$J/F^2_C\mathscr{O} = \left(-1 + \tfrac{m-1}{2}P^\sharp + Q^\sharp\right).$$

We may assume that (y_2, z_2) (resp. (y_3, z_3)) are ℓ-free ℓ-bases of $\left(-1+\tfrac{m-1}{2}P^\sharp + Q^\sharp\right)$ (resp. $(-1 + P^\sharp + Q^\sharp)$), $J^\sharp = (z_2, z_4, z_3^2)$ and

$$\beta \equiv z_1^2 z_4 + c_1 z_3^2 + c_2 z_2 z_3 + c_3 z_2^2 \mod (z_4, z_3^2, z_2 z_3, z_2^2) \cdot I_C$$

at Q for some c_1, c_2, $c_3 \in \mathbb{C}$. We note that $\beta \equiv z_1^2 z_4 + c_1 z_3^2 \mod J^\sharp I^\sharp$. The following Lemma 6.6 corresponds to Lemma 5.6. The fact that $(c_1, c_2, c_3) \neq (0, 0, 0)$ and the assertion on the type of Q follows from the classification of terminal 3-fold singularities [Rei87, (6.1)]. The assertion that $c_1 \neq 0$ for the case $m \geq 5$ is proved in the same way as Lemma 5.7. Thus Lemma 6.5 is proved. □

Lemma 6.6 ([KM92, (2.13.5)]). *Under the notation of the previous proof, assume either $m \geq 5$ and $c_1 \neq 0$ or $m = 3$ and $(c_1, c_2, c_3) \neq (0, 0, 0)$. Then for a general member E_X of $|-K_X|$, $\Delta(E_X, C)$ is*

$$\underbrace{\circ - \cdots - \circ}_{m-1} - \bullet - \underbrace{\circ - \cdots - \circ}_{2k-3} - \overset{\overset{\circ}{|}}{\circ} - \circ,$$

where $k(\geq 2)$ is the axial multiplicity of (X, Q).

Proof. The only difference from Lemma 5.6 is the analysis of the singularity $(E_X, Q) \simeq (z_1, z_3, z_4; \overline{\beta})/\boldsymbol{\mu}_2(1, 1, 0; 0)$, where $\overline{\beta}$ satisfies $\overline{\beta} \equiv z_1^2 z_4 + z_3^2 \mod (z_4, z_3^2)(z_4, z_3)$ and $\operatorname{ord}\overline{\beta}(0, 0, z_4) = k < \infty$. It is easy to see that $\overline{\beta} = z_1^2 z_4 + z_3^2 + z_4^k$ modulo formal $\boldsymbol{\mu}_m$-automorphisms in (z_1, z_3, z_4). Thus it is reduced to the following explicit computation (cf. [Rei87, (4.10)]). □

Computation 6.7 ([KM92, (2.13.6)]). *Let*

$$(E, Q) = (z_1, z_3, z_4; z_1^2 z_4 + z_3^2 + z_4^k)/\boldsymbol{\mu}_2(1, 1, 0; 0)$$

and $C = z_1$-axis$/\boldsymbol{\mu}_2$, where $k \geq 2$. Then (E, Q) is of type D_{2k} and $\Delta(E, C)$ is

$$\bullet - \underbrace{\circ - \cdots - \circ}_{2k-3} - \overset{\overset{\circ}{|}}{\circ} - \circ.$$

6.8 (Cf. [KM92, (2.13.7)]). In the rest of this chapter, we assume the case 6.4.1 unless otherwise mentioned.

We choose an ℓ-splitting $\operatorname{gr}^1_C \mathscr{O} \simeq \mathscr{L} \tilde{\oplus} \mathscr{M}$ as in [KM92, (2.8)] (see [Mor88, 9.1.7]) such that $\deg \mathscr{L} = 0$ and $\deg \mathscr{M} = -1$, see (6.4.1). Let J be the C-laminal ideal of width 2 such that $J/F^2_C\mathscr{O} = \mathscr{L}$. For an ℓ-invertible sheaf F with an ℓ-free ℓ-basis f at a point T of index n, we can give an equivalent definition of $\operatorname{qldeg}(F,T) \in [0,n)$ as $\operatorname{qldeg}(F,T) \equiv -\operatorname{wt} f \mod (n)$. (This is because $(C^\sharp, P^\sharp)$ and $(C^\sharp, Q^\sharp)$ are smooth.)

