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Partial Differential Equations — *A note on hybrid heteroclinic solutions for a class of semilinear elliptic PDEs*, by Sergey Bolotin[1] and Paul H. Rabinowitz, communicated on 14 January 2011.

Dedicated to the memory of Giovanni Prodi.

Abstract. — Variational methods will be used to find solutions of a semilinear elliptic PDE by ''glueing'' mountain pass critical points of the corresponding functional to minima or to other mountain pass critical points.

Key words: Variational methods, heteroclinic solutions, homoclinic solutions, hybrid solutions, mountain pass critical points.

Mathematics Subject Classification: 35J20, 35J61, 49J35, 49J40.

1. Introduction

Variational methods have long been used to find solutions of differential equations as critical points of corresponding functionals. This work was generally for bounded temporal and spatial domains, although some research used variational results for bounded temporal domains together with limit arguments to treat unbounded domains. See e.g. Morse [14] and Hedlund [11] for heteroclinic geodesics, and [5] for homoclinics to a critical point of a Hamiltonian system. However in the late 1980's, direct methods were developed for problems on unbounded domains, in particular to find heteroclinic and homoclinic solutions of Hamiltonian systems (see e.g. Coti Zelati, Ekeland, and Sèrè [8], Sèrè [18], and [9]) and of partial differential equations (see e.g. [10]). These solutions generally correspond to minima or mountain pass critical points of the associated functionals. More importantly, variational ''gluing'' arguments were discovered to find additional critical points of the functionals near sums of the basic ones just mentioned and corresponding solutions of the equations which are near formal concatenations of the basic heteroclinics/homoclinics. See e.g. Mather [13] for discrete Hamiltonian systems (area preserving maps), Sèrè [18, 19] for Hamiltonian systems, and [9, 10] for Hamiltonian systems and elliptic PDEs.

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See also Angenent [1] for another gluing method for elliptic PDEs based on the implicit function theorem.

Aside from the one dimensional case, the only work we know of using such variational gluing arguments treats critical points and solutions of the same type, i.e. minima are glued to minima and mountain pass solutions to mountain pass solutions. Our main goal in this note is to describe a situation in which minima and mountain pass critical points both exist and one can find "hybrid solutions" of the equations corresponding to critical values near the sum of a minimum and mountain pass critical value. Another novelty here is that we can give a simple variational characterization of the new solutions and also provide geometrical information on their location.

Historically our work is related to results of Poincaré and Birkhoff on homoclinic orbits of Hamiltonian systems. For example Poincaré showed that if a time periodic Hamiltonian system with 1 degree of freedom has an isolated homoclinic orbit to a hyperbolic periodic orbit, then there exists an infinite number of homoclinic orbits. Poincaré used geometrical methods. The approach we take is also geometrical, but instead of working in the phase space, we work in the configuration space and use variational methods. A key tool for us is the construction of invariant regions for the heat flow associated with our equation. Such an approach has been used by many authors such as Bessi [3] and de la Llave–Valdinoci [12], just to mention a couple, who work in settings related to ours. The heat flow enables us to carry out the variational arguments in these invariant regions and the shape of the region provides information on the form of the associated solution.

In §2, the family of equations we treat will be introduced and our main results will be presented in §3. A more detailed description of our results together with full proofs will appear in [7].

2. Preliminaries

Consider the equation

$$\Delta u = F_u(x, u), \quad x \in \mathbb{R}^n. \tag{2.1}$$

We assume that $F \in C^2(\mathbb{T}^{n+1}, \mathbb{R})$, where $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ is the torus, i.e. F is 1-periodic in its arguments. The study of this equation, and in fact a much more general family of equations, was initiated by Moser [15] as a step towards an Aubry–Mather theory for PDE's. A classical example of (2.1) is a pendulum with an oscillating suspension point:

$$\ddot{u} = f(t) \sin u$$

where $f > 0$ is periodic. For this example our results go back to Poincaré and they are strongly related to well known results in the dynamics of area preserving maps (see e.g. [13]).

