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Partial Differential Equations — *Propagation of analyticity for a class of nonlinear hyperbolic equations*, by Sergio Spagnolo, communicated on 14 January 2010.

A Giovanni Prodi,
matematico insigne, indimenticabile maestro ed amico.

Abstract. — We consider the semilinear hyperbolic equations of the form

$$\partial_t^m u + a_1(t)\partial_t^{m-1}\partial_x u + \cdots + a_m(t)\partial_x^m u = f(u)$$

with $f(u)$ entire analytic, where the characteristic roots satisfy

$$\lambda_i^2(t) + \lambda_j^2(t) \le M(\lambda_i(t) - \lambda_j(t))^2, \quad i \ne j.$$

We prove that, if the $a_h(t)$'s are analytic functions, all the solutions bounded in $\mathscr{C}^\infty$ enjoy the propagation of analyticity; while, if the $a_h(t)$'s are $\mathscr{C}^\infty$ functions, such a property holds for those solutions which are bounded in some Gevrey class.

Key words: Propagation of analyticity, weakly hyperbolic equations, semilinear systems.

AMS Subject Classification: 35L76, 35L80.

1. Introduction

The linear operator, on $[0, T] \times \mathbb{R}^n$

$$\mathscr{L}U = U_t + \sum_{h=1}^{n} A_h(t, x) U_{x_h} \tag{1}$$

where $U(t, x) \in \mathbb{R}^N$ and the A_h's are $N \times N$ matrices, is *hyperbolic* when, for all $\xi \in \mathbb{R}^n$, the matrix $\sum A_h(t, x)\xi_h$ has real eigenvalues $\lambda_j(t, x, \xi)$, $1 \le j \le N$.

Denoting by $\mu(\lambda_j)$ the multiplicity of λ_j, the integer (among 1 and N)

$$m = \max_j \max_{t, x, \xi} \mu(\lambda_j(t, x, \xi))$$

is called *multiplicity* of (1). When $m = 1$, the system is *strictly hyperbolic*.

We study the regularity of the solutions to a nonlinear weakly hyperbolic system, in particular, a semilinear system of the fom

$$\mathscr{L}U = f(t, x, U), \tag{2}$$

where $U : [0, T] \times \mathbb{R}^n \to \mathbb{R}^N$, and $f(t, x, U)$ is a $\mathbb{R}^N$-valued, entire analytic function (typically a polynomial) of the scalar components of U.

More precisely, assuming the coefficients of $\mathscr{L}$ analytic in x, we investigate under which additional assumptions a given solution $U(t, x)$ of (2), analytic at the initial time, keeps its analyticity, i.e.,

$$U(0, \cdot) \in \mathscr{A}(\mathbb{R}^n) \quad \Rightarrow \quad U(t, \cdot) \in \mathscr{A}(\mathbb{R}^n) \quad \forall t \in [0, T]. \tag{3}$$

Actually, we consider two versions of (3), the first weaker and the second one stronger than (3):

$$U(0, \cdot) \in \mathscr{A}_{L^2} \quad \Rightarrow \quad U(t, \cdot) \in \mathscr{A}_{L^2} \qquad \forall t \in [0, T], \tag{4}$$

$$U(0, \cdot) \in \mathscr{A}(\Gamma_0) \quad \Rightarrow \quad U(t, \cdot) \in \mathscr{A}(\Gamma_t) \quad \forall t \in [0, T], \tag{5}$$

where $\mathscr{A}_{L^2} \equiv \mathscr{A}_{L^2}(\mathbb{R}^n)$ is the class of analytic functions $\varphi \in H^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{H^j} \le C\Lambda^j j!$, while Γ is a *cone of determinacy* for the operator $\mathscr{L}$ with base Γ_0 (at $t = 0$) and sections $\{\Gamma_t\}$.

The propagation of analyticity is a natural property for nonlinear hyperbolic equations. Indeed, on one side, Cauchy-Kovalewsky ensures the validity of (3) on some time interval $[0, \tau[$, on the other side, after Bony and Schapira ([BS], 1972) we know that the Cauchy problem for any linear (weakly) hyperbolic system is globally well posed in the class of analytic functions.

The first results of analytic propagation go back to Lax ([L], 1953), who considered (2), with $n = 1$, in the strictly hyperbolic case, and proved (5) for those solutions which are a priori bounded in $\mathscr{C}^1$. Later on, Alinhac and Métivier ([AM], 1984) extended this results to several space dimensions, assuming now that $U(t, \cdot)$ is bounded in $H^s(\mathbb{R}^n)$ for $s \ge \bar{s}(n)$.

The first investigations in the weakly hyperbolic (nonlinear) case were devoted the second order equations of the form

$$L_0 u \equiv \sum_{i,j}^{1,n} \partial_{x_i}(a_{ij}(t, x)\partial_{x_j} u) = f(u), \quad \sum a_{ij}\xi_i\xi_j \ge 0, \tag{6}$$

where $f(u)$ and the $a_{ij}(t, x)$'s are analytic.

Theorem A ([S], 1988).

