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Partial Differential Equations — *Existence of ground states for nonlinear, pseudo-relativistic Schrödinger equations*, by VITTORIO COTI ZELATI and MARGHERITA NOLASCO, communicated on 10 December 2010.

Dedicated to Giovanni Prodi.

ABSTRACT. — We study existence and regularity of positive stationary solutions for a class of nonlinear pseudo-relativistic Schrödinger equations. Such equations are characterized by a nonlocal pseudo-differential operator closely related to the square-root of the Laplacian. We investigate such problems using critical point theory after transforming them to elliptic equations with nonlinear Neumann boundary conditions.

KEY WORDS: Nonlinear Schrödinger equation, solitary waves, pseudo-relativistic Hartree approximation.

AMS SUBJECT CLASSIFICATION: 35Q55, 35S05.

1. INTRODUCTION

The Hamiltonian for the motion of a free relativistic particle is given by

$$\mathscr{H} = \sqrt{p^2c^2 + m^2c^4}.$$

With the usual quantization rule $p \mapsto -i\hbar\nabla$ we get the so called *pseudo-relativistic* Hamiltonian operator and the associated Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{\mathscr{H}}\psi = \sqrt{-\hbar^2c^2\Delta + m^2c^4}\psi$$

We choose units so that $\hbar = 1$, $c = 1$. For a discussion of the main properties of the operator $\hat{\mathscr{H}}$ we refer to [10].

In the mean field limit description of a quantum relativistic Bose gas, one is lead to study the nonlinear mean field equation (see [4] for a rigorous derivation of the model)

$$i\frac{\partial\psi}{\partial t} = (\hat{\mathscr{H}} - m)\psi + V_{\text{eff}}(\psi)\psi = \hat{\mathscr{T}}\psi + V_{\text{eff}}(\psi)\psi \tag{1.1}$$

where $\hat{\mathscr{T}}$ denotes the kinetic energy operator and

$$V_{\text{eff}}(\psi) = -v \int_{\mathbb{R}^3} \Phi(|x-y|)|\psi(t,y)|^2\,dy$$

the effective potential operator, Φ being the two particles interaction potential. We will take attractive two body interaction, which means $\Phi > 0$. See [11] for a detailed analysis of this equation for gravitational interaction (and also of the corresponding equation for fermions). It has recently been proved that such an equation is locally well-posed in H^s, $s \geq 1/2$, and is global in time for small initial data in L^2 (see [8]). Blow up has been proved in [6, 7]. These results apply for Newton or Yukawa type two body interaction (i.e. $\Phi(x) = |x|^{-1}$ or $|x|^{-1}e^{-|x|}$). In these cases the estimates on the nonlinearity rely on the observation that

$$\frac{e^{-\mu|x|}}{4\pi|x|} * f = (\mu^2 - \Delta)^{-1} f \quad \text{for } f \in \mathscr{S}(\mathbb{R}^3),\ \mu \geq 0$$

and on some facts from potential theory.

Solitary waves solutions of (1.1) correspond to solutions of

$$\hat{\mathscr{T}}\phi + V_{\text{eff}}(\phi)\phi = \lambda\phi \tag{1.2}$$

of given L^2 norm equal to M. In the paper [11] Lieb and Yau have proved existence of such solutions (in the case $\Phi(x) = |x|^{-1}$) provided that $M < M_c$, M_c being the Chandrasekhar limit mass. More precisely they have shown the existence of a radial, real-valued non negative ground state in $H^{1/2}(\mathbb{R}^3)$. More recently (see [5, 9]) it has been proved that the solution is regular ($H^s(\mathbb{R}^3)$, for all $s \geq 1/2$), strictly positive and that it decays exponentially, more precisely that for every $0 < \delta < \min\{m, \lambda\}$ there exists $C > 0$ such that $|\phi(x)| \leq Ce^{-\delta|x|}$, for all $x \in \mathbb{R}^3$. Moreover the solution is unique, at least for small L^2 norm. Let us remark that all these results are heavily based on the specific form (i.e. of Newtonian or Yukawa type) of the two body interaction in the Hartree nonlinearity (regularity and uniqueness) and on the remarkable fact that the integral kernel of $\sqrt{-\Delta + m} - m + \lambda$ can be computed explicitly (strict positivity and exponential decay).

The main purpose of this paper is to prove existence and regularity results for a wider class of nonlinearities. In particular we will study such a problem exploiting the relation of equation (1.2) with an elliptic equation on $\mathbb{R}^{n+1}_+$ with a nonlinear Neumann boundary condition. Such a relation has been recently exploited to study several problems involving fractional powers of the laplacian, see in particular [2] from which we have learned it.

We will consider the pseudo-relativistic, static Schrödinger equation in $\mathbb{R}^N$, $N \geq 2$

$$\sqrt{-\Delta + m^2}\,u = \mu u + v|u|^{p-2}u + \sigma(W * u^2)u \tag{1.3}$$

(here $W * u^2$ denotes the convolution of W and u^2) where $p \in \left(2, \frac{2N}{N-1}\right)$, $\mu < m$, ν, $\sigma \geq 0$ (but not both 0), $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $W \geq 0$, $r > N/2$, $W(x) = W(|x|) \to 0$ as $|x| \to +\infty$. We will be interested in positive solutions of such an equation.

Remark 1.4. We can deal, in dimension 3, as in [11], with the Newton potential $|x|^{-1}$. When fixing (as in [11]) the L^2 norm to be M, the Newton potential is critical, in the sense that minimization is possible only for $M < M_c$ (i.e. smaller then the Chandrasekhar mass M_c). In contrast to [11], we are not fixing the L^2 norm of the solution. This allows us a wider range of variability for the nonlinear terms.

The operator

$$\sqrt{-\Delta + m^2}$$

can be defined for all $f \in L^2$ with Fourier transform $\mathscr{F}f$ satisfying

$$\int (m^2 + |k|^2)|\mathscr{F}f(k)|^2\, dk < +\infty \tag{1.5}$$

(i.e. for all functions in $H^1(\mathbb{R}^N)$) as

$$\mathscr{F}((\sqrt{-\Delta + m^2}f))(k) = \sqrt{m^2 + |k|^2}\mathscr{F}f(k).$$

See, for example, [10].

The associated energy is given as

$$\int_{\mathbb{R}^N} \sqrt{m^2 + |k|^2}\, |\mathscr{F}f(k)|^2\, dk$$

and is well defined for all functions in $H^{1/2}(\mathbb{R}^N)$, that is for all functions in $L^2(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (1 + |k|)|\mathscr{F}f(k)|^2\, dk < +\infty.$$

An alternative definition of the operator (1.3) can be obtained as follows. Given any function $u \in \mathscr{S}(\mathbb{R}^N)$ there is a unique function $v \in \mathscr{S}(\mathbb{R}_+^{N+1})$ (here $\mathbb{R}_+^{N+1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}^N \mid x > 0\}$) such that

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}_+^{N+1} \\ v(0, y) = u(y) & \text{for } y \in \mathbb{R}^N = \partial\mathbb{R}_+^{N+1} \end{cases}.$$

Setting

$$Tu(y) = -\frac{\partial v}{\partial x}(0, y)$$

we have that the equation

$$\begin{cases} -\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ w(0,y) = Tu(y) = -\frac{\partial v}{\partial x}(0,y) & \text{for } y \in \mathbb{R}^N \end{cases}$$

has the solution $w(x,y) = -\frac{\partial v}{\partial x}(x,y)$. From this we have that

$$T(Tu)(y) = -\frac{\partial w}{\partial x}(0,y) = \frac{\partial^2 v}{\partial x^2}(0,y) = (-\Delta_y v + m^2 v)(0,y)$$

and hence $T^2 = (-\Delta_y + m^2)$.