Set

$$L(x, u, \nabla u) = \frac{1}{2}|\nabla u|^2 + F(x, u).$$

Then equation (2.1) is the Euler–Lagrange equation for the functional

$$\mathscr{F}(u) = \int_{\mathbb{T}^n} L(x, u, \nabla u)\, dx, \quad u \in W^{1,2}(\mathbb{T}^n).$$

Standard results imply that $\mathscr{F}$ attains its minimum on $W^{1,2}(\mathbb{T}^n)$ and any minimizer is a classical periodic solution of (2.1). Moser further showed that, as in classical Aubry–Mather theory [2], the set $\mathscr{M}_0$ of minimizers is ordered: $v, w \in \mathscr{M}_0$ implies $v = w$, or $v < w$, or $v > w$. Thus either the graphs of minimizers foliate or laminate $\mathbb{T}^{n+1}$. The latter is the generic case and then there is a pair of minimizers $u_- < u_+$ such that there are no other minimizers between them.

We are interested in solutions of (2.1) lying in the gap between u_- and u_+ and periodic in all variables except the first one. Let $\mathscr{N} = \mathbb{R} \times \mathbb{T}^{n-1}$ and

$$\mathscr{W} = \{u \in W^{1,2}_{\mathrm{loc}}(\mathscr{N}) \,|\, u_- \le u \le u_+\}.$$

Let $\tau : \mathscr{N} \to \mathscr{N}$ be the right translation: $\tau(x) = (x_1 + 1, x_2, \dots, x_n)$. Then $\tau u = u \circ \tau^{-1}$ moves the graph of u to the right. A solution $u \in \mathscr{W}$ will be called *heteroclinic* from u_- to u_+ if $\tau^{\pm k} u \to u_\mp$ in the C^0_{loc} topology as $k \to \infty$. For us "solution" always means a solution of equation (2.1) and "heteroclinic" always refers to heteroclinics in x_1. Let $\mathscr{H}(u_-, u_+)$ be the set of heteroclinic solutions from u_- to u_+ and $\mathscr{H}(u_+, u_-)$ the analogous set of heteroclinic solutions from u_+ to u_-.

A solution $u \in \mathscr{W}$ will be called minimal if for all $\phi \in W^{1,2}(\mathscr{N})$ with compact support,

$$\int_{\mathscr{N}} (L(x, u, \nabla(u + \phi)) - L(x, u, \nabla u))\, dx \ge 0.$$

We say that $u \in \mathscr{W}$ is 1-monotone in x_1 or simply 1-monotone if $\tau u < u$ or $\tau u > u$. Equivalently, the graph of u in $\mathbb{T}^{n+1}$ is non self-intersecting. Moser and Bangert called such solutions non self-intersecting.

Bangert (in the more general setting of Moser) [4] proved the existence of minimal monotone solutions of (2.1) heteroclinic from u_- to u_+ and from u_+ to u_-. Let $\mathscr{M}_\pm \subset \mathscr{H}(u_\mp, u_\pm)$ be the sets of minimal monotone heteroclinic solutions. Then as Bangert showed, $\mathscr{M}_\pm$ is an ordered set. His argument is not variational. However for our gluing arguments, a direct variational characterization of minimal heteroclinics is needed. In particular, one appropriate for the current setting was given in [17]. To describe it, without loss of generality let $\min_{W^{1,2}(\mathbb{T}^n)} \mathscr{F} = 0$. Set $T_i = [i, i+1] \times \mathbb{T}^{n-1}$ and define a functional J on $\mathscr{W}$ by

$$J(u) = \sum_{i=-\infty}^{\infty} \int_{T_i} L(x, u, \nabla u)\, dx. \tag{2.2}$$