Lemma 6.9 ([KM92, (2.13.8)]). $\operatorname{qldeg}(\mathscr{M}, Q) = 1$.

Proof. We assume $\operatorname{qldeg}(\mathscr{M}, Q) = 0$. Then $\mathscr{M} \simeq (-1 + iP^\sharp)$ for $i = 0$, 1 or $(m-1)/2$ since y_2, y_3 and y_4 generate $\operatorname{gr}^1_C \mathscr{O}^\sharp$ at $P^\sharp$. It follows from $\operatorname{ql}_C(\operatorname{gr}^0_C \omega) = -1 + \frac{m-1}{2}P^\sharp + Q^\sharp$, that

$$\operatorname{gr}^1_C \omega \simeq \operatorname{gr}^1_C \mathscr{O} \tilde{\otimes} \operatorname{gr}^0_C \omega \simeq \mathscr{L} \tilde{\otimes} \operatorname{gr}^0_C \omega \tilde{\oplus} \left(-2 + \left(\tfrac{m-1}{2} + i\right) P^\sharp + Q^\sharp\right).$$

Since $(m-1)/2 + i \le m - 1 < m$, we have $H^1(\operatorname{gr}^1_C \omega) \neq 0$. This is a contradiction to $H^1(\omega/F^2_C\omega) = 0$ because of $H^1(\operatorname{gr}^0_C \omega) = 0$. Indeed, otherwise by [MP08a, (4.4)] we have $f^{-1}(o) \subset \operatorname{Spec} \mathscr{O}_X/F^2_C\mathscr{O}$ and $2 \le 3/(2m)$, which is a contradiction. □

Remark 6.10 ([KM92, (2.13.8.1)]). For comparison with [Mor88, (9)], it might be worthwhile to mention[1]

$$\operatorname{qldeg}(\mathscr{M}, Q) = 1 \qquad \text{iff } \ell(Q) + \operatorname{qldeg}(\mathscr{L}, Q) = 1,$$

$$\operatorname{qldeg}(\mathscr{M}, P) = \tfrac{m-1}{2} \qquad \text{iff } \ell(P) + \operatorname{qldeg}(\mathscr{L}, P) = 1.$$

Lemma 6.11 Corresponds to but different from [KM92, (2.13.9)]. $\operatorname{qldeg}(\mathscr{M}, P) \neq (m-1)/2$

Proof. We assume $\operatorname{qldeg}(\mathscr{M}, P) = (m-1)/2$ to the contrary. There is an ℓ-isomorphism $\mathscr{M} \simeq \operatorname{gr}^0_C \omega$. We may assume that y_2 is an ℓ-free ℓ-basis of $\mathscr{M}$ at P. Let $D = \{y_1 = 0\}/\boldsymbol{\mu}_m$. It is easy to see $D \in |-2K_X|$ by $(D \cdot C) = 1/m$. By $H^0(\mathscr{O}(-K_X)) = H^0(F^1_C(\omega^*))$, its general section s induces a section $\overline{s}$ of $\operatorname{gr}^1_C \omega^* \simeq \mathscr{L} \tilde{\otimes} (\operatorname{gr}^0_C \omega)^{\tilde{\otimes}(-1)} \tilde{\oplus} (0)$. Similar to arguments in 5.3.1 one can see that the projection of $\overline{s}$ to (0) is non-zero because y_2/Ω is an ℓ-free ℓ-basis of (0) at P and s induces an element of the form $y_2/\Omega + \cdots$ up to units, where Ω is an ℓ-free ℓ-basis of $\operatorname{gr}^0_C \omega$ at P. Thus $\overline{s}$ is nowhere vanishing, whence

[1] $q(-)$ in [KM92, 2,(13.8.1)] was $\operatorname{qldeg}(\mathscr{L}, -)$.