We will exploit this fact, and, in order to find solutions of (1.3) and to prove their regularity, we will look (following [2], see also [3] where a problem on a bounded domain is studied) for solutions of

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\frac{\partial v}{\partial x} = \mu v + \nu |v|^{p-2} v + \sigma (W * v^2) v & \text{on } \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+ \end{cases}.$$

Our main result is the following

Theorem 1.6. *Let $p \in \left(2, \frac{2N}{N-1}\right)$, $\mu < m$, $\nu, \sigma \geq 0$ (but not both 0), $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $W \geq 0$, $r > N/2$, $W(x) = \tilde{W}(|x|)$, $\tilde{W}(s) \to 0$ as $s \to +\infty$.*

Then equation (1.3) has a radially symmetric solution $u \in C^\infty(\mathbb{R}^N)$ such that

$$0 < u(y) \leq Ce^{-\delta|y|} \quad \text{for any } |y| \geq R \tag{1.7}$$

where $0 < \delta < m - \mu$, for $\mu \geq 0$ and $\delta = m$ for $\mu < 0$.

Notation. Let $(x,y) \in \mathbb{R} \times \mathbb{R}^N$. We have already introduced $\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^{N+1} \mid x > 0\}$. With $\|u\|_p$ we will always denote the norm of $u \in L^p(\mathbb{R}^{N+1}_+)$, with $\|u\|$ the norm of $u \in H^1(\mathbb{R}^{N+1}_+)$ and with $|v|_p$ the $L^p(\mathbb{R}^N)$ norm of $v \in L^p(\mathbb{R}^N)$.

2. Variational setting

We recall that for all $v \in H^1(\mathbb{R}^{N+1}) \cap C_0^\infty(\mathbb{R}^{N+1})$

$$\begin{aligned} \int_{\mathbb{R}^N} |v(0,y)|^p \, dy &= \int_{\mathbb{R}^N} dy \int_{+\infty}^0 \frac{\partial}{\partial x} |v(x,y)|^p \, dx \\ &\leq p \int_{\mathbb{R}^{N+1}_+} |v(x,y)|^{p-1} \left| \frac{\partial v}{\partial x}(x,y) \right| dx\, dy \\ &\leq p \left(\int_{\mathbb{R}^{N+1}_+} |v(x,y)|^{2(p-1)} \, dx\, dy \right)^{1/2} \left(\int_{\mathbb{R}^{N+1}_+} \left| \frac{\partial v}{\partial x}(x,y) \right|^2 dx\, dy \right)^{1/2} \end{aligned}$$

that is

$$|v(0,\cdot)|_p^p \le p\|v\|_{2(p-1)}^{p-1}\left\|\frac{\partial v}{\partial x}\right\|_2, \tag{2.1}$$

which, by Sobolev embedding, is finite for all $2 \le 2(p-1) \le 2(N+1)/((N+1)-2)$, that is $2 \le p \le \frac{2N}{N-1}$. By density of $H^1(\mathbb{R}^{N+1}) \cap C_0^\infty(\mathbb{R}^{N+1})$ in $H^1(\mathbb{R}_+^{N+1})$ such an estimates allows us to define the trace $\gamma(v)$ of v for all the functions $v \in H^1(\mathbb{R}_+^{N+1})$. The inequality

$$|\gamma(v)|_p^p \le p\|v\|_{2(p-1)}^{p-1}\left\|\frac{\partial v}{\partial x}\right\|_2, \tag{2.2}$$

holds then for all $v \in H^1(\mathbb{R}_+^{N+1})$.

It is known that traces of functions in $H^1(\mathbb{R}_+^{N+1})$ belongs to $H^{1/2}(\mathbb{R}^N)$ and that every function in $H^{1/2}(\mathbb{R}^N)$ is the trace of a function in $H^1(\mathbb{R}_+^{N+1})$.

Let us define, for all $v \in H^1(\mathbb{R}_+^{N+1})$,

$$\begin{aligned} I(v) &= \frac{1}{2}\iint_{\mathbb{R}_+^{N+1}} (|\nabla v|^2 + m^2 v^2)\,dx\,dy \\ &\quad - \int_{\mathbb{R}^N} \left(\frac{\mu}{2}|\gamma(v)|^2 + \frac{v}{p}|\gamma(v)|^p + \frac{\sigma}{4}(W * \gamma(v)^2)\gamma(v)^2\right) dy \end{aligned}$$

We have that, for all $p \in \left[2, \frac{2N}{N-1}\right]$

$$|\gamma(v)|_p \le \frac{(p-1)}{p}\|v\|_{2(p-1)} + \|\nabla v\|_2 \le C_p\|v\| \tag{2.3}$$

This is in fact equivalent to the well known fact that $\gamma(v) \in H^{1/2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ provided $q \in \left[2, \frac{2N}{N-1}\right]$, and shows that the terms $\int_{\mathbb{R}^N} |\gamma(v)|^2$ and $\int_{\mathbb{R}^N} |\gamma(v)|^p$ in our functional are well defined since $p \in \left(2, \frac{2N}{N-1}\right)$.

From Young's inequality we have that

$$|(W * \gamma(v)^2)\gamma(v)^2|_1 \le |W|_r|\gamma(v)^2|_q^2 = |W|_r|\gamma(v)|_{2q}^4 \quad \frac{1}{r} + \frac{2}{q} = 2.$$

Since $\gamma(v) \in L^{2q}$ for all $2q \in [2, 2N/(N-1)]$, we have that the norm is finite provided $W \in L^r$, $r \in [N/2, +\infty]$. Under our assumptions, $W = W_1 + W_2$, $W_1 \in L^r$, $r > N/2$, $W_2 \in L^\infty$. Hence

$$\int_{\mathbb{R}^N}(W*\gamma(v)^2)\gamma(v)^2\,dy = \int_{\mathbb{R}^N}(W_1*\gamma(v)^2)\gamma(v)^2\,dy + \int_{\mathbb{R}^N}(W_2*\gamma(v)^2)\gamma(v)^2\,dy \leq |W_1|_r|\gamma(v)|^4_{4r/(2r-1)} + |W_2|_\infty|\gamma(v)|_2^4 \leq C_W\|v\|^4 < +\infty \tag{2.4}$$

since $2 \leq 4r/(2r-1) < 2N/(N-1)$.

We will also need the following estimate:

$$\int_{\mathbb{R}^N}|W*\gamma(v)^2|\gamma(v)||^m \leq \Big(\int_{\mathbb{R}^N}|W*\gamma(v)^2|^{mq}\Big)^{1/q}\Big(\int_{\mathbb{R}^N}|\gamma(v)|^{mp}\Big)^{1/p} \leq C\Big(\int_{\mathbb{R}^N}|W|^r\Big)^{m/r}\Big(\int_{\mathbb{R}^N}|\gamma(v)|^{2s}\Big)^{m/s}\Big(\int_{\mathbb{R}^N}|\gamma(v)|^{mp}\Big)^{1/p}$$

where $p^{-1}+q^{-1}=1$ and $1+(mq)^{-1}=r^{-1}+s^{-1}$. Setting $mp=\alpha=2s$ we find that $1+m^{-1}=r^{-1}+3\alpha^{-1}$ so that

$$|(W*\gamma(v)^2)\gamma(v)|_m \leq C|W|_r|\gamma(v)|_\alpha^3 \tag{2.5}$$

hence for $\alpha \in \left[2, \frac{2N}{N-1}\right]$ and $r > \frac{N}{2}$ we can take $m \in \left(\frac{2N}{N+4}, \frac{2N}{N-3}\right)$.