(i) *In the special case when* $a_{ij} = \beta_0(t)\alpha_{ij}(x)$, *a solution of* (6) *enjoys* (5) *as long as* $u(t, \cdot)$ *remains bounded in* $\mathscr{C}^\infty$.
(ii) *In the general case, a solution enjoys* (5) *provided* $u(t, \cdot)$ *is bounded in some Gevrey class* γ^s *of order* $s < 2$.

We recall that the Cauchy problem for any strictly hyperbolic linear system is globally wellposed in $\mathscr{C}^\infty$. On the other hand, the Cauchy problem for the linear equation $L_0 u = 0$ is globally wellposed in $\mathscr{C}^\infty$ in the special case (i), while it is only globally wellposed in γ^s for $s < 2$ in the general case (ii). Thus, it is natural to formulate the following

CONJECTURE. *In order to get the analytic propagation for a given solution of a weakly hyperbolic system* $\mathscr{L}U = f(t,x,U)$, *it will be sufficient to assume* a priori *that* $U(t,\cdot)$ *is bounded in some functional class* $\mathscr{X}$ *in which the Cauchy problem for the linear systems* $\mathscr{L}U + B(t,x)U = f(t,x)$ *is globally well posed.*

Typically the space $\mathscr{X}$ is equal to $\mathscr{C}^\infty$ or to some Gevrey class γ^s.

If $\mathscr{L}$ is a weakly hyperbolic operator of the *general type* (1), the Conjecture says that a solution $U(t,\cdot)$ enjoys the analytic propagation as long as it remains bounded in some Gevrey class γ^s of order $s < m/(m-1)$, m being the multiplicity of $\mathscr{L}$. Indeed, Bronshtein's Theorem ([B], 1979) states that the Cauchy problem for any linear system $\mathscr{L}U + B(t,x)U = f(t,x)$, with coefficients analytic in x, is well-posed in each of these Gevrey classes.

Actually, this fact was proved in two special cases: time depending coefficients, and one space variable. More precisely:

THEOREM B ([DS] 1999; [J2] 2009). *A solution of*

$$U_t + \sum_{j=1}^{n} A_j(t)U_{x_j} = f(t,x,U), \quad x \in \mathbb{R}^n,$$

satisfies (4) *as long as* $U(t,\cdot)$ *remains bounded in some* γ^s *with* $s < m/(m-1)$.

THEOREM C ([ST], 2010). *A solution of*

$$U_t + A(t,x)U_x = f(t,x,U), \quad x \in \mathbb{R},$$

satisfies (5) *as long as* $U(t,\cdot)$ *remains bounded in some* γ^s *with* $s < m/(m-1)$.

The study of the general case, when $n \geq 2$ and the coefficients are depending on (t,x), is in progress.

OPEN PROBLEM. *To prove (or disprove) the sharpness of the bound* $s < m/(m-1)$ *in Theorems B and C. In particular, to construct a hyperbolic nonlinear system admitting a solution* $U \in \mathscr{C}^\infty(\mathbb{R}^{n+1})$ *which is analytic on the halfplane* $\{t < 0\}$ *but non-analytic at some point of the line* $\{t = 0\}$.

This questions are related to the "nonlinear Holmgren Theorem" (see [M]).

2. Main results

Here, we consider the scalar equations of the form

$$Lu \equiv \partial_t^m u + a_1(t)\partial_t^{m-1}\partial_x u + \cdots + a_m(t)\partial_x^m u = f(u), \tag{7}$$

on $[0,T] \times \mathbb{R}$, where $f(u) = \sum_{v=0}^{\infty} u^v$ is an entire analytic, real function on $\mathbb{R}$. We assume that the characteristic roots of the equation are real functions, say

$$\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_m(t),$$

which satisfy the condition

$$\lambda_i^2(t) + \lambda_j^2(t) \le M(\lambda_i(t) - \lambda_j(t))^2, \quad \forall t \in [0, T],\ i \ne j. \tag{8}$$

Remark 1 ([KS], 2006). Due to its symmetry w.r. to the roots this condition can be rewritten in term of the coefficients $\{a_h\}$ (Newton theorem).

In particular, for a second order equation (8) becomes, for some $c > 0$,

$$\Delta(t) \equiv a_1^2(t) - 4a_2(t) \ge c a_1^2(t)$$

while for a third order equation, it becomes

$$\Delta(t) \ge c(a_1(t)a_2(t) - 9a_3(t))^2,$$

the discriminant being now $\Delta = -4a_2^3 - 27a_3^2 + a_1^2a_2^2 - 4a_1^3a_3 + 18a_1a_2a_3$.

Particularly simple are the third order *traceless* equations. i.e., when $a_1 \equiv 0$: here we have $a_2 = -(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)/2$, $\Delta = -4a_2^3 - 27a_3^2$, so that (8) becomes $\Delta \ge -ca_2^3$, or equivalently $\Delta \ge ca_3^2$.

Condition (8) for the linear equation $Lu = 0$ was introduced in [CO] as a sufficient, and almost necessary, condition for the wellposedness in $\mathscr{C}^\infty$.