$E_X = \{s = 0\}$ is smooth outside of P and Q. The analysis of (E_X, P) and (E_X, Q) is the same as [Mor88, (9.9.3)]. Hence (E_Z, o') has a configuration:

$$\circ - \cdots - \circ - \bullet - \circ - \cdots - \circ.$$

The difference from [KM92] is that this implies that X is of index 2 by [MP08a, (11.2)] in our $\mathbb{Q}$-conic bundle germ case where the base is smooth. Since the index m of P is odd and > 1, this is a contradiction and we are done. □

Lemma 6.12 ([KM92, (2.13.10)]). *The point P is ordinary and $m \geq 5$. After changing coordinates, we may assume*

$$(X, P) = (y_1, y_2, y_3)/\boldsymbol{\mu}_m(1, (m+1)/2, -1) \supset (C, P) = y_1\text{-}axis/\boldsymbol{\mu}_m,$$

$$(X, Q) = (z_1, z_2, z_3, z_4; \beta)/\boldsymbol{\mu}_2(1, 1, 1, 0; 0) \supset (C, Q) = z_1\text{-}axis/\boldsymbol{\mu}_2;$$

y_2 and y_3 are ℓ-free ℓ-bases of $\mathscr{L}$ and $\mathscr{M}$ at P respectively; z_3 (resp. z_4) and z_2 are ℓ-free ℓ-bases of $\mathscr{L}$ and $\mathscr{M}$ at Q respectively,

$$\begin{aligned} &\mathscr{L} = (\tfrac{m-1}{2}P^\sharp + Q^\sharp) \text{ (resp. } \mathscr{L} = (\tfrac{m-1}{2}P^\sharp)), \\ &\mathscr{M} = (-1 + P^\sharp + Q^\sharp), \end{aligned} \tag{6.12.1}$$

$I \supset J$ has a $(1,2)$-monomializing ℓ-basis (y_3, y_2) at P, $I \supset J$ has a $(1,2)$-monomializing ℓ-basis (z_2, z_3) (resp. a $(1,2,2)$-monomializing ℓ-basis (z_2, z_4, z_3)) at Q, $\beta = z_4$ (resp. $\beta \equiv z_1 z_3 + z_2^2 \mod (z_2^2, z_3, z_4)(z_2, z_3, z_4)$) if $k = 1$ (resp. $k \geq 2$), where k is the axial multiplicity of Q. Furthermore, there is an ℓ-splitting

$$\mathrm{gr}^2(\mathscr{O}, J) \simeq \left(2P^\sharp\right) \tilde{\oplus} \left(-1 + \tfrac{m-1}{2}P^\sharp + Q^\sharp\right). \tag{6.12.2}$$

Proof. Proof will be given in a few steps. First by Lemma 6.11 we have $\mathrm{qldeg}(\mathscr{M}, P) \neq (m-1)/2$.

Step 6.12.3 ([KM92, (2.13.10.1)]). **Claim:** P is ordinary.

Assuming that P is not ordinary, we will derive a contradiction. We may assume $\alpha \equiv y_1 y_3 \mod (y_2, y_3, y_4)^2$ by Lemma 6.2. Thus y_2 and y_4 form an ℓ-free ℓ-basis of $\mathrm{gr}^1_C \mathscr{O}$ at P, and we may assume that they are ℓ-free ℓ-bases of $\mathscr{L}$ and $\mathscr{M}$, respectively because $\mathrm{qldeg}(\mathscr{M}, P) \neq (m-1)/2$. Hence $\mathscr{M} \simeq (-1+Q^\sharp)$. By the deformation $\alpha + t y_4^2$ [KM92, (2.9.2)], see also Remark 2.4, we may assume that $I \supset J$ has a $(1,2,2)$-monomializing ℓ-basis (y_4, y_2, y_3) at P. We may further assume that Q is an ordinary point by [KM92, (2.9.2)]. Hence

$\mathscr{L} \simeq (\frac{m-1}{2}P^\sharp + Q^\sharp)$ and $\mathrm{gr}^{2,1}(\mathscr{O}, J) \simeq \mathscr{M}^{\tilde{\otimes} 2} \tilde{\otimes} (P^\sharp) \simeq (-1 + P^\sharp)$. Therefore, by [Mor88, (8.12)(ii)]

$$\begin{aligned}
\mathrm{gr}^1(\omega, J) &\simeq \mathscr{M} \tilde{\otimes} \mathrm{gr}^0_C \omega \simeq (-1 + \tfrac{m-1}{2}P^\sharp),\\
\mathrm{gr}^{2,0}(\omega, J) &\simeq \mathscr{L} \tilde{\otimes} \mathrm{gr}^0_C \omega \simeq ((m-1)P^\sharp),\\
\mathrm{gr}^{2,1}(\omega, J) &\simeq \mathrm{gr}^{2,1}(\mathscr{O}, J) \tilde{\otimes} \mathrm{gr}^0_C \omega \simeq (-2 + \tfrac{m+1}{2}P^\sharp + Q^\sharp),\\
\mathrm{gr}^{3,0}(\omega, J) &\simeq \mathrm{gr}^{2,0}(\omega, J) \tilde{\otimes} \mathscr{M} \simeq (-1 + (m-1)P^\sharp + Q^\sharp),\\
\mathrm{gr}^{3,1}(\omega, J) &\simeq \mathrm{gr}^{2,1}(\omega, J) \tilde{\otimes} \mathscr{M} \simeq (-2 + \tfrac{m+1}{2}P^\sharp).
\end{aligned}$$