It was proved in [17] that for any $u \in \mathscr{W}$, the series (2.2) either converges or diverges to $+\infty$. Let

$$\Gamma_+ = \Gamma(u_-, u_+) = \left\{ u \in \mathscr{W} \mid \lim_{i \to \pm\infty} \|u - u_\pm\|_{L^2(T_i)} = 0 \right\},$$

$$\Gamma_- = \Gamma(u_+, u_-) = \left\{ u \in \mathscr{W} \mid \lim_{i \to \pm\infty} \|u - u_\mp\|_{L^2(T_i)} = 0 \right\}.$$

It was shown in [17] that the functional J attains its minimum, $c_\pm$, on $\Gamma_\pm$ with any minimizer a classical solution of (2.1) heteroclinic from $u_\mp$ to $u_\pm$. The set $\{u \in \Gamma_\pm \mid J(u) = c_\pm\}$ of minimizers is precisely the set $\mathscr{M}_\pm$ of minimal 1-monotone heteroclinics obtained by Bangert.

There are also mountain pass and multitransition solutions in $\mathscr{W}$ provided that we make a

(*) No foliation assumption: there are gaps in $\mathscr{M}_\pm$.

Assumption (*) is generic [17]. Its dynamical systems analogue is that the stable and unstable manifolds of a fixed point are not doubled. If (*) holds, every gap in $\mathscr{M}_\pm$ is bounded by a pair of minimal heteroclinics $v_\pm < w_\pm$ which we call a gap pair. In [17], it is proved that $w_\pm - v_\pm \in W^{1,2}(\mathscr{N})$. Let $E_\pm = v_\pm + W^{1,2}(\mathscr{N})$ be the affine space through $v_\pm$ and set

$$\Lambda_\pm = \{u \in E_\pm \mid v_\pm \le u \le w_\pm\}.$$

From [10, 6], we have that the functional J is C^1 on $E_\pm$ and it satisfies the Palais–Smale condition (PS) in $\Lambda_\pm$: if $(u_k) \subset \Lambda_\pm$ is a sequence such that $J(u_k)$ is bounded and $\|J'(u_k)\|_{W^{1,2}(\mathscr{N})} \to 0$, then u_k has a subsequence which is convergent in the $W^{1,2}$ norm to some $u \in \Lambda_\pm$.

With the aid of the (PS) condition, mountain pass heteroclinic solutions can be obtained between the gap pair, $v_\pm < w_\pm$. Set $I = [0, 1]$ and define

$$b_\pm = \inf_h \max_{h(I)} J,$$

where the infimum is taken over all continuous paths $h : I \to \Lambda_\pm$ connecting $v_\pm$ with $w_\pm$. Equivalently, $b_\pm$ is the supremum of all a such that $v_\pm$ and $w_\pm$ are in different path connected components of $\Lambda_\pm^a = \{u \in \Lambda_\pm \mid J(u) \le a\}$. It was shown in [6] that $b_\pm > c_\pm$ and there exists a critical point $u \in \Lambda_\pm$ of J with $J(u) = b_\pm$. Any u given by this minimax will be called a mountain pass critical point since it lies in the mountain pass critical level, $J^{-1}(b_\pm)$.

The mountain pass heteroclinic solution $u \in J^{-1}(b_\pm)$ is obtained in [6] by a variant of the usual Deformation Theorem [16]. An alternate approach is to use a heat flow argument. Since such an argument plays an important role in our main results, it will be sketched here. Let $u(t) = \Phi^t(u_0)$, $t \geq 0$, be the solution of the parabolic initial value problem

$$u_t = \Delta u - F_u(x, u), \quad u(0) = u_0.$$

By a parabolic comparison principle, $\Phi^t : \Lambda_\pm \to \Lambda_\pm$, $t \geq 0$. Since J decays along the heat flow Φ^t, the (PS) condition implies that for any $u_0 \in \Lambda_\pm$ and any sequence $t_k \to \infty$, there exists a subsequence such that $\Phi^{t_k}(u_0)$ converges in $W^{1,2}$ to a critical point $u \in \Lambda_\pm$ of J with $J(u) = \lim_{t\to\infty} J(\Phi^t(u_0))$. With the aid of these observations, that $b_\pm$ are critical values readily follows.