A different proof of such a result (based on the theory of quasi-symmetrizer) was given in [KS] where also the case of non-analytic coefficients was considered and it was proved that, if $a_h(t) \in \mathscr{C}^\infty([0, T])$ and (8) is fulfilled, then the Cauchy problem for $Lu = 0$ is well posed in each Gevrey class γ^s, $s \ge 1$.

By these existence results, it is natural to expect some kind of analytic propagation for the solutions bounded in $\mathscr{C}^\infty$ in case of analytic coefficients, and for those which are bounded in some Gevrey class in case of $\mathscr{C}^\infty$ coefficients. Before stating our results we introduce the following analytic and Gevrey classes:

$$\mathscr{A}_{L^2} = \{\varphi(x) \in H^\infty(\mathbb{R}) : \|\varphi\|_{H^j(\mathbb{R})} \le C\Lambda^j j!\},$$
$$\gamma_{L^2}^s = \{\varphi(x) \in H^\infty(\mathbb{R}) : \|\varphi\|_{H^j(\mathbb{R})} \le C\Lambda^j j!^s\}.$$

Theorem 1. *Assume that the $a_j(t)$'s are analytic functions on $[0, T]$. Then, for any solution of* (7) *satisfying*

$$\sup_{0 \le t \le T} \int_{\mathbb{R}} |\partial_t^h \partial_x^j u(t, x)|\, dx < \infty \quad \forall j \in \mathbb{N},\ 0 \le h \le m - 1, \tag{9}$$

$$\partial_t^h u(0, \cdot) \in \mathscr{A}_{L^2} \quad 0 \le h \le m - 1, \tag{10}$$

it holds

$$u \in \mathscr{C}^{m-1}([0, T], \mathscr{A}_{L^2}). \tag{11}$$

Under the same assumptions, we have also

$$u \in \mathscr{A}([0,T] \times \mathbb{R}). \tag{12}$$

THEOREM 2. *If the $a_j(t)$'s are $\mathscr{C}^\infty$ functions on $[0,T]$, the implication* (10) $\Rightarrow$ (11) *holds true for those solutions which belong to $\mathscr{C}^m([0,T], \gamma^s_{L^2})$ for some $s \geq 1$.*

PROOF OF THEOREM 1. For the sake of simplicity, we shall give the proof only in the case when the nonlinear term $f(u)$ is a monomial function, the general case requiring only minor additional computations. Thus, for a given integer $\nu \geq 1$, we consider the equation

$$\partial_t^m u + a_1(t)\partial_t^{m-1}\partial_x u + \cdots + a_m(t)\partial_x^m u = u^\nu. \tag{13}$$

By performing the partial Fourier transform

$$\hat{u}(t,\xi) = \int_{-\infty}^{+\infty} e^{-i\xi x} u(t,x)\,dx,$$

we transform (13) into the ODE's system

$$V' + i\xi A(t)V = F(t,\xi), \tag{14}$$

where

$$V(t,\xi) = \begin{pmatrix} (i\xi)^{m-1}\hat{u} \\ (i\xi)^{m-2}\hat{u}' \\ \vdots \\ \hat{u}^{(m-1)} \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & 0 & -1 \\ a_m(t) & \cdots & a_2(t) & a_1(t) \end{pmatrix}, \tag{15}$$

and

$$F(t,\xi) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t,\xi) \end{pmatrix} \quad \text{with } f(t,\xi) = \underbrace{\hat{u} * \cdots * \hat{u}}_{\nu}. \tag{16}$$

Our target is to prove that, if (for some constants K_j, K, and Λ) one has

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\xi|^j |V(t,\xi)|\,d\xi \leq K_j \qquad \forall j, \tag{17}$$

$$\int_{\mathbb{R}} |\xi|^j |V(0,\xi)|\,d\xi \leq K\Lambda^j j! \quad \forall j, \tag{18}$$

then, for some new constants $\tilde{K}$ and $\tilde{\Lambda}$, it holds

$$\sup_{0\le t\le T}\int_{\mathbb{R}} |\xi|^j |V(t,\xi)|\, d\xi \le \tilde{K}\tilde{\Lambda}^j j!, \quad \forall j. \tag{19}$$

Indeed, (9) gives

$$\sup_{0\le t\le T} |\xi|^j |V(t,\xi)| < \infty,$$

and hence (17). On the other hand (10) implies

$$\left\{\int_R |\xi|^{2j} |V(0,\xi)|^2\, d\xi\right\}^{1/2} \le K_0\Lambda_0^j j!,$$

and hence (18). Finally, (19) implies (11) since $|V(t,\xi)|$ is bounded.

To get the target, we firstly prove an apriori estimate for the *linear system* (14), without taking (16) into account. We follow [KS], but some modifications are needed in order to get an estimate suitable to the nonlinear case. The main tool is the theory of quasi-symmetrizer developed in [J1], [DS], and [J2].

REMARK 2. In the following we'll denote by C_j, C various positive constants depending only on the coefficients of the equation (13).