Hence, $H^i(\mathrm{gr}^1(\omega, J)) = 0$, $i = 1, 2$. From the exact sequences

$$0 \to \mathrm{gr}^{n,1}(\omega, J) \to \mathrm{gr}^n(\omega, J) \to \mathrm{gr}^{n,0}(\omega, J) \to 0, \quad n = 2,\ 3$$

we obtain $H^1(\mathrm{gr}^2(\omega, J)) = H^1(\mathrm{gr}^3(\omega, J)) = \mathbb{C}$. Finally, from the exact sequences (5.3.4) follows $H^i(\omega/F^2(\omega, J)) = 0$, $i = 1, 2$, $H^1(\omega/F^3(\omega, J)) = \mathbb{C}$, and $H^1(\omega/F^4(\omega, J)) \neq 0$. By [MP08a, Theorem (4.4)] we have $V := \mathrm{Spec}_X \mathscr{O}_X/F^4(\mathscr{O}, J) \supset f^{-1}(o)$. Hence $2 = (-K_X \cdot f^{-1}(o)) \le (-K_X \cdot V) = 6/(2m)$, a contradiction. Thus P is ordinary as claimed.

Step 6.12.4 ([KM92, (2.13.10.2)]). **Claim:** $m \ge 5$.

Assume that $m = 3$. Then $\mathrm{qldeg}(\mathscr{M}, P) = 1$ because $\mathrm{qldeg}(\mathscr{M}, P) \equiv -\mathrm{wt}\ y_3 \equiv 1$. This contradicts the original assumption that $\mathrm{qldeg}(\mathscr{M}, P) \neq (m-1)/2 = 1$. Thus $m \ge 5$ as claimed.

Step 6.12.5 ([KM92, (2.13.10.3)]). Since $\mathrm{gr}^1_C \mathscr{O}$ has an ℓ-free ℓ-basis $\{y_2, y_3\}$ at P, the assertions on ℓ-bases of $\mathscr{L}$ and $\mathscr{M}$ at P follow. Therefore (y_3, y_2) is a $(1,2)$-monomializing ℓ-basis for $I \supset J$ at P because $I^\sharp = (y_3, y_2)$ and $J^\sharp = (y_3^2, y_2)$ at P.

Since $\mathrm{gr}^1_C \mathscr{O}$ has an ℓ-free ℓ-basis $\{z_2, z_3\}$ (resp. $\{z_2, z_4\}$) at Q if $k = 1$ (resp. $k \ge 2$), the assertions on ℓ-bases of $\mathscr{L}$ and $\mathscr{M}$ at Q follow from $\mathrm{qldeg}(\mathscr{M}, Q) = 1$ (see Remark 6.10) possibly after a change of coordinates. Thus (6.12.1) is settled.

Assume $k = 1$. Then Q is ordinary, $I^\sharp = (z_2, z_3)$, and $J^\sharp = (z_2^2, z_3)$ at Q, whence (z_3, z_2) is a $(1,2)$-monomializing ℓ-basis. In particular, $\mathrm{gr}^{2,1}(\mathscr{O}, J) \simeq \mathscr{M}^{\tilde{\otimes} 2}$.