A similar argument holds for any Φ^t-invariant set $\Lambda \subset \mathscr{W}$ such that J is finite and differentiable at every point in Λ and satisfies the (PS) condition in Λ such as the setting of §3 that follows.

3. The main results

In this section, we turn to multitransition homoclinic solutions. The simplest cases are 2-transition homoclinics which will be discussed next. Up to permutations, there are three possibilities:

- gluing a minimal heteroclinic in $\mathscr{M}_+ = \mathscr{M}(u_-, u_+)$ corresponding to c_+ to one in $\mathscr{M}_- = \mathscr{M}(u_+, u_-)$ corresponding to c_- to obtain a locally minimal homoclinic solution in $\mathscr{H}(u_-, u_-)$.
- gluing a mountain pass heteroclinic in $\mathscr{H}(u_-, u_+)$ corresponding to b_+ to a minimizer in $\mathscr{M}(u_+, u_-)$ corresponding to c_- to obtain a mountain pass homoclinic solution in $\mathscr{H}(u_-, u_-)$.
- gluing a pair of mountain pass heteroclinics in $\mathscr{H}(u_-, u_+)$ and $\mathscr{H}(u_+, u_-)$ corresponding to b_+ and b_-.

The first case was already carried out in [17]. However in contrast to [17] where the goal was merely to construct a 2-transition homoclinic solution, here we need one that shadows two particular heteroclinics. This requires a more careful construction. To describe it, let $c = c_- + c_+$ and $w_k = \min(w_+, \tau^k w_-)$. Then with the aid of a result from [17], $J(w_k) < c$ and $J(w_k) \to c$ as $k \to \infty$. Set $m = (m_+, m_-) \in \mathbb{Z}^2$ and $r = (r_+, r_-)$ with

$$0 < r_\pm < \rho = \|u_+ - u_-\|_{L^2(T_0)}.$$

The parameters m and r must be chosen carefully. Postponing the choice for the moment, for $k \in \mathbb{N}$, define

$$Y_k = \{u \in \mathscr{W} \mid u \leq w_k, \|u - u_+\|_{L^2(T_{m_+})} \leq r_+, \|u - u_+\|_{L^2(T_{m_-+k})} \leq r_-\}$$

and

$$c_k = \inf_{u \in Y_k} J(u).$$

Then J attains its minimum c_k in Y_k and any minimizer $u_k \in Y_k$ is in the interior of Y_k. Thus u_k is a classical heteroclinic solution of (2.1).

To choose the parameters m, r, note that since the sets $\mathscr{M}_\pm$ of minimal heteroclinics are ordered,

$$S_\pm = \{\|v - u_+\|_{L^2(T_0)} \mid v \in \mathscr{M}_\pm\}$$

is an ordered set in $(0, \rho)$ which contains gaps. Since $\mathscr{M}_\pm = \tau \mathscr{M}_\pm$, is τ-invariant,

$$S_\pm = \{\|v - u_+\|_{L^2(T_i)} \mid v \in \mathscr{M}_\pm\}$$

for any $i \in \mathbb{Z}$. Choose $r_\pm$ in a gap, i.e.

$$r_\pm \in (0, \rho) \backslash S_\pm. \tag{3.1}$$

In fact we can choose $r_\pm$ as close to 0 as we please. By (3.1), there is a unique $m_\pm \in \mathbb{Z}$ such that

$$\|w_\pm - u_+\|_{L^2(T_{m_\pm})} < r_\pm < \|v_\pm - u_+\|_{L^2(T_{m_\pm})}.$$

By the choice of m, the functions w_+, $\tau^k w_-$ satisfy, respectively, the r_+, r_- inequalities in the definition of Y_k. Therefore for any large k, w_k belongs to Y_k so $Y_k \neq \emptyset$ and by slight modifications of the proof in [17], we have