Let us remark here that from inequality (2.1) we also deduce that for all $\lambda > 0$ we have

$$\int_{\mathbb{R}^N}|\gamma(v)|^p \leq \frac{\lambda p^2}{4}\int_{\mathbb{R}^{N+1}_+}|v|^{2(p-1)}\,dx\,dy + \frac{1}{\lambda}\int_{\mathbb{R}^{N+1}_+}\left|\frac{\partial v}{\partial x}\right|^2 dx\,dy. \tag{2.6}$$

In particular, we have that

$$\int_{\mathbb{R}^N}|\gamma(v)|^2 \leq \lambda\int_{\mathbb{R}^{N+1}_+}|v|^2\,dx\,dy + \frac{1}{\lambda}\int_{\mathbb{R}^{N+1}_+}\left|\frac{\partial v}{\partial x}\right|^2 dx\,dy. \tag{2.7}$$

As an easy consequence of the above discussion, we have that

PROPOSITION 2.8. *The functional I is C^1 on $H^1(\mathbb{R}^{N+1}_+)$.*

Let $v \in H^1(\mathbb{R}^{N+1}_+)$ be a critical point for I, then for all $w \in H^1$

$$\iint_{R^{N+1}_+}(\nabla v\nabla w + m^2 vw)\,dx\,dy = \int_{\mathbb{R}^N}(\mu\gamma(v)\gamma(w) + \nu|\gamma(v)|^{p-2}\gamma(v)\gamma(w) + \sigma(W*\gamma(v)^2)\gamma(v)\gamma(w))\,dy$$

and we say that v is a weak solution of

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\frac{\partial v}{\partial x} = \mu v + v|v|^{p-2}v + \sigma(W * v^2)v & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+ \end{cases}.$$

3. Regularity of critical points

To show that critical points of I are (classical) solutions of

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\frac{\partial v}{\partial x} = \mu v + v|v|^{p-2}v + \sigma(W * v^2)v & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+ \end{cases} \tag{3.1}$$

we are going to prove some regularity results for the critical points of I.

Theorem 3.2. *Suppose that $v \in H^1(\mathbb{R}^{N+1}_+)$ is a critical point for the functional I on $H^1(\mathbb{R}^{N+1}_+)$.*
Then $\gamma(v) \in L^p(\mathbb{R}^N)$ for all $p \in [2, +\infty]$ and $v \in L^\infty(\mathbb{R}^{N+1}_+)$.

Proof. We will follow a classical argument, see for example [2]. Since $v \in H^1(\mathbb{R}^{N+1}_+)$ is a critical point, we know that for all $w \in H^1(\mathbb{R}^{N+1}_+)$

$$\iint_{R^{N+1}_+} (\nabla v \nabla w + m^2 vw)\, dx\, dy$$
$$= \int_{\mathbb{R}^N} (\mu\gamma(v)\gamma(w) + v|\gamma(v)|^{p-2}\gamma(v)\gamma(w) + \sigma(W * \gamma(v)^2)\gamma(v)\gamma(w))\, dy.$$

Let $w = \phi_{\beta,T} = v v_T^{2\beta}$ where $v_T = \min\{v_+, T\}$ and $\beta > 0$. We have that $\phi_{\beta,T} \in H^1(\mathbb{R}^{N+1}_+)$, $\phi_{\beta,T} \geq 0$ and from $\langle I'(v), \phi_{\beta,T}\rangle = 0$ we deduce that (here we write v for $\gamma(v)$)

$$\iint_{R^{N+1}_+} v_T^{2\beta}(|\nabla v|^2 + m^2 v^2)\, dx\, dy + \iint_{D_T} 2\beta v_T^{2\beta}|\nabla v|^2\, dx\, dy$$
$$= \int_{\mathbb{R}^N} (\mu v^2 v_T^{2\beta} + v|v|^{p-2}v^2 v_T^{2\beta} + \sigma(W * v^2)v^2 v_T^{2\beta})\, dy$$

where $D_T = \{(x, y) \mid v_+(x, y) \leq T\}$.

Since

$$\iint_{\mathbb{R}^{N+1}_+} |\nabla(v v_T^\beta)|^2\, dx\, dy = \iint_{\mathbb{R}^{N+1}_+} v_T^{2\beta}|\nabla v|^2\, dx\, dy + \iint_{D_T} (2\beta + \beta^2) v_T^{2\beta}|\nabla v|^2\, dx\, dy$$

we find that, for $c_\beta = \max\{\frac{1}{m^2}, 1 + \frac{\beta}{2}\} > 0$

$$\begin{aligned}\|vv_T^\beta\|^2 &= \iint_{\mathbb{R}_+^{N+1}} (|\nabla(vv_T^\beta)|^2 + (vv_T^\beta)^2)\,dx\,dy \\ &\le c_\beta \int_{\mathbb{R}^N} (\mu v^2 v_T^{2\beta} + \nu|v|^{p-2}v^2 v_T^{2\beta} + \sigma(W * v^2)v^2 v_T^{2\beta})\,dy.\end{aligned}$$

By Young's inequality:

If $W_1 \in L^r$ with $r \in (N/2, N]$, we have, since $\gamma(v)^2 \in L^p$ with $p^{-1} + r^{-1} = 1 + N^{-1}$, that $W_1 * \gamma(v)^2 \in L^N$.

If $W_1 \in L^r$ with $r > N$, we have, since $\gamma(v)^2 \in L^p$ with $p^{-1} + r^{-1} = 1$, that $W_1 * \gamma(v)^2 \in L^\infty$.

Since $\gamma(v)^2 \in L^1$ and $W_2 \in L^\infty$ we have that $W_2 * \gamma(v)^2 \in L^\infty$.

So in any case we have that, for some constant $c_1 > 0$ and $g_1 \in L^N(\mathbb{R}^N)$

$$(W * \gamma(v)^2) \le c_1 + g_1$$

We also have that

$$|\gamma(v)|^{p-2} = |\gamma(v)|^{p-2}\chi_{\{|v|\le 1\}} + |\gamma(v)|^{p-2}\chi_{\{|v|>1\}} \le 1 + g_2$$

where $g_2 \in L^N(\mathbb{R}^N)$. Indeed, if $(p-2)N < 2$ we have that

$$\int_{\mathbb{R}^N} |\gamma(v)|^{N(p-2)}\chi_{\{|v|>1\}} \le \int_{\mathbb{R}^N} |\gamma(v)|^2\chi_{\{|v|>1\}} \le \int_{\mathbb{R}^N} |\gamma(v)|^2 < +\infty$$

while if $2 \le (p-2)N$ we have that $(p-2)N \in [2, 2N/(N-1)]$.

We have thus proved that, for some constant c and function $g \in L^N(\mathbb{R}^N)$, $g \ge 0$ and independent of T and β,

$$\begin{aligned}&\mu\gamma(v)^2\gamma(v_T)^{2\beta} + \nu|\gamma(v)|^{p-2}\gamma(v)^2\gamma(v_T)^{2\beta} + \sigma(W * \gamma(v)^2)\gamma(v)^2\gamma(v_T)^{2\beta} \\ &\quad \le (c + g)\gamma(v)^2\gamma(v_T)^{2\beta}.\end{aligned}$$

As a consequence

$$\iint_{\mathbb{R}_+^{N+1}} |\nabla(vv_T^\beta)|^2 + |vv_T^\beta|^2 \le cc_\beta \int_{\mathbb{R}^N} \gamma(v)^2\gamma(v_T)^{2\beta} + c_\beta \int_{\mathbb{R}^N} g\gamma(v)^2\gamma(v_T)^{2\beta}$$

and, using Fatou's lemma and monotone convergence, we can pass to the limit as $T \to +\infty$ to get

$$\iint_{\mathbb{R}_+^{N+1}} |\nabla(v_+^{1+\beta})|^2 + |v_+^{1+\beta}|^2 \le cc_\beta \int_{\mathbb{R}^N} \gamma(v_+)^{2(1+\beta)} + c_\beta \int_{\mathbb{R}^N} g\gamma(v_+)^{2(1+\beta)}. \tag{3.3}$$

For any $M > 0$, let $A_1 = \{g \le M\}$, $A_2 = \{g > M\}$.