RECALLS ON QUASI-SYMMETRIZER.

[DS] 1998, [J2] 2009: For any Sylvester matrix $A(t)$ (see (15)) with real eigenvalues, we can find a family of Hermitian matrices of the form

$$Q_\varepsilon(t) = \mathcal{Q}_0(t) + \varepsilon^2\mathcal{Q}_1(t) + \cdots + \varepsilon^{2(m-1)}\mathcal{Q}_{m-1}(t) \tag{20}$$

in such a way that the entries of each matrix $\mathcal{Q}_r(t)$ are polynomial functions of the coefficients $a_1(t),\dots,a_m(t)$ (hence inherit their regularity in t) and, for some constant $C_0 \ge 1$ and all $V \in \mathbb{C}^m$, $0 < \varepsilon \le 1$, it holds

$$C_0^{-1}\varepsilon^{2(m-1)}|V|^2 \le (Q_\varepsilon(t)V, V) \le |V|^2, \tag{21}$$

$$|(Q_\varepsilon(t)A(t) - A^*(t)Q_\varepsilon(t))V, V)| \le C_0\varepsilon(Q_\varepsilon(t)V, V). \tag{22}$$

[KS] 2006: If the eigenvalues of $A(t)$ satisfy condition (8), $Q_\varepsilon(t)$ is a *nearly diagonal* matrix, i.e. satisfies, for some constant C_1 independent on ε,

$$\sum_{j=1}^m q_{\varepsilon,jj}(t)|v_j|^2 \le C_1(Q_\varepsilon(t)V, V) \quad \forall V \in \mathbb{C}^m, \tag{23}$$

where $q_{\varepsilon,ij}$ are the entries of Q_ε, and v_j the scalar components of V. □

In our assumptions, the functions $a_h(t)$ are analytic on $[0, T]$, consequently also the entries $q_{r,ij}(t)$ of the matrix $\mathcal{Q}_r(t)$ will be analytic. Therefore, putting together all the isolated zeroes of all these functions, we form a partition of $[0, T]$ independent on ε, say

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T,$$

such that, for each $r = 0, 1, \ldots, m-1$, and $i, j = 1, \ldots, m$, it holds:

$$\text{either } q_{r,ij} \equiv 0, \quad \text{or} \quad q_{r,ij}(t) \neq 0 \quad \forall t \in I_h =]t_{h-1}, t_h[.$$

Now observe that, if a solution $u(t, \cdot)$ of (13) belongs to $\mathscr{A}_{L^2}$ together with its time derivatives of order $< m$, at some point t, then the same holds in a right neighborhood of t (Cauchy-Kovalewski). Thus, it will be sufficient to put ourselves inside one of the intervals $I_1, \ldots, I_N$; in other words it is not restrictive to assume that, for each r, i, j,

$$\text{either } q_{r,ij} \equiv 0, \quad \text{or} \quad q_{r,ij}(t) \neq 0 \quad \text{for } 0 \leq t < T.$$

Therefore, by the analyticity of $q_{r,ij}$ on $[0, T]$ we easily derive that

$$|q'_{r,ij}(t)| \leq \frac{C_2}{T-t}\, |q_{r,ij}(t)| \quad \text{on } [0, T[. \tag{24}$$

Next, following [KS], for any fixed $\xi \in \mathbb{R}$ we prove two different apriori estimates for a solution $V(t, \xi)$ of (14): a *Kovalewskian* estimate in a (small) left neighborhood of T, $[\tau, T[$, and a *hyperbolic* estimate on $[0, \tau]$.

Lemma 1. *Let $V(t, \xi)$ be a solution of* (14) *on $[0, T[$. Then, for any fixed $\xi \in \mathbb{R}$, the following estimates hold for some constant C:*

$$\partial_t\{|V(t, \xi)|\} \leq C|\xi|\, |V(t, \xi)| + |F(t, \xi)| \tag{25}$$

$$\partial_t\{\sqrt{E_\varepsilon(t, \xi)}\} \leq C\left\{\frac{1}{T-t} + \varepsilon|\xi|\right\}\sqrt{E_\varepsilon(t, \xi)} + |F(t, \xi)| \tag{26}$$

where

$$E_\varepsilon(t, \xi) = (Q_\varepsilon(t)V(t, \xi), V(t, \xi)). \tag{27}$$

In particular, defining

$$E_* = E_{\varepsilon_*} \quad \text{with } \varepsilon_* = \langle\xi\rangle^{-1}, \quad \langle\xi\rangle = 1 + |\xi|,$$

we have

$$\partial_t\{\sqrt{E_*(t, \xi)}\} \leq C\left\{\frac{1}{T-t} + 1\right\}\sqrt{E_*(t, \xi)} + |F(t, \xi)|. \tag{28}$$