Thus we only have to show that (z_2, z_4, z_3) is a $(1,2,2)$-monomializing ℓ-basis of $I \supset J$ assuming $k \ge 2$. Hence $J^\sharp = (z_2^2, z_3, z_4)$ and $\beta \equiv z_1 z_3 + c z_2^2$

mod $J^\sharp I^\sharp$ for some $c \in \mathbb{C}$. If $c = 0$, then $z_3 \in F^3(\mathscr{O}, J)$ and $\mathrm{gr}^{2,1}(\mathscr{O}, J) \simeq \mathscr{M}^{\tilde{\otimes} 2}$, whence

$$\mathrm{gr}^{2,0}(\omega, J) \simeq \mathscr{L} \tilde{\otimes} \mathrm{gr}^0_C \omega \simeq (-1 + (m-1)P^\sharp + Q^\sharp),$$

$$\mathrm{gr}^{2,1}(\omega, J) \simeq \mathscr{M}^{\tilde{\otimes} 2} \tilde{\otimes} \mathrm{gr}^0_C \omega \simeq (-2 + \tfrac{m+3}{2} P^\sharp + Q^\sharp).$$

As in the Step 6.12.3 we get $H^1(\omega/F^3(\omega, J)) \neq 0$ which implies a contradiction. Thus $c \neq 0$ and the assertion on ℓ-basis is proved. In particular, the assertion on β follows. So if $k \geq 2$, then $c \neq 0$ and z_3 is an ℓ-free ℓ-basis of $\mathrm{gr}^{2,1}(\mathscr{O}, J)$ and $\mathrm{gr}^{2,1}(\mathscr{O}, J) \simeq \mathscr{M}^{\tilde{\otimes} 2} \tilde{\otimes} (Q^\sharp)$.

Step 6.12.6 ([KM92, (2.13.10.4)]). Hence by (6.12.1), there are two cases:

$$\mathscr{L} = \begin{cases} \left(\frac{m-1}{2} P^\sharp + Q^\sharp\right) \\ \left(\frac{m-1}{2} P^\sharp\right) \end{cases} \qquad \mathrm{gr}^{2,1}(\mathscr{O}, J) = \begin{cases} (-1 + 2P^\sharp) & \text{if } k = 1, \\ (-1 + 2P^\sharp + Q^\sharp) & \text{if } k \geq 2, \end{cases}$$

Thus from the exact sequence

$$0 \to \mathrm{gr}^{2,1}(\mathscr{O}, J) \to \mathrm{gr}^2(\mathscr{O}, J) \to \mathscr{L} \to 0,$$

we have $\mathrm{gr}^2(\mathscr{O}, J) \simeq \mathscr{O}_C \oplus \mathscr{O}_C(-1)$ as $\mathscr{O}_C$-modules and one of the following holds [KM92, (2.8)]:

$$\mathrm{gr}^2(\mathscr{O}, J) = \begin{cases} \left(\frac{m-1}{2} P^\sharp + Q^\sharp\right) \tilde{\oplus} \left(-1 + 2P^\sharp\right) & (*_1) \\ \left(2P^\sharp + Q^\sharp\right) \tilde{\oplus} \left(-1 + \frac{m-1}{2} P^\sharp\right) & (*_2) \\ \left(\frac{m-1}{2} P^\sharp\right) \tilde{\oplus} \left(-1 + 2P^\sharp + Q^\sharp\right) & (*_3) \\ \left(2P^\sharp\right) \tilde{\oplus} \left(-1 + \frac{m-1}{2} P^\sharp + Q^\sharp\right) & (*_4) \end{cases}$$

Note also that $\mathrm{gr}^0_C \omega = (-1 + \frac{m-1}{2} P^\sharp + Q^\sharp)$. To determine the ℓ-splitting of $\mathrm{gr}^2(\mathscr{O}, J)$, it is enough to disprove the ℓ-isomorphisms $(*_1)$, $(*_2)$, $(*_3)$ when $m \geq 7$, and $(*_1)$, $(*_2)$ when $m = 5$. Then

$$\mathrm{gr}^2(\omega, J) = \begin{cases} \left((m-1)P^\sharp\right) \tilde{\oplus} (-2 + \frac{m+3}{2} P^\sharp + Q^\sharp) & (*_1) \\ \left(\frac{m+3}{2} P^\sharp\right) \tilde{\oplus} (-2 + (m-1)P^\sharp + Q^\sharp) & (*_2) \\ (-1 + (m-1)P^\sharp + Q^\sharp) \tilde{\oplus} (-1 + \frac{m+3}{2} P^\sharp) & (*_3) \end{cases}$$

Since $\mathrm{gr}^1(\omega, J) = \mathrm{gr}^1(\mathscr{O}, J) \tilde{\otimes} \omega = \left(-1 + \frac{m+1}{2} P^\sharp\right)$, $H^i(\mathrm{gr}^1(\omega, J)) = 0$ for $i = 0$, 1.