Then

$$\begin{aligned}\int_{\mathbb{R}^N} gv_+^{2(1+\beta)} &\le \int_{A_1} gv_+^{2(1+\beta)} + \int_{A_2} gv_+^{2(1+\beta)}\\ &\le M\int_{A_1} v_+^{2(1+\beta)} + \Big(\int_{A_2} g^N\Big)^{1/N}\Big(\int_{A_2} v_+^{2N(1+\beta)/(N-1)}\Big)^{(N-1)/N}\\ &\le M|v_+^{1+\beta}|_2^2 + \epsilon(M)|v_+^{1+\beta}|_{2^\#}^2\end{aligned}$$

where we have set $2^\# = 2N/(N-1)$. So we have that

$$\|v_+^{1+\beta}\|^2 \le c_\beta(c+M)|\gamma(v_+)^{1+\beta}|_2^2 + c_\beta\epsilon(M)|\gamma(v_+)^{1+\beta}|_{2^\#}^2.$$

Since by (2.3) $|\gamma(v_+)^{1+\beta}|_{2^\#} \le C_{2^\#}\|v_+^{1+\beta}\|$ we finally have (choosing M large so that $c_\beta\epsilon(M)C_{2^\#}^2 < 1/2$) that, for all weak solutions v,

$$\|v_+^{1+\beta}\|^2 \le 2c_\beta(c+M)|\gamma(v_+)^{1+\beta}|_2^2. \tag{3.4}$$

Remark that also M depends on β.

Using (2.3) we finally get that

$$|\gamma(v_+)^{\beta+1}|_{2^\#}^2 \le 2c_\beta(c+M)C_{2^\#}^2|\gamma(v_+)^{\beta+1}|_2^2. \tag{3.5}$$

Then a bootstrap argument can start: since $\gamma(v_+) \in L^{2N/(N-1)}$ we can apply (3.5) with $\beta_1 + 1 = N/(N-1)$ to deduce that $\gamma(v_+) \in L^{(\beta_1+1)2N/(N-1)} = L^{2N^2/(N-1)^2}$. We can then apply again (3.5) and, after k iterations, we deduce that $\gamma(v_+) \in L^{2N^k/(N-1)^k}$ and hence $\gamma(v_+) \in L^p(\mathbb{R}^N)$ for all $p \in [2, +\infty)$.

The same is clearly true for $\gamma(v_-)$ and hence for $\gamma(v)$.

We will now show that actually v is bounded in $\mathbb{R}_+^{N+1}$ and $\gamma(v)$ in $\mathbb{R}^N$.

We first of all observe that, since $\gamma(v) \in L^p$ for all $p \ge 2$, then $W * \gamma(v)^2 \in L^\infty$. Indeed this was already the case for $W_2 * \gamma(v)^2$, and for $W_1 * \gamma(v)^2$ if $W_1 \in L^r$ with $r > N$. The fact that $W_1 * \gamma(v)^2 \in L^\infty$ also when $W_1 \in L^r$ with $N/2 < r \le N$ follows from Young's inequality since we now know that $\gamma(v)^2 \in L^q$, $q^{-1} + r^{-1} = 1$ for all $r \in (N/2, N]$.

Then we remark that $\gamma(v)^{(p-2)} = \gamma(v)^{p-2}\chi_{\{|\gamma(v)|\le 1\}} + \gamma(v)^{p-2}\chi_{\{|\gamma(v)|>1\}}$ and now we have that $\gamma(v)^{p-2}\chi_{\{|\gamma(v)|>1\}} \in L^{2N}$. As a consequence we have now that, for some constant c and function $g \in L^{2N}(\mathbb{R}^N)$, $g \ge 0$ and independent of T and β,

$$\mu v^2 v_T^{2\beta} + \nu|v|^{p-2}v^2v_T^{2\beta} + \sigma(W * v^2)v^2v_T^{2\beta} \le (c+g)v^2v_T^{2\beta}.$$

So we have that (3.3) holds for v_+ but now $g \in L^{2N}$. Since

$$\int gv_+^{2(1+\beta)} \le |g|_{2N}|v_+^{1+\beta}|_2|v_+^{1+\beta}|_{2^\#} \le |g|_{2N}\Big(\lambda|v_+^{1+\beta}|_2^2 + \frac{1}{\lambda}|v_+^{1+\beta}|_{2^\#}^2\Big)$$

and

$$\|v_+^{1+\beta}\|^2 \le c_\beta(c + |g|_{2N}\lambda)|v_+^{1+\beta}|_2^2 + \frac{c_\beta|g|_{2N}}{\lambda}|v_+^{1+\beta}|_{2^\#}^2. \tag{3.6}$$

Taking λ such that

$$\frac{c_\beta|g|_{2N}}{\lambda} C_{2^\#}^2 = \frac{1}{2}$$

we find that

$$|v_+^{\beta+1}|_{2^\#}^2 \le 2c_\beta(c + |g|_{2N}\lambda)C_{2^\#}^2|v_+^{\beta+1}|_2^2 = M_\beta|v_+^{\beta+1}|_2^2 \tag{3.7}$$

and the advantage with respect to (3.5) is that now we control the dependence on β of the constant M_β. Indeed

$$M_\beta \le Cc_\beta^2 \le C(m^{-2} + 1 + \beta)^2 \le M_0^2 e^{2\sqrt{1+\beta}}.$$

Write (3.7) as

$$|v_+|_{2^\#(\beta+1)} \le M_0^{1/(1+\beta)} e^{1/\sqrt{1+\beta}}|v_+|_{2(\beta+1)}. \tag{3.8}$$

The same bootstrap argument of before shows, choosing $\beta_0 = 0$, $2(\beta_{n+1} + 1) = 2^\#(\beta_n + 1)$, that $u \in L^{2(\beta_n+1)}$ implies $u \in L^{2^\#(\beta_{n+1}+1)}$ and

$$|v_+|_{2^\#(1+\beta_n)} \le M_0^{\sum_{i=0}^n 1/(1+\beta_i)} e^{\sum_{i=0}^n 1/\sqrt{1+\beta_i}}|v_+|_{2(\beta_0+1)}.$$

Since $(1 + \beta_n) = (2^\#/2)^n = (N/(N-1))^n$ we have that

$$\sum_{i=0}^\infty \frac{1}{(1+\beta_i)} < +\infty, \quad \sum_{i=0}^\infty \frac{1}{\sqrt{1+\beta_i}} < +\infty$$

and from this we deduce that

$$|v_+|_\infty = \lim_{n\to+\infty} |v_+|_{2^\#(1+\beta_n)} < +\infty.$$

We can use the fact that $|v_+|_p \le C < +\infty$ for all p in (3.6) (with $\lambda = 1$) to deduce that, for all $\beta > 0$,

$$\|v_+^{1+\beta}\|^2 \le c_\beta(c + |g|_{2N})C^{2(1+\beta)} + c_\beta|g|_{2N}C^{2(1+\beta)}.$$

Since by Sobolev's embedding $\|v_+\|_{2^*(1+\beta)}^{1+\beta} = \|v_+^{1+\beta}\|_{2^*} \leq C_{2^*}\|v_+^{1+\beta}\|$ we deduce from the above inequality that

$$\|v_+\|_{2^*(1+\beta)}^{2(1+\beta)} \leq \tilde{c}c_\beta C^{2(1+\beta)}.$$

Since $\tilde{c}^{1/2(1+\beta)}c_\beta^{1/2(1+\beta)}C \leq \bar{c}$, as before we get that $v_+ \in L^\infty(\mathbb{R}_+^{N+1})$. □

Proposition 3.9. *Suppose that $v \in H^1(\mathbb{R}_+^{N+1}) \cap L^\infty(\mathbb{R}_+^{N+1})$ is a weak solution of*

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}_+^{N+1} \\ -\frac{\partial v}{\partial x} = g(y) & \text{for all } y \in \mathbb{R}^N \end{cases} \tag{3.10}$$

where $g \in L^p(\mathbb{R}^N)$ for all $p \in [2, +\infty]$.

