Rend. Lincei Mat. Appl. 22 (2011), 113–134
DOI 10.4171/RLM/590

Partial Differential Equations — *Hamiltonian formulation of the Klein-Gordon-Maxwell equations*, by Vieri Benci and Donato Fortunato, communicated on 14 January 2011.

Dedicated to the memory of Giovanni Prodi.

Abstract. — The nonlinear Klein-Gordon-Maxwell equations (NKGM) provide models for the interaction between the electromagnetic field and matter. The relevance of NKGM relies on the fact that they are the "simplest" gauge theory which is invariant under the group of Poincaré. These equations present the interesting phenomenon of solitons. In this paper, we show that NKGM present an Hamiltonian structure and hence they can be written as equations of the first order in t. This fact is not trivial since the Lagrangian does not depend on $\partial_t \varphi$ (see section 3.3) and a suitable analysis of its structure is necessary. In the last section, we recall a recent result which states the existence of solitons by using the particular structure of the Hamiltonian.

Key words: Maxwell equations, Klein-Gordon equation, Q-balls, Hylomorphic solitons, gauge theory.

AMS Subject Classification: 35Q61, 37K40, 35A15.

1. Introduction

The relevance of the Klein-Gordon-Maxwell equations (KGM) relies on the fact that they model the "simplest" gauge theory which is invariant under the group of Poincaré and which couples matter and field (see e.g. [24] section 2.7 and [26] section 1.4); the Klein-Gordon equation describes matter and the Maxwell equations describe the gauge invariant electromagnetic field.

Moreover, if you add a suitable nonlinear linear term, you get the so called Nonlinear-Klein-Gordon-Maxwell equations (NKGM). These equations present the interesting phenomenon of solitons. These solitons in the physics literature are called charged Q-balls (see e.g. [23], [12], [25]).

More recently also mathematicians became interested to the study of solitary waves and solitons in NKGM (see e.g. [5], [8], [4], [6], [7], [9], [14], [15], [16], [17]).

In this paper, we show that NKGM present an Hamiltonian structure and hence they can be written as equations of first order in t (see eq. (66)). This fact is not trivial since the Lagrangian does not depend on $\partial_t \varphi$ (see section 3.3) and a suitable analysis of its structure is necessary.

Finally, in the last section, we recall a recent result which states the existence of solitons by using the particular structure of the Hamiltonian.

2. The Maxwell equations

The D'Alembert equation and Maxwell equations can be considered as the "simplest" equations in $\mathbb{R}^4$ which are invariant both for the *Poincarè group* and for a *gauge group*. In the next sections we will show this fact.

2.1. The simplest gauge invariant equations

The first partial differential equation which has been written and studied is the D'Alembert equation

(D'Alembert) $\qquad \Box\psi = 0$ [1]

where

$$\Box\psi = \frac{\partial^2\psi}{\partial t^2} - \Delta\psi \quad \text{and} \quad \Delta\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}.$$

Hereafter $x = (x_1, x_2, x_3)$ and t will denote the space and time variables.

The D'Alembert equation is the simplest *variational* field equation which is *invariant under the Poincaré group*.

It is *variational* since it is the Euler-Lagrange equation relative to the functional

$$\mathscr{S}_0[\psi] = \frac{1}{2}\int [|\partial_t\psi|^2 - |\nabla\psi|^2]\, dx\, dt. \tag{1}$$

We recall that the *Poincaré group* can be defined as the subgroup of $GL(\mathbb{R}^4)$ which leaves invariant the Minkowski bilinear form, namely the form

$$\langle \xi, \eta \rangle_M = -\xi_0\eta_0 + \sum_{j=0}^{3} \xi_j\eta_j; \qquad \xi, \eta \in \mathbb{R}^4. \tag{2}$$

The Lagrangian $\mathscr{L}_0$ corresponding to the action (1) can be written as follows

$$\mathscr{L}_0 = \frac{1}{2}|\partial_t\psi|^2 - \frac{1}{2}|\nabla\psi|^2 = -\frac{1}{2}\langle d\psi, d\psi\rangle_M. \tag{3}$$

Then (1) takes the form

$$\mathscr{S}_0[\psi] = \int \mathscr{L}_0\, dx\, dt = -\frac{1}{2}\int \langle d\psi, d\psi\rangle_M\, dx\, dt. \tag{4}$$

Clearly $\mathscr{L}_0$ is invariant under the action of the Poincaré group. Observe that $\mathscr{S}_0[\xi]$ in (1) is invariant not only for the action of the Poincaré group, but also for

---

[1] Actually D'Alembert studied this equation only in one space diemensuin [13].

the action of the "trivial gauge group" $\xi \to \xi + c$, where c is a constant, namely $\mathscr{S}_0[\xi] = \mathscr{S}_0[\xi + c]$. Then, if ξ is a solution of (D'Alembert), also $\xi + c$ solves (D'Alembert).

The Maxwell equations in the empty space are the simplest generalization of equation (D'Alembert) in the sense explained below. In order to get this generalization we need to use the language of the differential forms and to regard the function ψ as a zero form. Then $d\psi$ is the exterior derivative of ψ and the equation (D'Alembert) becomes:

$$\delta\, d\psi = 0. \tag{5}$$

We recall that

$$\delta : \Lambda^k(\mathbb{R}^4) \to \Lambda^{k-1}(\mathbb{R}^4)$$ [2]

is the functional adjoint operator of

$$d : \Lambda^{k-1}(\mathbb{R}^4) \to \Lambda^k(\mathbb{R}^4),$$

namely it is the operator defined by the following equation:

$$\int \langle \xi, d\eta \rangle_M \, dx\, dt = -\int \langle \delta\xi, \eta \rangle_M \, dx\, dt$$

where we have assumed $\xi \in \Lambda^k(\mathbb{R}^4)$, $\eta \in \Lambda^{k-1}(\mathbb{R}^4)$, $k = 0, \ldots, 3$, and η with compact support. We recall that the Minkowski product between k-forms is defined as follows; if

$$\xi = \sum_{i_1,\ldots,i_k=0}^{3} \xi_{i_1,\ldots,i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \textit{and} \quad \eta = \sum_{i_1,\ldots,i_k=0}^{3} \eta_{i_1,\ldots,i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

then

$$\langle \xi, \eta \rangle_M = \sum_{\substack{i_1,\ldots,i_k=0 \\ j_1,\ldots,j_k=0}}^{3} g^{i_1 j_1} \ldots g^{i_k j_k} \xi_{i_1,\ldots,i_k} \eta_{i_1,\ldots,i_k},$$

where

$$[g^{ij}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

[2] As usual, Λ^k ($\not\leq$) denotes the space of the k-form defined in $\not\leq$.

One of the most natural generalization of $\mathscr{S}_0[\xi]$ defined by (4) is given by

$$\mathscr{S}_1[A] = \int \mathscr{L}_1 \, dx \, dt = -\frac{1}{2} \int \langle dA, dA \rangle_M \, dx \, dt \tag{6}$$

where A is a 1-form:

$$A = \sum_{j=0}^{3} A_j \, dx^j. \tag{7}$$

The variation of the action (6) gives the following Euler-Lagrange equation:

$$\delta \, dA = 0. \tag{8}$$

This simple generalization gives a much richer structure; in fact the action (6) is invariant for the gauge transformation $A \to A + d\chi$ where $\chi \in \mathscr{C}^2(\mathbb{R}^4)$; namely the gauge group $\mathscr{C}^2(\mathbb{R}^4)$ is an infinite dimensional group. However, in most of the physical interpretations of this theory, it is assumed that A and $A + d\chi$ give the same experimental results, namely χ has no physical meaning. For this reason, we can introduce the quantity

$$F = dA \tag{9}$$

which does not depend on χ (since $dd\chi = 0$) and which is considered the physically measurable quantity.