Theorem 3.2. *There is a $K > 0$ such that for any $k \in \mathbb{N}$ with $k \geq K$, there exists a homoclinic $u_k \in \mathscr{H}(u_-, u_-)$ such that*

- $u_k < w_k$.
- $J(u_k) < J(w_k)$.
- *u_k is locally minimizing, i.e. $J(v) \geq J(u_k)$ for any $v \in \mathscr{W}$ which is L^∞ close to u_k.*
- *As $k \to \infty$, $J(u_k) \to c$ and $\|u_k - w_k\|_{L^\infty(\mathscr{N})} \to 0$.*

Since u_k and $w_\pm$ are solutions of the elliptic PDE (2.1), the last item implies that $u_k \to w_+$ and $\tau^{-k} u_k \to w_-$ in C^2_{loc} as $k \to \infty$. In fact more can be proved: $u_k \to w_+$ in $W^{1,2}((-\infty, 0] \times \mathbb{T}^{n-1})$ and $\tau^{-k} u_k \to w_-$ in $W^{1,2}([0, \infty) \times \mathbb{T}^{n-1})$. Moreover under the nondegeneracy condition (ND^+) introduced below, $\|u_k - w_k\|_{W^{1,2}(\mathscr{N})} \to 0$. The details will be given in [7].

Now we turn to gluing a mountain pass heteroclinic to a minimizing heteroclinic. Define a region invariant under the heat flow Φ^t by e.g.

$$\Sigma_k = \{u \in \mathscr{W} \mid \tau u_{k-1} \leq u \leq w_k, u - w_k \in E\}.$$

Then $\underline{w}_k = \min(v_+, \tau^k w_-) \in \Sigma_k$. Let $d = b_+ + c_-$. Our first theorem is a consequence of a topological result:

Proposition 3.3. *For any $\delta \in (0, b_+ - c_-)$, there exists a $K > 0$ such that for $k \geq K$, $\underline{w}_k$ and w_k are in different path connected components of $\Sigma_k^{d-\delta}$ but in the same path connected component of $\Sigma_k^{d+\delta}$.*

Let d_k be the infimum of $a \in \mathbb{R}$ such that $\underline{w}_k$ and w_k are in the same path connected component of

$$\Sigma_k^a = \{u \in \Sigma_k \mid J(u) \le a\}.$$

Equivalently,

$$d_k = \inf_h \max_{h(I)} J,$$

where the infimum is taken over all continuous paths $h : I \to \Sigma_k$ connecting $\underline{w}_k$ and w_k. Then d_k is a mountain pass critical level. Indeed, since Σ_k is Φ^t-invariant, the heat flow argument implies:

THEOREM 3.4. *There exists a $K > 0$ such that for any $k \ge K$, the functional J has a critical point $U_k \in \Sigma_k$ with $J(U_k) = d_k$.*

We call $U_k \in \mathscr{H}(u_-, u_-)$ a hybrid homoclinic obtained by gluing a mountain pass and a minimal heteroclinic. To really justify this, the asymptotic behavior of U_k as $k \to \infty$ must be studied. By Proposition 3.3, $d_k \to d$ as $k \to \infty$. With the aid of Theorems 3.2 and 3.4, we have:

- $\tau^{-k} U_k \to w_-$ in C^2_{loc},
- there exists a heteroclinic V lying between τw_+ and w_+ and a subsequence $k \to \infty$ such that $U_k \to V$ in C^2_{loc}.

With a more complicated choice of Σ_k as in [7], we can conclude that U_k converges to $V \in \Lambda_+$. However for ease in exposition, we have chosen to simplify the construction here and therefore get a slightly weaker result. It seems probable that V is a mountain pass heteroclinic, with $J(V) = b_+$, but we are unable to prove this without a further mild nondegeneracy assumption:

$(ND^{\pm})$ The minimizer $u_\pm$ of the functional $\mathscr{F}$ on $W^{1,2}(\mathbb{T}^n)$ is a nondegenerate critical point of $\mathscr{F}$.