In the first two cases $(*_1)$ and $(*_2)$ one has $H^1(\mathrm{gr}^2(\omega, J)) \neq 0$. As in the Step 6.12.3 we get $H^1(\omega/F^3(\omega, J)) \neq 0$ which implies a contradiction. In the case $(*_3)$, one has $H^i(\mathrm{gr}^2(\omega, J)) = 0$ for $i = 0$, 1, and a computation similar to one in the Step 6.12.3 shows

$$\mathrm{gr}^3(\omega, J) \simeq \mathrm{gr}^2(\omega, J) \tilde{\otimes} \mathscr{M} \simeq (0) \tilde{\oplus} \left(-2 + \tfrac{m+5}{2} P^\sharp + Q^\sharp\right).$$

If $m \geq 7$, again as in the Step 6.12.3 we get $H^1(\omega/F^4(\omega, J)) \neq 0$ which implies a contradiction. Thus (6.12.2) holds. □

Lemma 6.13 ([KM92, (2.13.11)])**.** *We use the notation and assumptions of Lemma* 6.12. *Then $H^0(\mathscr{O}(-K_X)) = H^0(F^2(\omega^*, J))$ and a general section s of $H^0(\mathscr{O}(-K_X))$ induces a section $\overline{s}$ of $\mathrm{gr}^2(\omega^*, J)$ such that*

6.13.1 ([KM92, (2.13.11.1)])**.** $\overline{s}$ generates $\mathscr{L} \tilde{\otimes} \mathrm{gr}^0_C \omega^* \subset \mathrm{gr}^1_C \omega^*$ at P, and

6.13.2 ([KM92, (2.13.11.2)])**.** If $m \geq 7$ then $\overline{s}$ is a global generator of (0) in the ℓ-splitting of (6.12.2)

$$\mathrm{gr}^2(\omega^*, J) \simeq (0) \tilde{\oplus} \left(-1 + \tfrac{m+5}{2} P^\sharp + Q^\sharp\right).$$

If $m = 5$, the same assertion holds possibly after changing the ℓ-splitting of $\mathrm{gr}^2(\omega^*, J)$.

Proof. We see $H^0(\mathscr{O}(-K_X)) = H^0(F^2(\omega^*, J))$ by $H^0(\mathrm{gr}^0(\omega^*, J)) = H^0(\mathrm{gr}^1(\omega^*, J)) = 0$ (see Lemma 6.12). Let $D = \{y_1 = 0\}/\boldsymbol{\mu}_m \in |-2K_X|$ and let Ω be an ℓ-free ℓ-basis of $\mathrm{gr}^0 \omega$ at P. As in 5.3.1 by Corollary 2.2, $y_2/\Omega \in \mathscr{O}_D(-K_X)$ lifts modulo $\mathscr{O}_{D^\sharp}(y_2, y_3)^2 dy_2 \wedge dy_3$ to a section of $H^0(F^2(\omega^*, J))$. Since y_2 is a part of an ℓ-free ℓ-basis of $\mathrm{gr}^2(\mathscr{O}, J)$, we see that $\overline{s}$ is non-zero. If $m \geq 7$, then $\overline{s}$ must generate (0) because $H^0(C, (-1 + \frac{m+5}{2} P^\sharp + Q^\sharp)) = 0$. If $m = 5$, we see as above

$$H^0(\mathscr{O}(-K_X)) \twoheadrightarrow \mathrm{gr}^2(\omega^*, J) \otimes \mathbb{C}(P)$$

using $y_3^2/\Omega \in \mathscr{O}_D(-K_X)$. Then general s satisfies $\overline{s} \notin H^0(C, (Q^\sharp))$ in the ℓ-splitting of $\mathrm{gr}^2(\omega^*, J)$ and we have the same conclusion. □

Lemma 6.14 ([KM92, (2.13.12)])**.** *We assume the notation and assumptions of Lemma* 6.12. *In particular, we assume $m \geq 5$. Then the conclusions of the case* 1.3.3 *hold.*

Proof. Let $s \in H^0(\mathscr{O}(-K_X))$ be a general section. If $m = 5$, we change the ℓ-splitting of $\mathrm{gr}^2(\mathscr{O}, J)$ for which Lemma 6.13 holds. Depending on the value of k, we treat two cases.