Proof. The estimate (25), with $C = \sup_t \|A(t)\|$, can be easily derived by multiplying each term of (14) by $V(t,\xi)$ in the scalar product $\mathbb{C}^m$. To prove (26) we differentiate (27) in time. Then, by (21), we find

$$\begin{aligned} E'_\varepsilon(t,\xi) &= (Q'_\varepsilon V, V) + (Q_\varepsilon V', V) + (Q_\varepsilon V, V') \\ &= (Q'_\varepsilon V, V) - i\xi((Q_\varepsilon A - A^* Q_\varepsilon)V, V) + 2\Re(Q_\varepsilon F, V) \\ &\le K_\varepsilon(t,\xi)E_\varepsilon(t,\xi) + 2|F(t,\xi)|\sqrt{E_\varepsilon(t,\xi)} \end{aligned}$$

where

$$K_\varepsilon(t,\xi) = \frac{|(Q'_\varepsilon V, V)|}{(Q_\varepsilon V, V)} + |\xi| \frac{|((Q_\varepsilon A - A^* Q_\varepsilon)V, V)|}{(Q_\varepsilon V, V)}. \tag{29}$$

Hence, we have to prove that

$$K_\varepsilon(t,\xi) \le C\left\{\frac{1}{T-t} + \varepsilon|\xi|\right\}. \tag{30}$$

To this purpose let us firstly note that, thanks to (22), the second quotient in (29) is estimated by $C_0\varepsilon$. To estimate the first quotient, we apply to (24) and (23): recalling (20), and noting that $|q_{r,ij}| \le \{q_{r,ii} \cdot q_{r,jj}\}^{1/2}$ since $\mathcal{Q}_r(t) \ge 0$, we find

$$\begin{aligned} |(Q'_\varepsilon V, V)| &\le \sum_{r=0}^{m-1} \varepsilon^{2r} \sum_{i,j=1}^{m} |q'_{r,ij}|\,|v_i|\,|v_j| \le C_2(T-t)^{-1} \sum_r \varepsilon^{2r} \sum_{i,j} |q_{r,ij}|\,|v_i|\,|v_j| \\ &\le C_2(T-t)^{-1} \sum_r \varepsilon^{2r} m \sum_j q_{r,jj}|v_j|^2 = C_2 m(T-t)^{-1} \sum_j q_{\varepsilon,jj}|v_j|^2 \\ &\le C_2 m C_1 (T-t)^{-1}(Q_\varepsilon V, V). \end{aligned}$$

This completes the proof of (30), hence of Lemma 1. □

Next, putting

$$\tau(\xi) = \begin{cases} T - |\xi|^{-1} & \text{if } |\xi| > T^{-1} \\ 0 & \text{if } |\xi| \le T^{-1} \end{cases},$$

we define, for each fixed $\xi \in \mathbb{R}$, the phase function

$$\Phi(t,\xi) = 1 + \min\{(T-t)^{-1}, |\xi|\} = \begin{cases} 1 + (T-t)^{-1} & \text{on } [0,\tau(\xi)] \\ 1 + |\xi| & \text{on } [\tau(\xi), T[\end{cases}. \tag{31}$$

Therefore, (25) and (28) imply that

$$\begin{aligned} \partial_t\{|V(t,\xi)|\} &\le C\Phi(t,\xi)|V(t,\xi)| + |F(t,\xi)| \quad \text{on } [\tau(\xi), T[\\ \partial_t\{\sqrt{E_*(t,\xi)}\} &\le C\Phi(t,\xi)\sqrt{E_*(t,\xi)} + |F(t,\xi)| \quad \text{on } [0,\tau(\xi)[, \end{aligned}$$

and hence it follows

$$\partial_t\{e^{\rho(t,\xi)}|V(t,\xi)|\} \le e^{\rho(t,\xi)}|F(t,\xi)| \quad \text{for } \tau(\xi) \le t \le T$$
$$\partial_t\{e^{\rho(t,\xi)}\sqrt{E_*(t,\xi)}\} \le e^{\rho(t,\xi)}|F(t,\xi)| \quad \text{for } 0 \le t \le \tau(\xi)$$

where

$$\rho(t,\xi) = C\int_t^T \Phi(s,\xi)\,ds. \tag{32}$$

By integrating in time we find (we omit ξ everywhere)

$$e^{\rho(t)}|V(t)| \le e^{\rho(\tau)}|V(\tau)| + \int_\tau^t e^{\rho(s)}|F(s)|\,ds, \tag{33}$$

$$e^{\rho(\tau)}\sqrt{E_*(\tau)} \le e^{\rho(0)}\sqrt{E_*(0)} + \int_0^\tau e^{\rho(s)}|F(s)|\,ds. \tag{34}$$