By $(ND^{\pm})$, the second variation quadratic form is positive definite: there is a constant $\mu > 0$ such that

$$\mathscr{F}''(u_\pm)(\phi, \phi) \ge \mu \|\phi\|^2_{L^2(\mathbb{T}^n)} \quad \phi \in W^{1,2}(\mathbb{T}^n).$$

A dynamical systems analogue of this condition is that the fixed point of the map is hyperbolic.

With the aid of some additional estimates we now obtain:

THEOREM 3.5. *Suppose (ND^+) holds. Then there exists a mountain pass solution, V, lying between τw_+ and w_+ with $J(V) = b_+$ and a $K > 0$ such that for each $k \ge K$, there is a solution $U_k \in \Sigma_k$ such that $J(U_k) = d_k$ and along a subsequence of $k \to \infty$,*

$$\|U_k - V_k\|_{W^{1,2}(\mathcal{N})} \to 0, \quad V_k = \min(V, \tau^k w_-).$$

For our final result in this section, two mountain pass heteroclinics will be glued together. To begin, first note that the region in which the curves of Σ_k lie consists of two subregions. There is a "thick" part between τu_{k-1} and where $w_k = w_+$. This is coupled to a "thin" part between τu_{k-1} and where $w_k = \tau^k w_-$. Now we consider the heat flow invariant region between τu_{k-2} and w_k consisting of two "thick" parts in which a critical point of J can now be obtained. Set

$$\Omega_k = \{u \in \mathscr{W} \mid \tau u_{k-2} \leq u \leq w_k, u - w_k \in E\}.$$

Then $\Sigma_k \subset \Omega_k$ and $\underline{u}_k = \min(v_+, \tau_k v_-) \in \Omega_k$.

Let $I^2 = [0,1] \times [0,1]$ and $b = b_+ + b_-$. The topological basis for finding the critical point is provided by the next result.

Proposition 3.6. *Let $\delta \in (0, b - c)$. For any $\varepsilon \in (0, \delta)$, there is a $K > 0$ such that for any $k > K$, there exists a continuous map $g : (I^2, \partial I^2) \to (\Omega_k^{b+\varepsilon}, \Omega_k^{b-\delta})$ which is not homotopic to a map $(I^2, \partial I^2) \to (\Omega_k^{b-\varepsilon}, \Omega_k^{b-\delta})$ in the class of maps $(I^2, \partial I^2) \to (\Omega_k, \Omega_k^{b-\delta})$.*

As before, $\Omega_k^a = \{u \in \Omega_k \mid J(u) \leq a\}$. Following standard notation, we write $g : (I^2, \partial I^2) \to (X, Y)$ if $Y \subset X$, and $g : I^2 \to X$ is a continuous map such that $g(\partial I^2) \subset Y$. The map g in Proposition 3.3 is obtained by appropriately combining paths $h_\pm : [0,1] \to \Lambda_\pm$ joining v_+ with w_+ and $\tau^k v_-$ with $\tau^k w_-$ respectively. The idea goes back to Séré [19].

Now take $\delta \in (0, b - c)$ and define

$$a_k = \inf_h \sup_{h(I^2)} J,$$

where the infimum is taken over all maps $h : (I^2, \partial I^2) \to (\Omega_k, \Omega_k^{b-\delta})$ homotopic to g in the class of maps $(I^2, \partial I^2) \to (\Omega_k, \Omega_k^{b-\delta})$. By Proposition 3.6, $a_k \to b$ as $k \to \infty$ and the earlier heat flow argument gives:

Theorem 3.7. *There exists a $K > 0$ such that for any $k \geq K$, J has a critical point $V_k \in \Omega_k$ such that $J(V_k) = a_k$.*

We conclude this note with some remarks.

For each large k, the above arguments give seven homoclinics in Ω_k, four of them being local minimizers of J with J close to $c_+ + c_-$, two of mountain pass type with J close to $c_\pm + b_\mp$, and one has J close to $b_+ + b_-$.