By equation (8), and the fact $ddA = 0$, we have that F satisfies the following equations:

$$dF = 0 \tag{10}$$

$$\delta F = 0. \tag{11}$$

2.2. *The Maxwell equations as gauge theory*

In this section we will show that equations (10), (11) are nothing else but the Maxwell equations in the empty space.

In order to get the Maxwell equations in a more familiar form, let us write equations (10), (11) using the vector notation. We denote by

$$\mathfrak{j} : \mathbb{R}^{3+1} \to \Lambda^1(\mathbb{R}^{3+1})$$

the duality map which associate to a 4-vector $(v_0, \mathbf{v}) \in \mathbb{R}^{3+1}$ the 1-form $\mathfrak{j}(v_0, \mathbf{v}) \in (\mathbb{R}^{3+1})^* := \Lambda^1(\mathbb{R}^{3+1})$ defined by

$$\mathfrak{j}(v_0, \mathbf{v})[(w_0, \mathbf{w})] = -v_0 w_0 + \mathbf{v} \cdot \mathbf{w}.$$

Then, if A is as in (7), we set $(\varphi, \mathbf{A}) = \mathfrak{j}^{-1}(A)$, namely

$$\varphi := A^0 = -A_0, \quad \mathbf{A} := (A^1, A^2, A^3) = (A_1, A_2, A_3). \tag{12}$$

Then the Lagrangian $\mathscr{L}_1$ in the functional (6) becomes

$$\begin{aligned}\langle dA, dA\rangle_M &= \frac{1}{2}\left[\sum_{i,j=1}^{3}(\partial_i A_j - \partial_j A_i)^2 - \sum_{j=1}^{3}(\partial_0 A_j - \partial_j A_0)^2 - \sum_{i=1}^{3}(\partial_i A_0 - \partial_0 A_i)^2\right] \\ &= \frac{1}{2}\left[\sum_{i,j=1}^{3}(\partial_i A^j - \partial_j A^i)^2 - \sum_{j=1}^{3}(\partial_t A^j + \partial_j \varphi)^2 - \sum_{i=1}^{3}(\partial_i \varphi + \partial_t A^i)^2\right] \\ &= |\nabla \times \mathbf{A}|^2 - |\partial_t \mathbf{A} + \nabla\varphi|^2.\end{aligned}$$

Here and in the following $\nabla\times, \nabla$ and $\nabla\cdot$ will denote respectively the curl, the gradient and the divergence operators with respect to the x variable. So (6) takes the following aspect:

$$\mathscr{S}_1[(\varphi, \mathbf{A})] = \frac{1}{2}\int (|\partial_t \mathbf{A} + \nabla\varphi|^2 - |\nabla \times \mathbf{A}|^2)\, dx\, dt. \tag{13}$$

Making the variation of $\mathscr{S}_1$ with respect to φ and $\mathbf{A}$ we get

$$\nabla \cdot (\partial_t \mathbf{A} + \nabla\varphi) = 0 \tag{14}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}(\partial_t \mathbf{A} + \nabla\varphi) = 0. \tag{15}$$

Now we make the following change of variables:

$$\mathbf{E} = -(\partial_t \mathbf{A} + \nabla\varphi) \tag{16}$$

$$\mathbf{H} = \nabla \times \mathbf{A}. \tag{17}$$

From (16) and (17) we have that

$$\nabla \times \mathbf{E} + \partial_t \mathbf{H} = 0 \tag{18}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{19}$$

and (14), (15) become

$$\nabla \cdot \mathbf{E} = 0 \tag{20}$$

$$\nabla \times \mathbf{H} - \partial_t \mathbf{E} = 0. \tag{21}$$

Equations (18), (19), (20), (21) are respectively the Faraday's law, the no monopole law and the Gauss' and the Ampére's laws in the empty space. Thus we have obtained the Maxwell equations in the usual 3-vector notation.

3. The Nonlinear Klein-Gordon-Maxwell equations (NKGM)

3.1. The Klein-Gordon-Maxwell equations as Abelian gauge theory

The nonlinear Klein-Gordon equation for a complex valued field ψ, defined on the space-time $\mathbb{R}^4$, can be written as follows:

$$\Box\psi + W'(\psi) = 0 \tag{22}$$

where

$$\Box\psi = \frac{\partial^2\psi}{\partial t^2} - \Delta\psi, \quad \Delta\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

and, with some abuse of notation,

$$W'(\psi) = F'(|\psi|)\frac{\psi}{|\psi|}$$

for some smooth function $F : [0, \infty) \to \mathbb{R}$. The field $\psi : \mathbb{R}^4 \to \mathbb{C}$ will be called *matter field*. If $W'(s)$ is linear, $W'(s) = m_0^2 s$, $m_0 \neq 0$, equation (22) reduces to the Klein-Gordon equation.

Consider the Abelian gauge theory in $\mathbb{R}^4$ equipped with the Minkowski metric and described by the Lagrangian density (see e.g. [8], [26])

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 - W(\psi) \tag{23}$$

where

$$\mathscr{L}_0 = \frac{1}{2}(|D_\varphi\psi|^2 - |D_{\mathbf{A}}\psi|^2)$$

$$\mathscr{L}_1 = \frac{1}{2}(|\partial_t\mathbf{A} + \nabla\varphi|^2 - |\nabla\times\mathbf{A}|^2),$$

and

$$\mathbf{A} = (A_1, A_2, A_3) \in \mathbb{R}^3 \quad \text{and} \quad \varphi \in \mathbb{R}$$

are the gauge potentials (see (12)). Moreover

$$D_\varphi\psi = (\partial_t + iq\varphi)\psi$$

is the covariant derivative with respect to the t variable, and

$$D_{\mathbf{A}}\psi = (\nabla - iq\mathbf{A})\psi$$

is the covariant derivative with respect to the x variable (see e.g. [8] or [26]). Here q denotes a positive parameter.

Now consider the total action

$$\mathscr{S} = \int (\mathscr{L}_0 + \mathscr{L}_1 - W(\psi))\, dx\, dt. \tag{24}$$

Making the variation of $\mathscr{S}$ with respect to ψ, φ and $\mathbf{A}$ we get the system of the Nonlinear Klein-Gordon-Maxwell equations (NKGM)

$$D_\varphi^2\psi - D_{\mathbf{A}}^2\psi + W'(\psi) = 0 \tag{25}$$

$$\nabla \cdot (\partial_t \mathbf{A} + \nabla\varphi) = q\,\mathrm{Re}(iD_\varphi\psi\bar{\psi}) \tag{26}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \partial_t(\partial_t \mathbf{A} + \nabla\varphi) = q\,\mathrm{Re}(iD_{\mathbf{A}}\psi\bar{\psi}). \tag{27}$$