Then $v \in C^{0,\alpha}([0, +\infty) \times \mathbb{R}^N) \cap W^{1,q}((0, R) \times \mathbb{R}^N)$ for all $q \in [2, +\infty)$ and $R > 0$.

If, in addition, $g \in C^\alpha(\mathbb{R}^N)$ then $v \in C^{1,\alpha}([0, +\infty) \times \mathbb{R}^N) \cap C^2(\mathbb{R}_+^{N+1})$ is a classical solution of (3.10).

Proof. By a weak solution we mean a function $v \in H^1(\mathbb{R}_+^{N+1})$ such that

$$\iint_{R_+^{N+1}} (\nabla v \nabla w + m^2 v w)\, dx\, dy = \int_{\mathbb{R}^N} g w\, dy \quad \text{for all } w \in H^1(\mathbb{R}_+^{N+1}) \tag{3.11}$$

Following [2] we let

$$u(x, y) = \int_0^x v(t, y)\, dt.$$

We clearly have that $u \in H^1((0, R) \times \mathbb{R}^N)$ for all $R > 0$. We will show that u satisfies

$$\iint_{\mathbb{R}_+^{N+1}} (\nabla u \nabla \eta + m^2 u \eta - g\eta)\, dx\, dy = 0 \quad \text{for all } \eta \in C_0^1(\mathbb{R}_+^{N+1}) \tag{3.12}$$

so that u is a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u + m^2 u = g & \text{in } \mathbb{R}_+^{N+1} \\ u = 0 & \text{for all } y \in \mathbb{R}^N \end{cases}$$

where $g(x, y) = g(y)$ for all $(x, y) \in \mathbb{R}_+^{N+1}$.

Take any $\eta \in C_0^1(\mathbb{R}_+^{N+1})$ and set, for all $t \geq 0$ $w_t(x, y) = \eta(x + t, y) \in H^1(\mathbb{R}_+^{N+1})$. From (3.11) we get

$$\iint_{R_+^{N+1}} (\nabla v \nabla w_t + m^2 v w_t)\, dx\, dy = \int_{\mathbb{R}^N} g w_t\, dy \quad \text{for all } \eta \in C_0^1(\mathbb{R}_+^{N+1}),\ t \geq 0.$$

Integrating such an equation in t from 0 to $+\infty$ we get that (3.12) holds.

Indeed

$$\begin{aligned}\int_0^{+\infty} dt \int_0^{+\infty} dx \int_{\mathbb{R}^N} \nabla v(x,y)\nabla\eta(x+t,y)\,dy\\ = \int_0^{+\infty} dx \int_x^{+\infty} ds \int_{\mathbb{R}^N} \nabla v(x,y)\nabla\eta(s,y)\,dy\\ = \int_0^{+\infty} ds \int_0^{s} dx \int_{\mathbb{R}^N} \nabla v(x,y)\nabla\eta(s,y)\,dy\\ = \int_0^{+\infty} ds \int_{\mathbb{R}^N} \nabla\Big(\int_0^{s} v(x,y)\,dx\Big)\nabla\eta(s,y)\,dy.\end{aligned}$$

Let us define $u_{\text{odd}} \in H^1((-R,R)\times\mathbb{R}^N)$ and $g_{\text{odd}} \in L^q((-R,R)\times\mathbb{R}^N)$ (for all $q \in [2,+\infty]$ and $R>0$) setting

$$u_{\text{odd}}(x,y) = \begin{cases} u(x,y) & x \geq 0\\ -u(-x,y) & x<0\end{cases} \quad\text{and}\quad g_{\text{odd}}(x,y) = \begin{cases} g(y) & x\geq 0\\ -g(y) & x<0\end{cases}.$$

It is easy to check that

$$\iint_{\mathbb{R}^{N+1}} (\nabla u_{\text{odd}}\nabla\eta + m^2 u_{\text{odd}}\eta - g_{\text{odd}}\eta)\,dx\,dy = 0 \quad \text{for all } \eta\in C_0^1(\mathbb{R}^{N+1}) \tag{3.13}$$

so that u_{odd} is a weak solution of the Dirichlet problem

$$-\Delta u_{\text{odd}} + m^2 u_{\text{odd}} = g_{\text{odd}} \quad \text{in } \mathbb{R}^{N+1}.$$

Since $g_{\text{odd}} \in L^q((-R,R)\times\mathbb{R}^N)$ for all $q\in[2,+\infty]$ and $R>0$ we deduce by standard elliptic regularity that

$$u_{\text{odd}} \in W^{2,q}((-R,R)\times\mathbb{R}^N) \quad \text{for all } q\in[2,+\infty),\ R>0$$

and hence by Sobolev's embedding $u_{\text{odd}} \in C^{1,\alpha}(\mathbb{R}^{N+1})$ for all $\alpha\in(0,1)$, $u \in C^{1,\alpha}([0,+\infty)\times\mathbb{R}^N)$ and $v(x,y) = \frac{\partial}{\partial x}u(x,y) \in C^{0,\alpha}([0,+\infty)\times\mathbb{R}^N)$.

If $g\in C^\alpha(\mathbb{R}^N)$, we can apply classical elliptic boundary regularity for Dirichlet problems and deduce that $u\in C^{2,\alpha}([0,+\infty)\times\mathbb{R}^N)$, showing that $v\in C^{1,\alpha}([0,+\infty)\times\mathbb{R}^N)$. The last statement follows again from classical interior elliptic regularity applied directly to v. □

THEOREM 3.14. *Suppose that $v\in H^1(\mathbb{R}^{N+1}_+)$ is a strictly positive critical point for the functional I on $H^1(\mathbb{R}^{N+1}_+)$.*

Then $v\in C^\infty([0,+\infty)\times\mathbb{R}^N)$ and satisfies

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+\\ -\frac{\partial v}{\partial x} = \mu v + v|v|^{p-2}v + \sigma(W * v^2)v & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+\end{cases}. \tag{3.15}$$

Moreover $v(x,y)e^{\lambda x}\to 0$, as $x+|y|\to+\infty$, for any $\lambda<m$.

PROOF. We know from Theorem 3.2 that $\gamma(v) \in L^q(\mathbb{R}^N)$ for all $q \in [2, +\infty]$. Then also

$$g(y) = \mu v + v|v|^{p-2}v + \sigma(W * v^2)v \in L^q(\mathbb{R}^N) \quad \text{for all } q \in [2, +\infty].$$

From Theorem 3.9 we then deduce that $\gamma(v) \in C^{0,\alpha}(\mathbb{R}^N)$, and then that $g \in C^{0,\alpha}(\mathbb{R}^N)$. Again Theorem 3.9 tells us that v is a classical solution. A bootstrap argument allows to deduce that $v \in C^\infty([0, +\infty) \times \mathbb{R}^N)$.

To prove the decay at infinity, let us remark that v is a classical, bounded solution of

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v(0, y) \equiv v_0(y) \in L^2(\mathbb{R}^N) & \text{for } y \in \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+ \end{cases}.$$

Then by using the Fourier transform with respect to the variable $y \in \mathbb{R}^N$ we get

$$\mathscr{F}v(x, k) = e^{-\sqrt{|2\pi k|^2 + m^2}x}\mathscr{F}v_0(k)$$

and hence

$$\sup_{y \in \mathbb{R}^N} |v(x, y)| \leq C|v_0|_2 \mathrm{e}^{-mx}. \tag{3.16}$$

Since by Theorem 3.9 $v \in W^{1,q}((0, R) \times \mathbb{R}^N)$ for all $q \in [2, +\infty)$ and $R > 0$, we have that $v(x, y) \to 0$ as $|y| \to +\infty$ for any x and we conclude that $v(x, y)\mathrm{e}^{\lambda x} \to 0$, as $x + |y| \to +\infty$, for any $\lambda < m$. □

4. Existence of a critical point

We will look for solutions in the following space of symmetric functions

$$H^1_\# = \{u \in H^1(\mathbb{R}^{N+1}_+) \mid u(x, Ry) = u(x, y) \text{ for all } R \in O(N)\}.$$

We start by analyzing the geometric structure of the functional

LEMMA 4.1. *The functional I has the Mountain Pass structure, that is:*

- $I(0) = 0$ *and there exist* $r, \alpha > 0$ *such that* $I(v) \geq \alpha > 0$ *for all* $\|v\| = r$*;*
- $I(\lambda v) \to -\infty$ *as* $\lambda \to +\infty$ *for all* $v \in H^1_\#$, $\gamma(v) \not\equiv 0$.