Now, by (21) with $\varepsilon = \langle\xi\rangle^{-1}$, we know that (for some $C_0 \ge 1$)

$$C_0^{-1}\langle\xi\rangle^{-2(1-m)}|V(t,\xi)|^2 \le E_*(t,\xi) \le |V(t,\xi)|^2,$$

hence from (33) and (34) it follows

$$\begin{aligned} e^{\rho(t)}|V(t)| &= C_0\langle\xi\rangle^{m-1}e^{\rho(\tau)}\sqrt{E_*(\tau)} + \int_\tau^t e^{\rho(s)}|F(s)|\,ds \\ &\le C_0\langle\xi\rangle^{m-1}\left\{e^{\rho(0)}\sqrt{E_*(0)} + \int_0^\tau e^{\rho(s)}|F(s)|\,ds\right\} + \int_\tau^t e^{\rho(s)}|F(s)|\,ds \\ &\le C_0\langle\xi\rangle^{m-1}\left\{e^{\rho(0)}\sqrt{E_*(0)} + \int_0^t e^{\rho(s)}|F(s)|\,ds\right\} \\ &\le C_0\langle\xi\rangle^{m-1}\left\{e^{\rho(0)}|V(0)| + \int_0^t e^{\rho(s)}|F(s)|\,ds\right\}. \end{aligned}$$

Recalling the definitions of Φ and ρ, we get

$$\rho(0,\xi) = C\int_0^T \Phi(s,\xi)\,ds \le C\int_0^{\tau(\xi)}\left\{1+\frac{1}{T-t}\right\}dt + C(T-\tau(\xi))\langle\xi\rangle,$$

and hence we derive, since $\partial_t\rho < 0$ and $\tau(\xi) = \max\{T - |\xi|^{-1}, 0\}$,

$$\rho(t,\xi) \le C\log\langle\xi\rangle + C_3 \quad \text{for all } t \in [0,T],\ \xi \in \mathbb{R}. \tag{35}$$

Therefore, if $N \ge (m-1) + C$, we get the wished *linear estimate*

$$e^{\rho(t,\xi)}|V(t,\xi)| \le C_0\langle\xi\rangle^N|V(0,\xi)| + C_0\langle\xi\rangle^{m-1}\int_0^t e^{\rho(s,\xi)}|F(s,\xi)|\,ds. \tag{36}$$

By the way, we note that (36) ensures the $\mathscr{C}^\infty$ wellposedness of the Cauchy problem for the linear system (14).

Now we go back to the nonlinear equation $Lu = u^\nu$. More precisely, we consider the more general equation

$$Lu = u_1 \dots u_\nu,$$

where u_j, $1 \le j \le \nu$, are given functions of (t, ξ) (actually, the u_j's will be x-derivatives of u). Such an equation can be regarded as a linear equation of type (14), where the function F is given by (16) with

$$f(t, \xi) = \hat{u}_1 * \dots * \hat{u}_\nu,$$

the convolutions being effected w.r. to ξ. Thus we have

$$|F(t, \xi)| = |f(t, \xi)| \le \int_{\{\xi_1 + \dots + \xi_\nu = \xi\}} |\hat{u}_1(t, \xi_1) \dots \hat{u}_\nu(t, \xi_\nu)| \, d\sigma_{(\xi_1, \dots, \xi_\nu)}. \tag{37}$$

The function $\xi \mapsto \min\{C, |\xi|\}$ is sub-additive; consequently for each fixed t (see (31), (32)) the function $\Phi(t, \xi)$, hence also $\rho(t, \xi)$, are sub-additive in ξ.

On the other hand, $\xi \to \langle \xi \rangle$ is sub-multiplicative. Thus, for $\xi = \xi_1 + \dots + \xi_\nu$,

$$\rho(t, \xi) \le \rho(t, \xi_1) + \dots + \rho(t, \xi_\nu), \quad \langle \xi \rangle^{m-1} \le \langle \xi_1 \rangle^{m-1} \dots \langle \xi_\nu \rangle^{m-1},$$
$$e^{\rho(t, \xi)} \langle \xi \rangle^{m-1} \le e^{\rho(t, \xi_1)} \langle \xi_1 \rangle^{m-1} \dots e^{\rho(t, \xi_\nu)} \langle \xi_\nu \rangle^{m-1},$$

and hence, by (37), it follows the pointwise estimate

$$e^\rho \langle \xi \rangle^{m-1} |F| \le (e^\rho \langle \xi \rangle^{m-1} |\hat{u}_1|) * \dots * (e^\rho \langle \xi \rangle^{m-1} |\hat{u}_\nu|).$$

Now, if $V_j(t, \xi)$ is the vector formed as $V(t, \xi)$ in (15), with u_j in place of u, we have

$$\langle \xi \rangle^{m-1} |\hat{u}_j(t, \xi)| \le |V_j(t, \xi)|, \quad j = 1, \dots, \nu,$$

and thus the linear estimate (36) gives

$$e^{\rho(t, \xi)} |V(t, \xi)| \le C_0 \langle \xi \rangle^N |V(0, \xi)| + C_0 \int_0^t (e^\rho |V_1| * \dots * e^\rho |V_\nu|)(s, \xi) \, ds.$$

Finally, we integrate in $\xi \in \mathbb{R}$ to define the $\mathscr{C}^\infty$-energy

$$\mathscr{E}(t, u) = \int_{\mathbb{R}} e^{\rho(t, \xi)} |V(t, \xi)| \, d\xi; \tag{38}$$

hence the last inequality gives

$$\mathscr{E}(t, u) \le C_0 \int_{\mathbb{R}} \langle \xi \rangle^N |V(0, \xi)| \, d\xi + C_0 \int_0^t \mathscr{E}(s, u_1) \dots \mathscr{E}(s, u_\nu) \, ds. \tag{39}$$