3.2. *General features of the Klein-Gordon-Maxwell equations*

Now we make the following change of variables:

$$\mathbf{E} = -(\partial_t \mathbf{A} + \nabla\varphi) \tag{28}$$

$$\mathbf{H} = \nabla \times \mathbf{A} \tag{29}$$

$$\rho = -q\,\mathrm{Re}(iD_\varphi\psi\bar{\psi}) \tag{30}$$

$$\mathbf{j} = q\,\mathrm{Re}(iD_{\mathbf{A}}\psi\bar{\psi}). \tag{31}$$

By this change of variables, we see that (26) and (27) are the second couple of the Maxwell equations (respectively the Gauss and Ampére's laws) with respect to a matter distribution whose electric charge and current densities are respectively ρ and $\mathbf{j}$:

$$\nabla \cdot \mathbf{E} = \rho \tag{GAUSS}$$

$$\nabla \times \mathbf{H} - \partial_t \mathbf{E} = \mathbf{j}. \tag{AMPERE}$$

As for (18), (19), equations (28) and (29) give rise to the first couple of the Maxwell equations (respectively the Faraday and no monopole laws):

$$\nabla \times \mathbf{E} + \partial_t \mathbf{H} = 0 \tag{FARADAY}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{NOMONOPOLE}$$

Sometimes it is useful to give a different form to these equations; if we write ψ in polar form

$$\psi(x,t) = u(x,t)e^{iS(x,t)}, \quad u \geq 0,\ S \in \mathbb{R}/2\pi\mathbb{Z} \tag{32}$$

equation (25) can be split in the two following ones

$$\Box u + W'(u) + [|\nabla S - q\mathbf{A}|^2 - (\partial_t S + q\phi)^2]\, u = 0 \tag{33}$$

$$\frac{\partial}{\partial t}[(\partial_t S + q\phi)u^2] - \nabla \cdot [(\nabla S - q\mathbf{A})u^2] = 0. \tag{34}$$

Observe that, using the polar form (32), the charge and the current densities (30) and (31) become

$$\rho = -q(\partial_t S + q\phi)u^2, \quad \mathbf{j} = q(\nabla S - q\mathbf{A})u^2. \tag{35}$$

Then equations (33) and (34), using the variables $\mathbf{j}$ and ρ, can be written as follows:

$$\Box u + W'(u) + \frac{\mathbf{j}^2 - \rho^2}{q^2 u^3} = 0 \tag{MATTER}$$

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0. \tag{CONTINUITY}$$

Equation (CONTINUITY) is the charge continuity equation.

Notice that equation (CONTINUITY) is also a consequence of (GAUSS) and (AMPERE) and hence it can be eliminated. Thus equations (25), (26), (27) are equivalent to equations (GAUSS), (AMPERE), (FARADAY), (NOMONOPOLE), (MATTER).

In conclusion, an Abelian gauge theory, via equations (GAUSS), (AMPERE), (FARADAY), (NOMONOPOLE), (MATTER), provides a model of interaction of the matter field ψ with the electromagnetic field $(\mathbf{E}, \mathbf{H})$.

Observe that the Lagrangian (23) is invariant with respect to the gauge transformations

$$\psi \to e^{iq\chi}\psi \tag{36}$$

$$\varphi \to \varphi - \partial_t \chi \tag{37}$$

$$\mathbf{A} \to \mathbf{A} + \nabla\chi \tag{38}$$

where $\chi \in C^2(\mathbb{R}^4)$.

So our equations are gauge invariant; if we use the variables u, ρ, $\mathbf{j}$, $\mathbf{E}$, $\mathbf{H}$, this fact can be checked directly since these variables are gauge invariant. In fact, equations (GAUSS), (AMPERE), (FARADAY), (NOMONOPOLE), (MATTER) are the gauge invariant formulation of equations (25), (26), (27).

3.3. *The modified Lagrangian*

The Lagrangian (23) has the following form:

$$\begin{aligned}\mathscr{L}(\psi, \partial_t\psi, \mathbf{A}, \partial_t\mathbf{A}, \varphi) = {} & \frac{1}{2}(|D_\varphi \psi|^2 - |D_{\mathbf{A}}\psi|^2) - W(\psi) \\ & + \frac{1}{2}(|\partial_t \mathbf{A} + \nabla\varphi|^2 - |\nabla \times \mathbf{A}|^2).\end{aligned}$$

Since $\mathscr{L}$ does not depend on $\partial_t\varphi$ it is not possible to make the Legendre transformation with respect to $\partial_t\varphi$ and hence to get an Hamiltonian formulation of the dynamics.

To overcome this difficulty, we set

$$M_L = \{(\mathbf{Q}, \dot{\mathbf{Q}}) : \partial_t \varphi + \nabla \cdot \mathbf{A} = 0, \nabla \cdot (\partial_t \mathbf{A} + \nabla \varphi) = q \operatorname{Re}(iD_\varphi \psi \bar{\psi})\} \tag{39}$$

where

$$\mathbf{Q} = (\psi, \mathbf{A}, \varphi), \quad \dot{\mathbf{Q}} = (\partial_t \psi, \partial_t \mathbf{A}, \partial_t \varphi)$$

and we consider the *modified Lagrangian*

$$\hat{\mathscr{L}}(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2}[|D_\varphi \psi|^2 - |D_\mathbf{A} \psi|^2] - W(\psi) + \frac{1}{2}[|\partial_t \mathbf{A}|^2 - |\nabla \mathbf{A}|^2 - (\partial_t \varphi)^2 + |\nabla \varphi|^2]. \tag{40}$$

The dynamics induced by $\hat{\mathscr{L}}$ is given by the following equations:

$$D_\varphi^2 \psi - D_\mathbf{A}^2 \psi + W'(\psi) = 0 \tag{41}$$

$$\Box \mathbf{A} - q \operatorname{Re}(iD_\mathbf{A} \psi \bar{\psi}) = 0 \tag{42}$$

$$\Box \varphi + q \operatorname{Re}(iD_\varphi \psi \bar{\psi}) = 0 \tag{43}$$

THEOREM 1. *The set M_L is invariant for the dynamics induced by equations* (41), (42), (43)*. Moreover, if the initial data are in M_L and $(\psi, \mathbf{A}, \varphi)$ is a smooth solution of eqs.* (41), (42), (43) *then it is also a solution of* (25), (26), (27).

PROOF. As in (30) and (31) we set

$$\mathbf{j} = +q \operatorname{Re}(iD_\mathbf{A} \psi \bar{\psi})$$
$$\rho = -q \operatorname{Re}(iD_\varphi \psi \bar{\psi}).$$

Let us first show that M_L is invariant for the dynamics induced by (41), (42), (43).

Let

$$\mathbf{Q}(x,t) = (\psi(x,t), \mathbf{A}(x,t), \varphi(x,t))$$

be the solution of (41), (42), (43) with initial data

$$(\mathbf{Q}(x,0), \dot{\mathbf{Q}}(x,t))_{t=0} \in M_L. \tag{44}$$

So

$$\partial_t \varphi(x,t)_{t=0} + \nabla \cdot \mathbf{A}(x,0) = 0, \quad \nabla \cdot (\partial_t \mathbf{A}(x,t)_{t=0} + \nabla \varphi(x,0)) = -\rho_0 \tag{45}$$

where

$$\rho_0 = \rho(0) = -q \operatorname{Re}(iD_\varphi \psi \bar{\psi})_{t=0}.$$

We want to show that for all $t \geq 0$

$$(\mathbf{Q}(x,t), \dot{\mathbf{Q}}(x,t)) \in M_L$$

namely that for all $t \geq 0$

$$\partial_t \varphi(x,t) + \nabla \cdot \mathbf{A}(x,t) = 0, \tag{46}$$

$$\nabla \cdot (\partial_t \mathbf{A}(x,t) + \nabla \varphi(x,t)) = q \operatorname{Re}(iD_\varphi \psi \bar{\psi}). \tag{47}$$