PROOF. Using (2.7) with $\lambda = m$, (2.3) and (2.4) we have

$$\begin{aligned} I(v) &= \frac{1}{2}\iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2)\, dx\, dy \\ &\quad - \int_{\mathbb{R}^N} \left(\frac{\mu}{2}|\gamma(v)|^2 + \frac{v}{p}|\gamma(v)|^p + \frac{\sigma}{4}(W * \gamma(v)^2)\gamma(v)^2\right) dy \\ &\geq \frac{1}{2}\iint_{\mathbb{R}^{N+1}_+} \left(\left(1 - \frac{\mu}{m}\right)|\nabla v|^2 + m(m - \mu)v^2\right) dx\, dy - \frac{v}{p} C_p^p \|v\|^p - \frac{\sigma}{4} C_W \|v\|^4. \end{aligned}$$

Hence we can find $c > 0$ such that

$$I(v) \geq c\|v\|^2 - \frac{\nu}{p} C_p^p \|v\|^p - \frac{\sigma}{4} C_W \|v\|^4.$$

We immediately deduce that there exist r and $\alpha > 0$ such that

$$I(v) \geq \alpha > 0 \quad \text{for all } \|v\| = r.$$

Moreover for $v \in H^1_\#$, $\gamma(v) \not\equiv 0$, it is immediate to check that $I(\lambda v) \to -\infty$ as $\lambda \to +\infty$. □

Lemma 4.2. *The functional I satisfies the Palais-Smale condition, that is:*
For all sequences $v_n \in H^1_\#$ such that $I(v_n) \to c$ and $I'(v_n) \to 0$ there is a convergent subsequence.

Proof. We have that

$$\begin{aligned} c + 1 + \|v_n\| &\geq I(v_n) - \frac{1}{2}\langle I'(v_n), v_n \rangle \\ &= \left(\frac{\nu}{2} - \frac{\nu}{p}\right) \int_{\mathbb{R}^N} |\gamma(v_n)|^p \, dy + \left(\frac{\sigma}{2} - \frac{\sigma}{4}\right) \int_{\mathbb{R}^N} (W * |\gamma(v_n)|^2)\gamma(v_n)^2 \, dy. \end{aligned}$$

We can then find $c_1, c_2 > 0$ such that

$$\frac{\nu}{p} \int_{\mathbb{R}^N} |\gamma(v_n)|^p \, dy + \frac{\sigma}{4} \int_{\mathbb{R}^N} (W * |\gamma(v_n)|^2)\gamma(v_n)^2 \, dy \leq c_1\|v_n\| + c_2.$$

It follows then from

$$c + 1 \geq I(v_n) \geq c_0\|v_n\|^2 - c_1\|v_n\| - c_2$$

that v_n is bounded in $H^1(\mathbb{R}^{N+1}_+)$. Then v_n converges weakly to some v in $H^1_\#$. We want to prove that $\gamma(v_n) \to \gamma(v)$ strongly in $L^q(\mathbb{R}^N)$ for all $q \in \left(2, \frac{2N}{N-1}\right)$. Setting $w_n = v_n - v$, by (2.2) it is enough to prove that $w_n \to 0$ strongly in $L^{2(q-1)}(\mathbb{R}^{N+1}_+)$. Let us remark that also w_n belongs to $H^1_\#$.

By a result of P. L. Lions [12] (see also [14, Lemma 1.21]), it is enough to prove that, for some $r > 0$,

$$\sup_{z \in \mathbb{R}^{N+1}_+} \int_{B(z,r)} |w_n|^2 \to 0 \quad \text{as } n \to +\infty.$$

Suppose this is not the case. Then there are r and $\alpha > 0$ and a sequence $z_n = (x_n, y_n) \in \mathbb{R}^{N+1}_+$ such that (up to a subsequence)

$$\int_{B(z_n,r)} |w_n|^2 \geq \alpha \quad \text{for all } n \in \mathbb{N}.$$

If z_n is bounded, say $z_n \to \bar{z}$, we get a contradiction since from the compactness of the embedding of $H^1(B(\bar{z}, r))$ in $L^2(B(\bar{z}, r))$ we get that $w_n \to 0$ strongly in $L^2(B(\bar{z}, r))$.

If $z_n = (x_n, y_n)$ is unbounded and $|y_n| \to +\infty$, we can find an increasing number k_n of rotations $R_i \in O(N)$ such that

$$B((x_n, R_i y_n), r) \neq B((x_n, R_j y_n), r) \quad \text{for } i \neq j,\ i, j \in \{1, 2, \dots, k_n\}.$$

Then

$$\int_{\mathbb{R}_+^{N+1}} (|\nabla w_n|^2 + w_n^2) \geq \sum_{i=1}^{k_n} \int_{B((x_n, R_i y_n), r)} (|\nabla w_n|^2 + w_n^2) \geq k_n \alpha \to +\infty$$

a contradiction.

So we can assume that $x_n \to +\infty$ and $|y_n|$ bounded. We will show that in such a case

$$\int_{B(z_n, r)} |v_n|^2 \to 0.$$

First of all let us remark that we can assume $z_n = (x_n, 0)$, eventually taking r larger. Since, clearly

$$\int_{B(z_n, r)} |v|^2 \to 0,$$

we will immediately deduce that

$$\int_{B(z_n, r)} |v_n - v|^2 \to 0.$$

Let $\bar{\alpha}$ be such that

$$\int_{B(z_n, r)} |v_n|^2 \geq \bar{\alpha} > 0.$$

For all n let $m_n \in \mathbb{N}$ be the smallest integer such that

$$\int_{C(z_n, r+m_n, r+m_n+1)} (|\nabla v_n|^2 + v_n^2) < \bar{\alpha},$$

where $C(z, r_1, r_2)$ denotes the annulus of radii $r_1 < r_2$ and center z. Since v_n is bounded in $H^1(\mathbb{R}_+^{N+1})$,

$$m_n \leq \frac{1}{\bar{\alpha}} \int_{\mathbb{R}_+^{N+1}} (|\nabla v_n|^2 + v_n^2) \leq \bar{m}.$$

We can assume that $x_n > r + \bar{m} + 1$ for all n (so that $C_n = C(z_n, r + m_n, r + m_n + 1) \subset \mathbb{R}_+^{N+1}$), and that $r > 2$.

Let $\phi_R : \mathbb{R} \to [0,1]$ be defined as follows

$$\phi_R(s) = \begin{cases} 1 & |s| \le R \\ 0 & |s| > R+1 \\ \text{linear} & \text{elsewhere} \end{cases}.$$

We finally let $\psi_n(z) = \phi_{r+m_n}(|z - z_n|)$. Since $z_n = (x_n, 0)$ then $\psi_n v_n \in H^1_{\#}$, and we have that

$$\begin{aligned} \langle I'(v_n), v_n\psi_n \rangle &= \int_{\mathbb{R}^{N+1}_+} (|\nabla v_n|^2 + v_n^2)\psi_n + \int_{\mathbb{R}^{N+1}_+} v_n \nabla v_n \nabla \psi_n \\ &\ge \bar{\alpha} - \int_{C_n} |\nabla v_n|\, |v_n| \ge \bar{\alpha} - \frac{1}{2}\int_{C_n} (|\nabla v_n|^2 + v_n^2) \ge \frac{\bar{\alpha}}{2}, \end{aligned}$$

a contradiction with the fact that $|\langle I'(v_n), v_n\psi_n\rangle| \le \|I'(v_n)\|\, \|v_n\psi_n\| \to 0$.