Case 6.14.1 ($k = 1$, [KM92, (2.13.12.1)]). We claim that the image of $\overline{s}$ in $\operatorname{gr}^1_C \omega^*$ generates $\mathscr{L} \tilde{\otimes} \operatorname{gr}^0_C \omega^* \simeq (1)$ ($\subset \operatorname{gr}^1_C \omega^*$) at P and Q and vanishes at some point R ($\neq P, Q$). Indeed, the generation at P is proved in Lemma 6.13. If $\overline{s}$ does not generate $\mathscr{L} \tilde{\otimes} \operatorname{gr}^0_C \omega^* = \operatorname{gr}^{2,0}(\omega^*, J)$ at Q, $\overline{s}$ is not a part of an ℓ-free ℓ-basis of $\operatorname{gr}^{2,0}(\omega^*, J)$ at Q because

$$\operatorname{qldeg}(\operatorname{gr}^{2,1}(\omega^*, J), Q) = \operatorname{qldeg}\left(\mathscr{M}^{\tilde{\otimes} 2} \tilde{\otimes} \operatorname{gr}^0_C \omega^*, Q\right) = 1 \neq 0.$$

This contradicts Lemma 6.13 and our claim is proved.

Then it is easy to see that $E_X = \{s = 0\} \in |-K_X|$ is smooth outside of P, Q and R. Moreover, $(E_X, P) \simeq (y_1, y_3)/\boldsymbol{\mu}_m(1, -1)$ and $(E_X, Q) \simeq (z_1, z_2)/\boldsymbol{\mu}_2(1, 1)$. We choose coordinates at R so that $(X, R) = (w_1, w_2, w_3) \supset (C, R) = w_1$-axis, and $J = (w_2, w_3^2)$ at R. Using a generator Ω of $\mathscr{O}(K_X)$ at R, we see $\Omega s \equiv u w_1 w_2 \mod (w_2, w_3)^2$ for some unit u because $\overline{s}$ vanishes at R to order 1. Since Ωs is a part of a free basis of $\operatorname{gr}^2(\mathscr{O}, J)$ at R, we have

$$\Omega s \equiv u w_1 w_2 + v w_3^2 \mod (w_2, w_3^2)(w_2, w_3)$$

for some unit v. Thus (E_X, Q) is an A_1 point and we are done in case $k = 1$.

Case 6.14.2 ($k \geq 2$, [KM92, (2.13.12.2)]). We see that the image of $\overline{s}$ in $\operatorname{gr}^1_C \omega^*$ generates $\mathscr{L} \tilde{\otimes} \operatorname{gr}^0_C \omega^* \simeq (Q^\sharp)$ outside of Q by Lemma 6.13. Then $E_X = \{s = 0\} \in |-K_X|$ is smooth outside of P and Q, $(E_X, P) \simeq (y_1, y_3)/\boldsymbol{\mu}_m(1, -1)$. Using an ℓ-free ℓ-basis Ω of $\mathscr{O}(K_X)$ at Q, we see the image of $\overline{s}$ in $\operatorname{gr}^1_C \omega^*$ is $u z_1 z_4/\Omega$ at Q, where u is a unit. Since s is a part of an ℓ-free ℓ-basis of $\operatorname{gr}^2(\omega^*, J)$ at Q, we have $\Omega s \equiv u z_1 z_4 + v z_3 \mod J^\sharp I^\sharp$ at Q for some unit v. Eliminating z_3, we see $(E_X, Q) \simeq (z_1, z_2, z_4; \overline{\beta})/\boldsymbol{\mu}_2(1, 1, 0; 0)$, where $\overline{\beta}$ satisfies $\overline{\beta} \equiv z_1^2 z_4 + z_2^2 \mod (z_2^2, z_4)(z_2, z_4)$ and $\operatorname{ord} \overline{\beta}(0, 0, z_4) = k$. Then we can apply Computation 6.7. □

Remark 6.15. We note that the case 1.3.3 ([KM92, (2.2.3)], [Mor07]) comes out of two sources: Lemma 6.5 where $k \geq 2$, $m \geq 3$ and Q is of type $cA/2$, $cAx/2$ or $cD/2$, and Lemma 6.12 where $m \geq 5$ and Q is of type $cA/2$.

We note that Lemma 6.5 assumes the case 6.4.2, where $(X^\sharp, Q^\sharp)$ is not smooth by $\ell(Q) > 0$ and hence the axial multiplicity $k \geq 2$.

Thus the proof of Theorem 1.3 is completed in the case (IA)+(IA).

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