We set for all x, t

$$F = F(x,t) = \partial_t \varphi(x,t) + \nabla \cdot \mathbf{A}(x,t).$$

Then

$$\Box F = \partial_t(\Box \varphi) + \nabla \cdot (\Box \mathbf{A}) = (\text{by } 42, 43) = \partial_t \rho + \nabla \cdot \mathbf{j} \tag{48}$$

Since (41) is equivalent to (33) and (34), we have by (34) that

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0. \tag{49}$$

Then by (48) and (49) we get

$$\Box F = 0. \tag{50}$$

By (45)

$$F(x,0) = \partial_t \varphi(x,t)_{t=0} + \nabla \cdot \mathbf{A}(x,0) = 0. \tag{51}$$

Moreover

$$\partial_t F = \partial_t^2 \varphi + \nabla \cdot (\partial_t \mathbf{A}) = (\text{by } 43) = \nabla \cdot (\nabla \varphi) + \rho + \nabla \cdot (\partial_t \mathbf{A}(x,t)). \tag{52}$$

Then

$$\partial_t F(x,t)_{t=0} = \nabla \cdot (\nabla \varphi(x,0) + \partial_t \mathbf{A}(x,t)_{t=0}) + \rho_0 = (\text{by } (45)) = 0. \tag{53}$$

By (50), (51), (53) we get and

$$\begin{aligned} \Box F &= 0 \\ F &= 0 \quad \text{for } t = 0 \\ \partial_t F &= 0 \quad \text{for } t = 0. \end{aligned}$$

So,

$$F = 0.$$

Then (46) is proved. Now in order to prove (47), we observe that by (46) we get

$$\partial_t F = 0 \quad \text{for all } t.$$

From which, using (52), we have

$$\nabla \cdot (\nabla\varphi) + \rho + \nabla \cdot (\partial_t \mathbf{A}(x,t)) = 0 \quad \text{for all } t.$$

Then (47) clearly holds.

Let us now prove the second part of the theorem. Let $(\psi, \mathbf{A}, \varphi)$ be a smooth solution of eqs. (41), (42), (43) with initial data in M_L. We show that it is a solution of (25), (26), (27).

Since (41) coincides with (25), we are reduced to show that $(\psi, \mathbf{A}, \varphi)$ satisfies (26), (27).

Clearly we have

$$\begin{aligned}
\mathbf{j} &= (\text{by } 42) = \partial_t^2 \mathbf{A} - \Delta \mathbf{A} \\
&= \partial_t^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) + \nabla \times (\nabla \times \mathbf{A}) \qquad (\text{by } (46)) \\
&= \partial_t(\partial_t \mathbf{A}) + \nabla(\partial_t \varphi) + \nabla \times (\nabla \times \mathbf{A}) \\
&= \partial_t(\partial_t \mathbf{A} + \nabla\varphi) + \nabla \times (\nabla \times \mathbf{A}).
\end{aligned}$$

So (27) is satisfied. On the other hand

$$\begin{aligned}
\rho &= (\text{by } (43)) = \partial_t(\partial_t \varphi) - \nabla \cdot \nabla\varphi \\
&= (\text{by } (46)) = -\partial_t \nabla \cdot \mathbf{A} - \nabla \cdot \nabla\varphi \\
&= -\nabla \cdot (\partial_t \mathbf{A} + \nabla\varphi).
\end{aligned}$$

Then also (26) is satisfied. □

Remark 2. *By the proof of the above theorem, we can easily deduce that any smooth solution* $(\psi, \mathbf{A}, \varphi)$ *of eqs.* (25), (26), (27) *belonging to* M_L, *is also a solution of* (41), (42), (43).

3.4. Hamiltonian formulation

The Lagrangian (40) depends on the configuration variable $\mathbf{Q} = (\psi, \mathbf{A}, \varphi)$ and its time derivative $\dot{\mathbf{Q}} = (\partial_t\psi, \partial_t\mathbf{A}, \partial_t\varphi)$; moreover it is convex in $\dot{\mathbf{Q}}$. Then it is possible to define the conjugate variable of $\mathbf{Q}$ via the Legendre transform. These conjugate variables will be denoted by $\mathbf{P} = (\hat{\psi}, \hat{\mathbf{A}}, \hat{\varphi})$. We have:

$$\hat{\psi} = \frac{\partial\hat{\mathscr{L}}}{\partial(\partial_t\psi)} = \partial_t\psi + iq\varphi\psi = D_\varphi\psi \tag{54}$$

$$\hat{\mathbf{A}} = \frac{\partial\hat{\mathscr{L}}}{\partial(\partial_t\mathbf{A})} = \partial_t\mathbf{A} \tag{55}$$

$$\hat{\varphi} = \frac{\partial\hat{\mathscr{L}}}{\partial(\partial_t\varphi)} = -\partial_t\varphi. \tag{56}$$

We denote by $\mathbf{u}$ the state of our dynamical system described by the canonical variables

$$\mathbf{u} = (\mathbf{Q}, \mathbf{P}) = (\psi, \mathbf{A}, \varphi, \hat{\psi}, \hat{\mathbf{A}}, \hat{\varphi}).$$

The invariant set M_L (see (39)), expressed in the canonical variables $(\mathbf{Q}, \mathbf{P})$ is given by

$$M_H = \{(\mathbf{Q}, \mathbf{P}) : -\hat{\varphi} + \nabla \cdot \mathbf{A} = 0, \nabla \cdot (\hat{\mathbf{A}} + \nabla\varphi) = q \operatorname{Re}(i\hat{\psi}\overline{\psi})\}. \tag{57}$$

We notice that the equation

$$-\hat{\varphi} + \nabla \cdot \mathbf{A} = 0 \tag{58}$$

defines the Lorentz gauge in the canonical variables, while

$$\nabla \cdot (\hat{\mathbf{A}} + \nabla\varphi) = q \operatorname{Re}(i\hat{\psi}\overline{\psi}) \tag{59}$$

is nothing else but the equation (GAUSS) in these variables.

The duality between $\mathbf{P}$ and $\dot{\mathbf{Q}}$ is given by

$$\langle \mathbf{P}, \dot{\mathbf{Q}} \rangle = \operatorname{Re}(\hat{\psi}\overline{\partial_t \psi}) + \hat{\mathbf{A}} \cdot \partial_t \mathbf{A} + \hat{\varphi}\partial_t \varphi, \tag{60}$$

then, the Hamiltonian density takes the form

$$H(\mathbf{Q}, \mathbf{P}) = [\langle \mathbf{P}, \dot{\mathbf{Q}} \rangle - \hat{\mathscr{L}}(\mathbf{Q}, \dot{\mathbf{Q}})]_{\dot{\mathbf{Q}} = \dot{\mathbf{Q}}(\mathbf{u})}. \tag{61}$$

By the definition of covariant derivative and (54), we have

$$\partial_t \psi = D_\varphi \psi - iq\varphi\psi = \hat{\psi} - iq\varphi\psi \tag{62}$$

and hence, inserting (62), (55) and (56) in (60), we have

$$\langle \mathbf{P}, \dot{\mathbf{Q}} \rangle_{\dot{\mathbf{Q}} = \dot{\mathbf{Q}}(\mathbf{u})} = |\hat{\psi}|^2 + q\varphi \operatorname{Re}(i\hat{\psi}\overline{\psi}) + |\hat{\mathbf{A}}|^2 - \hat{\varphi}^2. \tag{63}$$