Hence $w_n \to 0$ in $L^{2(q-1)}(\mathbb{R}^{N+1}_+)$, and $\gamma(v_n) \to \gamma(v)$ strongly in $L^q(\mathbb{R}^N)$ for all $q \in \left(2, \frac{2N}{N-1}\right)$.

We can now prove strong convergence of $v_n \to v$. (Here we write v for $\gamma(v)$.) For $g(v) = v|v|^{p-2} + \sigma(W * v^2)$, using (2.7) as in Lemma 4.1 we have

$$\begin{aligned} \epsilon_n \ge \langle I'(v_n) - I'(v), v_n - v\rangle &= \iint_{\mathbb{R}^{N+1}_+} (|\nabla(v_n - v)|^2 + m^2|v_n - v|^2)\, dx\, dy \\ &- \mu \int_{\mathbb{R}^N} |v_n - v|^2\, dy - \int_{\mathbb{R}^N} (g(v_n)v_n - g(v)v)(v_n - v)\, dy \\ &\ge \iint_{\mathbb{R}^{N+1}_+} \left(\left(1 - \frac{\mu}{m}\right)|\nabla(v_n - v)|^2 + m(m-\mu)|v_n - v|^2\right) dx\, dy \\ &- \nu \int_{\mathbb{R}^N} (|v|^{p-1} + |v_n|^{p-1})|v_n - v|)\, dy \\ &- \sigma \int_{\mathbb{R}^N} ((W * v^2)|v| + (W * v_n^2)|v_n|)|v_n - v|\, dy. \end{aligned}$$

As a consequence

$$\begin{aligned} \iint_{\mathbb{R}^{N+1}_+} (|\nabla(v_n - v)|^2 + |v_n - v|^2)\, dx\, dy &\le \epsilon_n \\ &+ c_1 \int_{\mathbb{R}^N} (|v|^{p-1} + |v_n|^{p-1})|v_n - v|)\, dy \\ &+ c_2 \int_{\mathbb{R}^N} ((W * v^2)|v| + (W * v_n^2)|v_n|)|v_n - v|\, dy. \end{aligned}$$

By Hölder inequality we have

$$\int_{\mathbb{R}^N} |v|^{p-1}|v_n - v|\,dy \le |v|_p^{p-1}|v_n - v|_p,$$

$$\int_{\mathbb{R}^N} |v_n|^{p-1}|v_n - v|\,dy \le |v_n|_p^{p-1}|v_n - v|_p.$$

In the term

$$\int_{\mathbb{R}^N}(W * v^2)|v|\,|v_n - v| = \int_{\mathbb{R}^N}(W_1 * v^2)|v|\,|v_n - v| + \int_{\mathbb{R}^N}(W_2 * v^2)|v|\,|v_n - v|$$

involving convolutions we have to estimate the two integrals on the right hand side separately. Take $\epsilon_R = \sup\{W_2(x)\mid |x| \ge R\}$. From our assumptions, $\epsilon_R \to 0$ as $R \to +\infty$. Then $W_2\chi_{B(0,R)} \in L^r(\mathbb{R}^N)$ and $|W_2\chi_{\mathbb{R}^N\setminus B(0,R)}|_\infty < \epsilon_R$ (χ_E being the characteristic function of the set E). This shows that we can always assume that $|W_2|_\infty < \epsilon_R$ modifying W_1. We take R so large that $\epsilon_R c_2 d^3 < \frac{C_2}{4}$, d being a bound for the L^2 norm of v_n and C_2 is given by (2.3).

Then

$$\begin{aligned} c_2\int_{\mathbb{R}^N}(W_2 * v^2)|v|\,|v_n - v| &= c_2\int_{\mathbb{R}^N} v^2(W_2 * (|v|\,|v_n - v|)) \\ &\le c_2|W_2 * (|v|\,|v_n - v|)|_\infty |v|_2^2 \\ &\le \epsilon_R c_2|v|_2^3|v_n - v|_2 \le \epsilon_R c_2 d^3|v_n - v|_2 \le \frac{1}{4}\|v_n - v\|. \end{aligned}$$

Let us now estimate the term with W_1. Recalling (2.5) for $1 + m^{-1} = r^{-1} + 3q^{-1}$, we have

$$|(W_1 * v^2)v|_m \le C|W_1|_r|v|_q^3$$

then choosing $m = \frac{q}{q-1}$ (the Hölder conjugate of q) we get by Hölder inequality

$$\int_{\mathbb{R}^N}(W_1 * v^2)|v|\,|v_n - v|\,dy \le |(W_1 * v^2)v|_{q'}|v_n - v|_q \le C|W_1|_r|v|_q^3|v_n - v|_q$$

where $q = \frac{4r}{2r-1}$. Analogously

$$\int_{\mathbb{R}^N}(W_1 * v_n^2)|v_n|\,|v_n - v|\,dy \le C|W_1|_r|v_n|_q^3|v_n - v|_q.$$

Hence we get

$$\begin{aligned} &\iint_{\mathbb{R}^{N+1}_+}(|\nabla(v_n - v)|^2 + |v_n - v|^2)\,dx\,dy \\ &\qquad \le \epsilon_n + c_1|\gamma(v_n) - \gamma(v)|_p + c_2|\gamma(v_n) - \gamma(v)|_q + \frac{1}{2}|\gamma(v_n) - \gamma(v)|_2. \end{aligned}$$

Since $2 \leq q = 4r/(2r-1) < 2N/(N-1)$, by the strong convergence of $\gamma(v_n) \to \gamma(v)$ in $L^s(\mathbb{R}^N)$, for $s \in (2, 2N/(N-1))$ we may conclude that $v_n \to v$ strongly in $H^1(\mathbb{R}^{N+1}_+)$. □

Using the two above lemmas it follows immediately from the Mountain Pass Lemma (see [1]) that

Theorem 4.3. *There is a critical point $v_0 \in H^1_\#$ for the functional $I(v)$. Such a critical point is a weak solution of* (3.1).

Moreover $v_0 \geq 0$.

Proof. By the Mountain Pass Theorem it follows immediately that there is a critical point v_0 for I on $H^1_\#$. Since the problem under study is invariant by rotation around the x-axis, follows from Palais principle of symmetric criticality [13] that v_0 is also a critical point for I on $H^1(\mathbb{R}^{N+1}_+)$, and hence a weak solution of (3.1).

From the mountain pass Theorem we know that

$$I(v_0) = c_\# = \inf_{g \in \Gamma_\#} \max_{t \in [0,1]} I(g(t))$$

where $\Gamma_\# = \{g \in C([0,1]; H^1_\#) \mid g(0) = 0, I(g(1)) < 0\}$.

To show that $v_0 \geq 0$ we start by observing that, given any critical point w for I on $H^1(\mathbb{R}^{N+1}_+)$, the function $\lambda \mapsto I(\lambda w)$ has only one strict maximum at $\lambda = 1$.

We then observe that $I(|v|) \leq I(v)$ for all $v \in H^1(\mathbb{R}^{N+1}_+)$.

As a consequence we have that for all $\lambda > 0$, $\lambda \neq 1$

$$I(\lambda|v_0|) \leq I(\lambda v_0) < I(v_0).$$

The path $\lambda \mapsto \lambda|v_0|$ is in $\Gamma_\#$ and hence

$$c_\# \leq \sup_{\lambda > 0} I(\lambda|v_0|) \leq I(v_0) = c_\#.$$

If $|v_0|$ is not a critical point, one can deform, using the gradient flow, the path $\lambda \mapsto \lambda|v_0|$ into a path $g(\lambda) \in \Gamma_\#$ such that $I(g(\lambda)) < c_\#$ for all λ, a contradiction with the definition of $c_\#$ which proves that there is always a non-negative critical point at the mountain pass level. □

5. Properties of the Mountain Pass solution

Theorem 5.1. *Suppose that v is the critical point of I found via Theorem* 4.3.