At this point we notice that, thanks to our assumption (17) (and to (35)),

(40) $$\sup_{0\le t\le T} \mathscr{E}(t,u) \equiv M_0 < \infty.$$

Next, differentiating in x our originary equation $Lu = u^\nu$, we get the equation

$$L(\partial^j u) = j! \sum_{h_1+\cdots+h_\nu=j} \frac{\partial^{h_1}u}{h_1!}\cdots\frac{\partial^{h_\nu}u}{h_\nu!} \quad (\text{where } \partial = \partial_x),$$

to which we apply (39) with $u_j = \partial^j u$. Thus, putting

$$\mathscr{E}_j(t) = \mathscr{E}(t, \partial^j u),$$

we obtain:

$$\mathscr{E}_j(t) \le C_0 \int_{\mathbb{R}} \langle\xi\rangle^N |V_j(0,\xi)|\, d\xi + C_0 j! \sum_{|h|=j} \int_0^t \frac{\mathscr{E}_{h_1}(s)}{h_1!}\cdots\frac{\mathscr{E}_{h_\nu}(s)}{h_\nu!}\, ds,$$

or also

(41) $$\mathscr{E}_j(t) \le C_0 \alpha_j(t),$$

with the position

$$\alpha_j(t) = \int_{\mathbb{R}} \langle\xi\rangle^N |V_j(0,\xi)|\, d\xi + j! \sum_{|h|=j} \int_0^t \frac{\mathscr{E}_{h_1}(s)}{h_1!}\cdots\frac{\mathscr{E}_{h_\nu}(s)}{h_\nu!}\, ds.$$

Then, following [DS], we introduce the *super-energies*

(42) $$\mathscr{F}(t) = \sum_{j=0}^\infty \mathscr{E}_j(t) \frac{r(t)^j}{j!},$$

(43) $$\mathscr{G}(t) = \sum_{j=0}^\infty \alpha_j(t)\frac{r(t)^j}{j!}, \quad \mathscr{G}^1(t) = \sum_{j=1}^\infty \alpha_j(t)\frac{r(t)^{j-1}}{(j-1)!},$$

where $r(t)$ is a decreasing, positive function on $[0,T]$ to be defined later.

By differentiating in time, we find

$$\mathscr{G}' = \sum_{j=0}^\infty \alpha_j' \frac{r^j}{j!} + \sum_{j=1}^\infty \alpha_j \frac{r^{j-1}}{(j-1)!} r' = \sum_{j=0}^\infty \sum_{|h|=j} \mathscr{E}_{h_1}\frac{r^{h_1}}{h_1!}\cdots \mathscr{E}_{h_\nu}\frac{r^{h_\nu}}{h_\nu!} + r'\mathscr{G}^1(t)$$

$$= \left\{\sum_{h=0}^\infty \mathscr{E}_h \frac{r^h}{h!}\right\}^\nu + r'\mathscr{G}^1 = \mathscr{F}^\nu + r'\mathscr{G}^1,$$

and hence, noting that $\mathscr{F}(t) \le C_0 \mathscr{G}(t)$ by (41), it follows

$$\mathscr{G}'(t) \le C_0^\nu \mathscr{G}(t)^\nu + r'(t)\mathscr{G}^1(t). \tag{44}$$

Now, by (17) and (40),

$$\alpha_0(t) = \int_R \langle\xi\rangle^N |V(0,\xi)|\, d\xi + \int_0^t \mathscr{E}(s)\, ds \le K_N + M_0 \equiv M,$$

and hence, by the definitions (43),

$$\mathscr{G}(t) \le \alpha_0(t) + r(t)\mathscr{G}^1(t) \le M + r(t)\mathscr{G}^1(t).$$

From this inequality it follows, arguing by induction w.r. to k,

$$\mathscr{G}(t)^k \le M^k + r(t)\mathscr{G}^1(t)\{M + \mathscr{G}(t)\}^{k-1}, \quad k = 1, 2, \dots,$$

and consequently by (44) we obtain, for $\phi(\mathscr{G}) = C_0^\nu\{M + \mathscr{G}\}^{\nu-1}$,

$$\mathscr{G}'(t) \le \mathscr{G}^1(t)\{r'(t) + r(t)\phi(\mathscr{G}(t))\} + C_0^\nu M^\nu. \tag{45}$$