Moreover the Lagrangian $\hat{\mathscr{L}}$ (see (40), expressed in the canonical variables $\mathbf{u} = (\mathbf{Q}, \mathbf{P})$, takes the form

$$\begin{aligned} \hat{\mathscr{L}}(\mathbf{Q}, \mathbf{P}) &= \hat{\mathscr{L}}((\mathbf{Q}, \dot{\mathbf{Q}})(\mathbf{u})) \\ &= \frac{1}{2}(|\hat{\psi}|^2 - |D_{\mathbf{A}}\psi|^2) + \frac{1}{2}(|\hat{\mathbf{A}}|^2 - |\nabla \mathbf{A}|^2 - \hat{\varphi}^2 + |\nabla\varphi|^2) - W(\psi). \end{aligned} \tag{64}$$

Then by (63) and (64) the Hamiltonian density (61) becomes

$$\begin{aligned} H(\mathbf{Q}, \mathbf{P}) = \frac{1}{2}(|\hat{\psi}|^2 + |\hat{\mathbf{A}}|^2 - \hat{\varphi}^2 + |D_{\mathbf{A}}\psi|^2 + |\nabla \mathbf{A}|^2 - |\nabla\varphi|^2) \\ + q\varphi \operatorname{Re}(i\hat{\psi}\overline{\psi}) + W(\psi). \end{aligned} \tag{65}$$

The action (in a bounded domain Ω) is given by:

$$\begin{aligned}
\mathscr{S}(\mathbf{Q},\mathbf{P}) &= \int_\Omega (\langle \mathbf{P}, \dot{\mathbf{Q}}\rangle - H(\mathbf{Q},\mathbf{P}))\,dx\,dt \\
&= \int_\Omega (\operatorname{Re}(\hat{\psi}\overline{\partial_t\psi}) + \hat{\mathbf{A}}\cdot\partial_t\mathbf{A} + \hat{\varphi}\partial_t\varphi)\,dx\,dt \\
&\quad - \int_\Omega \left[\frac{1}{2}(|\hat{\psi}|^2 + |\hat{\mathbf{A}}|^2 - \hat{\varphi}^2 + |D_\mathbf{A}\psi|^2 + |\nabla\mathbf{A}|^2 - |\nabla\varphi|^2)\right] dx\,dt \\
&\quad - \int_\Omega [q\varphi\operatorname{Re}(i\hat{\psi}\overline{\psi}) + W(\psi)]\,dx\,dt.
\end{aligned}$$

Making the variations of $\mathscr{S}(\mathbf{Q},\mathbf{P})$ with respect $\hat{\psi}$, ψ, $\hat{\mathbf{A}}$, $\mathbf{A}$, $\hat{\varphi}$, φ respectively, we get the canonical equations of motion:

$$\begin{aligned}
\partial_t\psi &= \hat{\psi} - iq\varphi\psi \\
\partial_t\hat{\psi} &= -iq\varphi\hat{\psi} + D_\mathbf{A}^2\psi - W'(\psi) \\
\partial_t\mathbf{A} &= \hat{\mathbf{A}} \\
\partial_t\hat{\mathbf{A}} &= \nabla^2\mathbf{A} + q\operatorname{Re}(iD_\mathbf{A}\psi\overline{\psi}) \\
\partial_t\varphi &= -\hat{\varphi} \\
\partial_t\hat{\varphi} &= -\nabla^2\varphi - q\operatorname{Re}(i\hat{\psi}\overline{\psi}).
\end{aligned} \tag{66}$$

REMARK 3. *If we use the covariant derivative with respect to time D_φ the Hamilton equations take a simplerg form:*

$$D_\varphi\psi = \hat{\psi} \tag{67}$$

$$D_\varphi\hat{\psi} = D_\mathbf{A}^2\psi - W'(\psi) \tag{68}$$

$$\partial_t\mathbf{A} = \hat{\mathbf{A}} \tag{69}$$

$$\partial_t\hat{\mathbf{A}} = \nabla^2\mathbf{A} + q\operatorname{Re}(iD_\mathbf{A}\psi\overline{\psi}) \tag{70}$$

$$\begin{aligned}
\partial_t\varphi &= -\hat{\varphi} \\
\partial_t\hat{\varphi} &= -\nabla^2\varphi - q\operatorname{Re}(iD_\varphi\psi\overline{\psi})
\end{aligned} \tag{71}$$

3.5. NKGM as a dynamical system

We assume that M_H (or equivalently M_L) consist of C^∞ functions; we would like to define a *natural* metric and to take the completion of M_H with respect to this metric.

The choice of the "right" metric is a delicate problem which depends on mathematical and physical considerations. In many problems, we have that

$$\{\text{energy}\} = \{\text{positive quadratic form}\} + \{\text{higher order terms}\}. \tag{72}$$

In these problems, usually, the "norm of the energy" is a good choice:

$$\|\cdot\| := \sqrt{\{\text{positive quadratic form}\}}. \tag{73}$$

In NKGM, the energy is just the Hamiltonian; namely (if the energy is finite), by (65)

$$\begin{aligned} \mathscr{H}(\mathbf{Q},\mathbf{P}) &= \int H(\mathbf{Q},\mathbf{P})\,dx \\ &= \frac{1}{2}\int (|\hat{\psi}|^2 + |\hat{\mathbf{A}}|^2 - \hat{\varphi}^2 + |D_{\mathbf{A}}\psi|^2 + |\nabla\mathbf{A}|^2 - |\nabla\varphi|^2)\,dx \\ &\quad + \int [q\varphi\,\mathrm{Re}(i\hat{\psi}\bar{\psi}) + W(\psi)]\,dx. \end{aligned} \tag{74}$$

We observe that $\mathscr{H}(\mathbf{Q},\mathbf{P})$ is not positive. However, if $(\mathbf{Q},\mathbf{P}) \in M_H$ and if $W \geq 0$, the energy is positive as the following proposition shows:

Proposition 4. *If $(\mathbf{Q},\mathbf{P}) \in M_H$, and the energy is finite, then*

$$\mathscr{H}(\mathbf{Q},\mathbf{P}) = \frac{1}{2}\int (|\hat{\psi}|^2 + |D_{\mathbf{A}}\psi|^2 + |\nabla\varphi + \hat{\mathbf{A}}|^2 + |\nabla\times\mathbf{A}|^2) + \int W(\psi). \tag{75}$$

Proof. We recall that, if $(\mathbf{Q},\mathbf{P}) \in M_H$, then by eq. (58), we have that

$$|\hat{\varphi}|^2 = |\nabla\cdot\mathbf{A}|^2$$

and, recalling that

$$|\nabla\mathbf{A}|^2 = |\nabla\cdot\mathbf{A}|^2 + |\nabla\times\mathbf{A}|^2,$$

we get

$$\begin{aligned} \mathscr{H}(\mathbf{Q},\mathbf{P}) &= \frac{1}{2}\int (|\hat{\psi}|^2 + |\hat{\mathbf{A}}|^2 + |D_{\mathbf{A}}\psi|^2 + |\nabla\times\mathbf{A}|^2 - |\nabla\varphi|^2) \\ &\quad + \int q\varphi\,\mathrm{Re}(i\hat{\psi}\bar{\psi}) + W(\psi). \end{aligned} \tag{76}$$