Then $v(x,y) > 0$ in $[0,+\infty) \times \mathbb{R}^N$ and, for any $0 \leq \alpha \in (\mu, m)$, there exists $C > 0$ such that

$$0 < v(x,y) \leq Ce^{-(m-\alpha)\sqrt{x^2+|y|^2}} e^{-\alpha x} \tag{5.2}$$

for any $(x,y) \in [0,+\infty) \times \mathbb{R}^N$.

Hence in particular

$$0 < v(0, y) \le Ce^{-\delta|y|} \quad \textit{for any } y \in \mathbb{R}^N \tag{5.3}$$

where $0 < \delta < m - \mu$, *for* $\mu \ge 0$ *and* $\delta = m$ *for* $\mu < 0$.

PROOF. The strict positivity of v follows immediately from the maximum principle: since $v \ge 0$, if $v(\bar{x}, \bar{y}) = 0$, then $\bar{x} = 0$. From the equation we deduce that $\frac{\partial v}{\partial n}(0, \bar{y}) = 0$ and we reach a contradiction applying the Hopf lemma.

For $R > 0$ let us define the following sets:

$$\begin{aligned} B_R^+ &= \{(x, y) \in \mathbb{R}_+^{N+1} \mid \sqrt{x^2 + |y|^2} < R\} \\ \Omega_R^+ &= \{(x, y) \in \mathbb{R}_+^{N+1} \mid \sqrt{x^2 + |y|^2} > R\} \\ \Gamma_R &= \{(0, y) \in \partial\mathbb{R}_+^{N+1} \mid |y| \ge R\} \end{aligned}$$

and the auxiliary function

$$f_R(x, y) = C_R e^{-\alpha x} e^{-(m-\alpha)\sqrt{x^2+|y|^2}} \quad \text{for } (x, y) \in \overline{\Omega}_R^+$$

with $0 \le \alpha \in (\mu, m)$ and C_R a positive constant to be fixed later.

We have

$$\Delta f_R = \left(\alpha^2 + (m-\alpha)^2 + \frac{2\alpha(m-\alpha)x}{\sqrt{x^2 + |y|^2}} - \frac{N(m-\alpha)}{\sqrt{x^2 + |y|^2}}\right) f_R$$

$$\begin{cases} -\Delta f_R + m^2 f_R \ge 0 & \text{in } \Omega_R^+ \\ -\frac{\partial f_R}{\partial x} = \frac{\partial f_R}{\partial n} = \alpha f_R(0, y) & \text{on } \Gamma_R^+. \end{cases}$$

Let us define $w(x, y) = f_R(x, y) - v(x, y)$ for $(x, y) \in \overline{\Omega}_R^+$.

We have $-\Delta w + m^2 w \ge 0$ in Ω_R^+ and choosing $C_R = e^{mR} \max_{\partial B_R^+} v$ we get $w(x, y) \ge 0$ on ∂B_R^+ and $w(x, y) \to 0$ as $x + |y| \to +\infty$.

Now we claim that $w(x, y) \ge 0$ in $\overline{\Omega}_R^+$. Indeed, let us suppose by contrary that

$$\inf_{\overline{\Omega}_R^+} w < 0.$$

By the strong maximum principle there exists $(0, y_0) \in \Gamma_R$ such that

$$w(0, y_0) = \inf_{\overline{\Omega}_R^+} w < w(x, y) \quad \text{for any } (x, y) \in \Omega_R^+.$$

Let us define $z(x, y) = w(x, y)e^{\lambda x}$ for some $\lambda \in (\alpha, m)$. By Theorem 3.14 we have that $z(x, y) \to 0$ as $x + |y| \to +\infty$ and $z(x, y) \ge 0$ on ∂B_R^+. Moreover,

$$-\Delta w + m^2 w = e^{-\lambda x}(-\Delta z + 2\lambda \partial_x z + (m^2 - \lambda^2) z)$$

and we may conclude that $-\Delta z + 2\lambda \partial_x z + (m^2 - \lambda^2) z \ge 0$.

Then by the strong maximum principle $\inf_{\Gamma_R} z = \inf_{\overline{\Omega}_R^+} z < z(x, y)$ for all $(x, y) \in \Omega_R^+$ and hence $z(0, y_0) = \inf_{\Gamma_R} z = \inf_{\Gamma_R} w = w(0, y_0) < 0$. Finally by the Hopf lemma we may conclude that $\frac{\partial z}{\partial n}(0, y_0) < 0$ and hence

$$\frac{\partial w}{\partial n}(0, y_0) - \lambda w(0, y_0) < 0.$$

On the other hand,

$$\frac{\partial w}{\partial n}(0, y) = \alpha f_R - \mu v - g(v)v \quad \text{on } \Gamma_R$$

where $g(v) = v|\gamma(v)|^{p-2} + \sigma(W * \gamma(v)^2)$. Hence

$$\frac{\partial w}{\partial n}(0, y_0) - \lambda w(0, y_0) = (\alpha - \lambda)w(0, y_0) + (\alpha - \mu - g(v)(y_0))v(0, y_0).$$

From our choiche of λ follows that the term $(\alpha - \lambda)w(0, y_0) > 0$. Let us show that also $(\alpha - \mu - g(v)(y_0))v(0, y_0)$ is positive by showing that $g(v)(y_0)$ is small for R large enough.

Recalling that $v(0, y) \to 0$ as $|y| \to +\infty$ and $W(y) \to 0$ as $|y| \to +\infty$, we have that for any $\epsilon > 0$ there exists $R > 0$ such that

$$\sup_{|y| \geq R} g(v)(y) \leq \epsilon$$

(to show that $(W * \gamma(v)^2)(y_0) \to 0$ as $|y_0| \to +\infty$, take $\rho > 0$ such that $\sup\{W(y) \mid |y| > \rho\} < \epsilon/2$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} W(y_0 - y)v^2(0, y)\, dy &= \int_{B(y_0,\rho)} W(y_0 - y)v^2(0, y)\, dy \\ &\quad + \int_{\mathbb{R}^N \setminus B(y_0,\rho)} W(y_0 - y)v^2(0, y)\, dy \\ &\leq |W|_r \Big(\int_{B(y_0,\rho)} v^{2r'}(0, y)\, dy \Big)^{1/r'} + \frac{\epsilon}{2}|v|_2^2\end{aligned}$$

and the claim follows)

Therefore since $\lambda \in (\alpha, m)$ and $0 \leq \alpha \in (\mu, m)$, taking $0 < \varepsilon \leq \alpha - \mu$ we get

$$\frac{\partial w}{\partial n}(0, y_0) - \lambda w(0, y_0) \geq 0$$

a contradiction. Namely, we get

$$0 < v(x, y) \leq f_R(x, y) = C_R e^{-\alpha x} e^{-(m-\alpha)\sqrt{x^2+|y|^2}} \quad \text{for } (x, y) \in \overline{\Omega}_R^+.$$

Hence setting $0 < \delta = m - \alpha$ we finally get

$$0 < v(0, y) \leq C_R e^{-\delta|y|} \quad \text{for any } |y| \geq R.$$

Since v is a regular solution, the theorem follows. □

PROOF OF THEOREM 1.6. It is a direct consequence of Theorems 4.3, 3.14 and 5.1. □

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Coti Zelati
Dipartimento di Matematica Pura e Applicata "R. Caccioppoli"
Università di Napoli "Federico II"
via Cintia, M.S. Angelo
80126 Napoli (NA), Italy
zelati@unina.it

Margherita Nolasco
Dipartimento di Matematica Pura e Applicata
Università dell'Aquila
via Vetoio, Loc. Coppito
67010 L'Aquila AQ, Italia
nolasco@univaq.it