On the other hand, by virtue of our assumption (18), we know that

$$\mathscr{G}(0) = \sum_{j=0}^\infty \left\{\int_{\mathbb{R}} \langle\xi\rangle^N |V_j(0,\xi)|\, d\xi\right\} \frac{r(0)^j}{j!} < \infty.$$

provided $r(0) \equiv r_0$ is small enough. Therefore, taking

$$\theta = \mathscr{G}(0) + (CM)^\nu T, \quad r(t) = r_0 e^{-\phi(\theta)t}, \tag{46}$$

we can derive from (45) the estimate

$$\mathscr{G}(t) < \theta \quad \text{for all } t \in [0, T]. \tag{47}$$

Proof of (47). Since $\theta > \mathscr{G}(0)$, (47) holds true in a right neighborhood of $t = 0$ by Cauchy-Kovalewsky. Therefore, assuming that (47) holds for all $t < \tau_*$, for some $\tau_* < T$, but fails at $t = \tau_*$, we have $\mathscr{G}(\tau_*) = \theta$. Hence it follows, taking $r(t)$ as in (46),

$$r'(t) + r(t)\phi(\mathscr{G}(t)) \le r'(t) + r(t)\phi(\theta) \le 0 \quad \text{on } [0, \tau_*[,$$

whence, going back to (45), we derive a contradiction:

$$\mathscr{G}(t) \le \mathscr{G}(0) + C_0^\nu M^\nu \tau_* < \theta \quad \text{on } [0, \tau_*].$$

Conclusion of the Proof of Theorem 1. Recalling that $\mathscr{F}(t) \le C_0\mathscr{G}(t)$, (47) says that $\mathscr{F}(t) < C_0\theta$ on $[0, T]$. Therefore, by (42), we get our goal (19):

$$\int_{\mathbb{R}} |\xi|^j |V(t,\xi)|\, d\xi \le \int_{\mathbb{R}} e^{\rho(t,\xi)} |\xi|^j |V(t,\xi)|\, d\xi = \mathscr{E}_j(t) \le \mathscr{F}(t) r(t)^{-j} j!$$
$$\le C_0\theta\{r_0^{-1} e^{\phi(\theta)T}\}^j j! = \tilde{K}\tilde{\Lambda}^j j!.$$

To prove (12), i.e., the analyticity of the solution u in (t, x), it is sufficient to apply to Cauchy-Kovalewski. □

Remark 3. The previos proof of (47) is somewhat formal, since it assumes not only that $\mathscr{G}(t) < \infty$, but also that $\mathscr{G}^1(t) < \infty$ on $[0, \tau_*[$. To make the proof more precise we must replace the radius function $r(t)$ by $r_\eta(t) = \eta r_0 \exp(-\phi(\theta)t)$, $\eta < 1$, and apply the previous computation to the corresponding functions $\mathscr{G}_\eta(t)$ and $\mathscr{G}^1_\eta(t)$. Finally we let $\eta \to 1$ (see [ST] for the details).

Proof of Theorem 2. The proof is not very different from that of Thm. 1, thus we give only a sketch of it.

The main difference is that now the entries $q_{r,ij}(t)$ are not analytic but only $\mathscr{C}^\infty$ functions, hence (24) fails. However, for any function $f \in \mathscr{C}^k([0, T])$ it holds

$$|f'(t)| \le \Lambda(t)|f(t)|^{1-1/k}(\|f\|_{\mathscr{C}^k([0,T])})^{1/k},$$

for some $\Lambda \equiv \Lambda_f \in L^1(0, T)$. This was proved in [CJS] in the case $f(t) \ge 0$, and in [T] in the general case.

Therefore, using that $Q_\varepsilon(t)$ is a nearly diagonal matrix, and proceeeding as in [KS], for all integer $k \ge 1$ there is some function $\Lambda_k \in L^1(0, T)$, independent of ε, such that

$$(48) \quad |(Q'_\varepsilon(t)V(t,\xi), V(t,\xi))| \le \Lambda_k(t)(Q_\varepsilon(t)V(t,\xi), V(t,\xi))^{1-1/k}|V(t,\xi)|^{2/k}.$$

Differently from Theorem 1, we can now consider only the *hyperbolic energy*

$$E_*(t,\xi) = (Q_{\varepsilon_*}(t)V, V) \quad \text{with } \varepsilon_* = \langle\xi\rangle^{-1}.$$

Thanks to (48) we prove, for each integer $k \ge 1$, an estimate

$$\partial_t\{\sqrt{E_*(t,\xi)}\} \le C_k\Phi_k(t,\xi)\sqrt{E_*(t,\xi)} + |F(t,\xi)|$$

on the interval $[0, T]$, where

$$\Phi_k(t,\xi) = \Lambda_k(t)|\xi|^{2(m-1)/k} + 1.$$

The phase function Φ_k is sub-additive w.r. to ξ as soon as $k \ge 2(m-1)$.

Next, putting

$$\rho_k(t,\xi) = C_k \int_t^T \Phi_k(t,\xi)\,d\xi = C_k|\xi|^{2(m-1)/k} \int_t^T \Lambda_k(s)\,ds + C_k(T-t),$$

we define the *Gevrey-energy*

$$\mathscr{E}_{(k)}(t,u) = \int_{\mathbb{R}} e^{\rho_k(t,\xi)} \sqrt{E_*(t,\xi)}\,d\xi,$$

and we conclude as in the proof of Theorem 1. □

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Dipartimento di Matematica "L. Tonelli"
Largo Pontecorvo 5, 56127 Pisa
spagnolo@dm.unipi.it