Moreover, by (59), multiplying by φ and integrating by parts, we get

$$\int |\nabla\varphi|^2 + \nabla\varphi\cdot\hat{\mathbf{A}} = q\int \varphi\,\mathrm{Re}(i\hat{\psi}\bar{\psi}).$$

Thus, replacing $q\int \varphi\,\mathrm{Re}(i\hat{\psi}\bar{\psi})$ in eq. (76), we have

$$\begin{aligned}\mathscr{H}(\mathbf{Q},\mathbf{P}) &= \frac{1}{2}\int (|\hat{\psi}|^2 + |\hat{\mathbf{A}}|^2 + |D_{\mathbf{A}}\psi|^2 + |\nabla\times\mathbf{A}|^2 + |\nabla\varphi|^2 + 2\nabla\varphi\cdot\hat{\mathbf{A}}) + \int W(\psi)\\ &= \frac{1}{2}\int (|\hat{\psi}|^2 + |D_{\mathbf{A}}\psi|^2 + |\nabla\varphi+\hat{\mathbf{A}}|^2 + |\nabla\times\mathbf{A}|^2) + \int W(\psi). \qquad \square\end{aligned}$$

Now we want to express the energy in the gauge invariant variables (GIV) u, v, ρ, $\mathbf{j}$, $\mathbf{E}$, $\mathbf{H}$ defined by

$$\text{(GIV)} \qquad \begin{aligned} u &= |\psi|\\ v &= \frac{\mathrm{Re}(\hat{\psi}\bar{\psi})}{|\psi|}\\ \rho &= -q\,\mathrm{Re}(i\hat{\psi}\bar{\psi})\\ \mathbf{j} &= q\,\mathrm{Re}(iD_{\mathbf{A}}\psi\bar{\psi})\\ \mathbf{E} &= -(\nabla\varphi+\hat{\mathbf{A}})\\ \mathbf{H} &= \nabla\times\mathbf{A}.\end{aligned}$$

We notice that $\mathbf{E}$ and $\mathbf{H}$ coincide with the electromagnetic field defined by eqs. (28) and (29), u is consistent with (32) while ρ and $\mathbf{j}$ are the electric and the current density.

We now set

$$X_0 = \{\mathbf{u}\in C^\infty(\mathbb{R}^3,\mathbb{R}^{12}) \mid \nabla\cdot\mathbf{E} = \rho, \nabla\cdot\mathbf{H} = 0, E(\mathbf{u}) < +\infty\}$$

where $\mathbf{u} = (u, v, \rho, \mathbf{j}, \mathbf{E}, \mathbf{H})$ denotes the generic state and $E(\mathbf{u})$ is the energy as function of $\mathbf{u}$.

We observe that X_0 is invariant for the dynamics given by the equations (GAUSS, AMPERE, FARADAY, NOMONOPOLE, MATTER).

PROPOSITION 5. *If $\mathbf{u}\in X_0$, then the energy takes the following expression:*

$$E(\mathbf{u}) = \frac{1}{2}\int\left(v^2 + |\nabla u|^2 + \frac{\rho^2+\mathbf{j}^2}{q^2u^2} + \mathbf{E}^2 + \mathbf{H}^2\right)dx + \int W(u)\,dx.$$

PROOF. By using the polar form $\psi = ue^{iS}$ and by the first eq. of (66), we have

$$\begin{aligned} v &= \frac{\mathrm{Re}(\hat{\psi}\bar{\psi})}{|\psi|} = \frac{\mathrm{Re}[(\partial_t\psi + iq\varphi\psi)\bar{\psi}]}{u}\\ &= \frac{\mathrm{Re}[\partial_t\psi\bar{\psi}]}{u} = \frac{\mathrm{Re}[(\partial_t ue^{iS} + i\partial_t Sue^{iS})ue^{-iS}]}{u}\\ &= \partial_t u.\end{aligned}$$

Then

$$v = \partial_t u. \tag{77}$$

Using again the first eq. of (66) and the polar form of ψ, we have

$$\hat{\psi} = \partial_t \psi + iq\varphi\psi = [\partial_t u + i(\partial_t S + q\varphi)u]e^{iS}. \tag{78}$$

Moreover, by (35),

$$\rho = -q(\partial_t S + q\phi)u^2. \tag{79}$$

Thus, by (78), (79) and (77), we have

$$|\hat{\psi}|^2 = |\partial_t u|^2 + |\partial_t S + q\varphi|^2 u^2 = |v|^2 + \frac{\rho^2}{q^2 u^2}. \tag{80}$$

Similarly, writing ψ in polar form, we have

$$D_{\mathbf{A}}\psi = \nabla\psi - iq\mathbf{A}\psi = [\nabla u - i(\nabla S - q\mathbf{A})u]e^{iS}$$

and using (35)

$$\mathbf{j} = q(\nabla S - q\mathbf{A})u^2,$$

we get

$$|D_{\mathbf{A}}\psi|^2 = |\nabla u|^2 + |\nabla S - q\mathbf{A}|^2 u^2 = |\nabla u|^2 + \frac{\mathbf{j}^2}{q^2 u^2}. \tag{81}$$

Thus, by (75), (80), (81) and the last two equations of GIV

$$\begin{aligned} E(\mathbf{u}) &= \frac{1}{2}\int (|\hat{\psi}|^2 + |D_{\mathbf{A}}\psi|^2 + |\nabla\varphi + \hat{\mathbf{A}}|^2 + |\nabla \times \mathbf{A}|^2) + \int W(\psi) \\ &= \frac{1}{2}\int \left[|v|^2 + \frac{\rho^2}{q^2u^2} + |\nabla u|^2 + \frac{\mathbf{j}^2}{q^2u^2} + \mathbf{E}^2 + \mathbf{H}^2\right] dx + \int W(u)\, dx. \qquad \square \end{aligned}$$

We write W as follows

$$W(s) = \frac{m^2}{2}s^2 + N(s), \quad N(s) = o(s^2) \tag{82}$$

and we will assume $m > 0$. Then we have

$$E(\mathbf{u}) = \frac{1}{2}\int \left[v^2 + |\nabla u|^2 + m^2u^2 + \frac{\rho^2 + \mathbf{j}^2}{q^2u^2} + \mathbf{E}^2 + \mathbf{H}^2\right] dx + \int N(u)\, dx.$$

The term $(\rho^2 + \mathbf{j}^2)/u^2$ is singular and the energy does not have the form (72). In order to avoid this problem, it is convenient to introduce new gauge invariant variables which eliminate this singularity:

$$\theta = \frac{-\rho}{qu}; \quad \Theta = \frac{\mathbf{j}}{qu}. \tag{83}$$

Using these new variables the energy takes the form:

$$E(\mathbf{u}) = \frac{1}{2}\int [v^2 + |\nabla u|^2 + m^2u^2 + \theta^2 + \Theta^2 + \mathbf{E}^2 + \mathbf{H}^2] + \int N(u). \tag{84}$$

Thus, we can take a norm having the form (73), namely

$$\|\mathbf{u}\| = \left(\int [v^2 + |\nabla u|^2 + m^2u^2 + \theta^2 + \Theta^2 + \mathbf{E}^2 + \mathbf{H}^2]\, dx\right)^{1/2}. \tag{85}$$

If $\mathbf{Q} = (\psi, \mathbf{A}, \varphi)$ is a solution of the Cauchy problem relative to equations (41), (42), (43) with initial data in M_L and with finite energy, after the change of variables (83) and (GIV), $\mathbf{u} = (u, v, \theta, \Theta, \mathbf{E}, \mathbf{H})$ solves the Cauchy problem relative to equations

$$\begin{aligned} \Box u + W'(u) &= \frac{\theta^2 - \Theta^2}{u} \\ \nabla \cdot \mathbf{E} &= -q\theta u \\ \nabla \times \mathbf{H} - \partial_t \mathbf{E} &= q\Theta u \\ \nabla \times \mathbf{E} + \partial_t \mathbf{H} &= 0 \\ \nabla \cdot \mathbf{H} &= 0 \end{aligned} \tag{86}$$

with initial data in X_0.