By the arguments of Theorems 3.5, when (ND^+) holds, as $k \to \infty$, V_k of Theorem 3.7 converges to V, a solution of (2.1) between τw_+ and w_+ that is heteroclinic from u_- to u_+ and $\tau^{-k} V_k$ converges to W, a solution between $\tau^{-1} w_-$ and w_- that is heteroclinic from u_+ to u_-. Moreover $J(V) + J(W) = b_+ + b_-$. However we are unable at this point to prove that V, W are mountain pass solutions with $J(V) = b_+$ and $J(W) = b_-$.

The existence arguments giving the hybrid solution, U_k, and the solution, V_k, involve the construction of invariant regions for the associated heat flow. In particular the invariant region for the former consisted of a "thick" part and a "thin" part while its analogue for the latter consisted of two "thick" parts. In a related but more elaborated fashion, invariant regions can be constructed and variational arguments given that yield hybrid multitransition homoclinic solutions with critical values near $C_a = \sum_{i=1}^{p} a_i$ where in the sequence $a = (a_i)_{i=1}^{p}$ alternate elements belong to our choice of c_+ or b_+ and c_- or b_-. When $a_i = b_\pm$, the regions will have a "thick" part corresponding roughly to a phase shift of the region between $v_\pm$ and $w_\pm$. Likewise when $a_i = c_\pm$, there is a "thin" part corresponding to a phase shift of $w_\pm$.

For a precise statement, suppose for definiteness that p is even and $a_i \in \{c_+, b_+\}$ for odd i and $a_i \in \{c_-, b_-\}$ for even i. For an increasing integer sequence $q = (q_i)_{i=1}^{p}$ define

$$k_i = q_{i+1} - q_i, \quad \bar{v}_q = \max_{i \text{ odd}} (\tau^{q_i} w_{k_i}),$$

where $w_k = \min(w_+, \tau^k w_-)$ is as in Theorem 3.2. From the results of [17], it follows that for an integer sequence $m = (m_i)_{i=1}^{p}$ with sufficiently large separation parameter $k = \min_i k_i$, there exists a locally minimizing homoclinic $v_q \in \mathscr{H}(u_-, u_-)$ such that $v_q < \bar{v}_q$. Moreover by arguments from the proof of Theorem 3.2, as $k \to \infty$, $\|v_q - \bar{v}_q\|_{L_\infty(\mathscr{N})} \to 0$ and $J(v_q) \to (c_- + c_+)p/2$.

The homoclinics v_q can now be used to construct an invariant region containing hybrid multitransition homoclinics. To do so, take another integer sequence $q' = (q'_i)_{i=1}^{p}$ with $q_i = q_i$ if $a_i = c_\pm$, $q'_i = q_i + 1$ if $a_i = b_+$ and $q'_i = q_i - 1$ if $a_i = b_-$. Then one can show there is a locally minimizing homoclinic $v_{q'} \in \mathscr{H}(u_-, u_-)$ with $v_{q'} < v_q$. The region

$$\Lambda_q = \{u \in \mathscr{W} \mid v_{q'} < u < v_q\}$$

is invariant under the parabolic semiflow Φ_t, $t \geq 0$. A topological property similar to Proposition 3.6 shows that for large k, the functional J has a critical point $u_q \in \Lambda_q$ which is a hybrid multitransition homoclinic solution and as $k \to \infty$, $J(u_q) \to C_a$.

As in Theorem 3.5, whenever we have a "thin" region, i.e. $a_i = c_\pm$, for an appropriate phase shift $\tau^{-q_i} u_q$ of u_q we have $\tau^{-q_i} u_q \to w_\pm$. If the nondegeneracy conditions $(ND^\pm)$ hold, we expect that for "thick" regions, i.e. $a_i = b_\pm$, a phase shift $\tau^{-q_i} u_q$ converges to some v, a heteroclinic solution of (2.1) with $J(v) = b_\pm$ as the case may be. However this has not yet been established.

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