We will denote by V the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^{12})$ with respect to the norm (85) so that

$$\mathbf{u} = (u, v, \theta, \Theta, \mathbf{E}, \mathbf{H}) \in V \cong H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3, \mathbb{R}^{11}).$$

Finally $X \subset V$ will denote the closure of X_0 with respect to the norm (85).

In the following we shall assume that the Cauchy problem for NKGM is globally well posed in X so that we can consider NKGM as a dynamical system. Actually in the literature there are not many results relative to this problem (we know only [18], [19], [21]). In any case X seems to be an appropriate space to study the Cauchy problem.

In the following we will denote by (X, γ) the dynamical system relative to NKGM.

4. Solitary waves and solitons in NKGM

An interesting fact concerning NKGM is the existence of solitary waves and solitons, provided that W satisfies suitable assumptions. Roughly speaking a *solitary wave* is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A *soliton* is a solitary wave which exhibits some form of stability so that it has a particle-like behavior (see e.g. [1], [8], [22], [25]). In theoretical physics, the solitons occurring in NKGM are called charged *Q-balls*.

In this section we will describe some recent results concerning the existence and the nature of such solitons.

4.1. Definition of solitary waves and solitons

Now we will give a rigorous definition of solitary waves and solitons in NKGM. Solitary waves and solitons are particular *states* of the dynamical system (X, γ) described by the nonlinear Klein-Gordon-Maxwell equations (see (86) and the end of section 3.5).

Definition 6. A state $\mathbf{u}(x) \in X$ is called solitary wave if

$$|\gamma_t \mathbf{u}(x)| = f(x - vt).$$

In particular, if $v = 0$, then $\mathbf{u}(x)$ is called standing wave.

The solitons are solitary waves characterized by some form of stability. To define them, we need to recall some well known notions in the theory of dynamical systems.

A set $\Gamma \subset X$ is called *invariant* if $\forall \mathbf{u} \in \Gamma$, $\forall t \in \mathbb{R}$, $\gamma_t \mathbf{u} \in \Gamma$.

Definition 7. Let (X, d) be a metric space and let (X, γ) be a dynamical system. An invariant set $\Gamma \subset X$ is called stable, if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall \mathbf{u} \in X$,

$$d(\mathbf{u}, \Gamma) \leq \delta,$$

implies that

$$\forall t \in \mathbb{R}, \quad d(\gamma_t \mathbf{u}, \Gamma) \leq \varepsilon.$$

Let G be the group induced by the translations in $\mathbb{R}^N$, namely, for every $\tau \in \mathbb{R}^N$, the transformation $g_\tau \in G$ is defined as follows:

$$(g_\tau \mathbf{u})(x) = \mathbf{u}(x - \tau). \tag{87}$$

A subset $\Gamma \subset X$ is called G-invariant if

$$\forall \mathbf{u} \in \Gamma, \quad \forall \tau \in \mathbb{R}^N, \quad g_\tau \mathbf{u} \in \Gamma.$$

DEFINITION 8. A closed set $\Gamma \subset X$ is called G-compact if it is G-invariant and for any sequence $\mathbf{u}_n(x)$ in Γ there is a sequence $\tau_n \in \mathbb{R}^N$, such that $\mathbf{u}_n(x - \tau_n)$ has a converging subsequence.

Now we are ready to give the definition of soliton:

DEFINITION 9. A standing wave $\mathbf{u}(x)$ is called (standing) soliton if there is a closed set Γ such that

- (i) $\forall t$, $\gamma_t \mathbf{u}(x) \in \Gamma$,
- (ii) Γ is stable,
- (iii) Γ is G-compact.

Some times, in the literature, the kind of stability described by the above definition is called *orbital stability*.

REMARK 10. *The above definition needs some explanations. For simplicity, we assume that* Γ *is a manifold* (*actually, it is possible to prove that this is the generic case if the problem is formulated in a suitable function space*). *Then* (iii) *implies that* Γ *is finite dimensional. Since* Γ *is invariant,* $\mathbf{u}_0 \in \Gamma \Rightarrow \gamma_t \mathbf{u}_0 \in \Gamma$ *for every time. Thus, since* Γ *is finite dimensional, the evolution of* $\mathbf{u}_0$ *is described by a finite number of parameters. Thus the dynamical system* (Γ, γ) *behaves as a point in a finite dimensional phase space. By the stability of* Γ, *a small perturbation of* $\mathbf{u}_0$ *remains close to* Γ. *However, in this case, its evolution depends on an infinite number of parameters. Thus, this system appears as a finite dimensional system with a small perturbation, namely as a "particle" perturbed by a field.*

REMARK 11. *We recall that* (*NKGM*) *are defined by a Lagrangian which is invariant under the action of the Lorentz group. If* $\mathbf{u}_0$ *is a standing wave, it is possible to obtain a travelling wave just making a Lorentz boost* (*see e.g.* [8] *or* [4]). *More precisely, let* $T_{\mathbf{v}}$ *be the representation of a Lorentz boost relative to our system and let*

$$\mathbf{u}(t, x) = \gamma_t \mathbf{u}_0(x)$$

be the evolution of our standing wave $\mathbf{u}_0(x)$; *then*

$$\mathbf{u}'(t', x') := T_{\mathbf{v}} \mathbf{u}(t, x)$$

is a solution of our equation which moves in time with velocity $\mathbf{v}$. *In* [4] *you can see the details and how this principle works in some particular cases. Obviously, if* $\mathbf{u}_0$ *is a standing soliton,* $\mathbf{u}_{\mathbf{v}}$ *is orbitally stable and hence it is a travelling soliton.*

4.2. The charge and hylomorphic solitons

In recent papers (see e.g. [2], [3], [4], [7], [9], [10]), the notion of *hylomorphic soliton* has been introduced and analyzed. The existence and the properties of hylomorphic solitons are guaranteed by the interplay between *energy* E and another integral of motion which is called *hylenic charge* and it will be denoted by C. More precisely, a soliton $\mathbf{u}_0 \in X$ is hylomorphic if

$$E(\mathbf{u}_0) = \min\{E(\mathbf{u}) \mid C(\mathbf{u}) = C(\mathbf{u}_0)\}.$$

In NKGM, the hylenic charge coincides with the electric charge and we will refer to it just using the word *charge*. The charge, by definition, is the quantity which is preserved by the gauge action (36), (37), (38). Using (34), we see that it has the following expression

$$C = \int \rho \, dx. \tag{88}$$

Using the variables u and θ, by (83), the charge becomes:

$$C(\mathbf{u}) = -q \int \theta u \, dx. \tag{89}$$

We make the following assumptions on W:

- (W-i) (Positivity) $W(s) \geq 0$
- (W-ii) (Nondegeneracy) $W = W(s)$ $(s \geq 0)$ is C^2 near the origin with $W(0) = W'(0) = 0$; $W''(0) = m^2 > 0$
- (W-iii) (Hylomorphy) $\exists \bar{s} > 0$ and $\alpha \in (0, m)$ such that $W(\bar{s}) \leq \frac{1}{2}\alpha^2 \bar{s}^2$
- (W-iiii) (Growth condition) There are constants $a, b > 0$, $6 > p > 2$ s.t. $|N'(s)| \leq as^{p-1} + bs^{2-2/p}$ where N is defined by eq. (82).

Here there are some comments on assumptions (W-i), (W-ii), (W-iii), (W-iiii).

(W-i) Clearly (see (84)) (W-i) implies that the energy is positive; if this condition does not hold, it is possible to have solitary waves, but not hylomorphic waves (cf. the discussion in section 4.2 of [7]).

(W-ii) In order to have solitary waves it is necessary to have $W''(0) \geq 0$. There are some results also when $W''(0) = 0$ (null-mass case, see e.g. [11]), however the most interesting situation occurs when $W''(0) > 0$.

(W-iii) This is the crucial assumption which characterizes the potentials which might produce hylomorphic solitons. By this assumption there exists s_0 such that $N(s_0) < 0$.

(W-iiii) This assumption contains the usual growth condition at infinity which guarantees the C^1 regularity of the functional. Moreover it implies that $|N'(s)| = O(s^{2-2/p})$ for s small.

We have the following results (for the proof see ([10])):

Theorem 12. *Assume that* (W-i), (W-ii), (W-iii), (W-iiii) *hold, then there exists* $\bar{q}$ *such that for every* $q \in [0, \bar{q}]$, *equations* (86) *have a continuous family* $\mathbf{u}_\delta$ $(\delta \in (0, \bar{\delta}(q)))$ *of independent, hylomorphic solitons* (*two solitons* $\mathbf{u}_{\delta_1}$, $\mathbf{u}_{\delta_2}$ *are called independent if* $\mathbf{u}_{\delta_1} \neq g\mathbf{u}_{\delta_2}$ *for every* $g \in G$).

THEOREM 13. *The solitons* $\mathbf{u}_\delta = (u_\delta, \hat{u}_\delta, \theta_\delta, \Theta_\delta, \mathbf{E}_\delta, \mathbf{H}_\delta)$ *in Theorem* 12 *are stationary solutions of* (86), *this means that* $\hat{u}_\delta = \Theta_\delta = \mathbf{H}_\delta = 0$, $\mathbf{E}_\delta = -\nabla\varphi_\delta$ *and* u_δ, θ_δ, φ_δ *solve the equations*

$$-\Delta u_\delta + W'(u_\delta) - \frac{\theta_\delta^2}{u_\delta} = 0$$
$$-\Delta\varphi_\delta + q\theta_\delta u_\delta = 0.$$

References

[1] M. Badiale - V. Benci - S. Rolando, *Solitary waves: physical aspects and mathematical results*, Rend. Sem. Math. Univ. Pol. Torino 62 (2004), 107–154.

[2] J. Bellazzini - V. Benci - C. Bonanno - A. M. Micheletti, *Solitons for the nonlinear Klein-Gordon-Equation*, Advanced Nonlinear Studies 10 (2010), 481–500, (arXiv:0712.1103).

[3] J. Bellazzini - V. Benci - C. Bonanno - E. Sinibaldi, *Hylomorphic solitons in the nonlinear Klein-Gordon equation*, Dynamics of Partial Differential Equations 6 (2009), 311–336.

[4] V. Benci, *Hylomorphic solitons*, Milan J. Math. 77 (2009), 271–332.

[5] V. Benci - D. Fortunato, *Solitary waves of the nonlinear Klein-Gordon field equation coupled with the Maxwell equations*, Rev. Math. Phys. 14 (2002), 409–420.

[6] V. Benci - D. Fortunato, *Solitary waves in Abelian Gauge Theories*, Adv. Nonlinear Stud. 3 (2008), 327–352.

[7] V. Benci - D. Fortunato, *Existence of hylomorphic solitary waves in Klein-Gordon and in Klein-Gordon-Maxwell equations*, Rend. Lincei Mat. Appl. 20 (2009).

[8] V. Benci - D. Fortunato, *Solitary waves in the Nonlinear Wave equation and in Gauge Theories*, Journal of fixed point theory and Applications 1 n.1 (2007), 61–86.

[9] V. Benci - D. Fortunato, *Spinning Q-balls for the Klein-Gordon-Maxwell equations*, Commun. Math. Phys. 295 (2010), 639–668.

[10] V. Benci - D. Fortunato, *On the existence of stable charged Q-balls*, arXiv:1011 .5044.

[11] H. Berestycki - P. L. Lions, *Nonlinear scalar field equations, I—Existence of a ground state*, Arch. Rational Mech. Anal. 82 (1983), 313–345.

[12] S. Coleman, *Q-Balls*, Nucl. Phys. B262 (1985) 263–283; erratum: B269 (1986) 744–745.

[13] J. D'Alembert, *Recherches sur les cordes vibrantes*, Histoire de l'Académie des Sciences et Belles-Lettres de Berlin (1747).

[14] T. D'Aprile - D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. of Royal Soc. of Edinburgh, section A Mathematics 134 (2004), 893–906.

[15] T. D'Aprile - D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Advanced Nonlinear studies 4 (2004), 307–322.

[16] P. D'Avenia - L. Pisani, *Nonlinear Klein-Gordon-Maxwell equations coupled with Born-Infeld type equations*, Electron. J. Differential equations 26 (2002), 1–13.

[17] O. Druet - E. Hebey, *Existence and a priori bounds for electrostatic Klein-Gordon-Maxwell systems in fully inhomogeneous spaces*, Communications in Contemporary Mathematics, to appear.

[18] D. Eardley - V. Moncrief, *The global existence of Yang-Mills-Higgs fields in* $\mathbf{R}^{3+1}$, Comm. Math. Phys. 83 (1982), 171–212.
[19] S. Klainerman - M. Machedon, *On the Maxwell-Klein-Gordon equation with finite energy*, Duke Math. Journal 74 (1994), 19–44.
[20] E. Long, *Existence and stability of solitary waves in nonlinear Klein-Gordon-Maxwell equations*, Rev. Math. Phys. 18 (2006), 747–779.
[21] D. M. Petrescu, *Time decay of solutions of coupled Maxwell-Klein-Gordon equations*, Comm. Math. Phys. 179 (1996), 11–24.
[22] R. Rajaraman, *Solitons and instantons*, North-Holland, Amsterdam 1989.
[23] G. Rosen, *Particlelike solutions to nonlinear complex scalar field theories with positive-definite energy densities*, J. Math. Phys. 9 (1968), 996–998.
[24] V. Rubakov, *Classical theory of Gauge fields*, Princeton University Press, Princeton (2002).
[25] A. Vilenkin - E. P. S. Shellard, *Cosmic strings and other topological defects*, Cambridge monographs on mathematical physics, 1994.
[26] Y. Yang, *Solitons in field theory and nonlinear analysis*, Springer, New York, Berlin, 2000.

Received 14 December 2010,
and in revised form 20 December 2011.

Vieri Benci
Dipartimento di Matematica Applicata "U. Dini"
Università di Pisa
via F. Buonarroti 1/c 56127 Pisa, Italy
56126, Pisa, Italy
benci@dma.unipi.it

Donato Fortunato
Dipartimento di Matematica
Università di Bari and INFN sezione di Bari
Via Orabona 4, 70125 Bari, Italy
fortunat@dm.uniba.it