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Partial Differential Equations — *Some recent results concerning a class of forward-backward parabolic equations*, by Flavia Smarrazzo and Alberto Tesei, communicated on 14 January 2011.

Dedicated to the memory of Professor Giovanni Prodi.

Abstract. — We study a quasilinear parabolic equation of forward-backward type in one space dimension, under assumptions on the nonlinearity satisfied by important mathematical models (e.g., the one-dimensional Perona–Malik equation). We first address a degenerate pseudoparabolic regularization of the equation, which takes time delay effects into account, proving existence and uniqueness of positive solutions of the regularized problem in a space of Radon measures, as well as qualitative properties of such solutions. Then we address the vanishing viscosity limit of the regularized problem, proving that the limiting points (in a suitable topology) of the family of solutions of the regularized problem can be regarded as solutions of the original problem. In particular, we characterize the disintegration of the narrow limit of the family of Young measures associated with the absolutely continuous part of the solutions of the regularized problem, proving that it is a superposition of two Dirac masses with support on the branches of the graph of the nonlinearity φ. In the above study, the existence of a family of infinitely many entropy inequalities plays an important role.

Key words: Forward-backward parabolic equations, degenerate pseudoparabolic regularization, bounded Radon measures, Young measures, entropy inequalities.

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1. Introduction

In this paper we describe some recent results concerning the initial-boundary value problem

$$\begin{cases} U_t = [\varphi(U)]_{xx} & \text{in } \Omega \times (0, T] =: Q \\ \varphi(U) = 0 & \text{in } \partial\Omega \times (0, T] \\ U = U_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

referring the reader to [ST1, ST2] for proofs. Here $\Omega \subseteq \mathbb{R}$ is a bounded interval, $T > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a *nonmonotone* odd function, which satisfies the following assumptions:

$$(H_1) \qquad \begin{cases} \text{(i)} & \varphi \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R}),\ \varphi \text{ odd};\ \varphi(s) > 0 \quad \text{for } s > 0; \\ \text{(ii)} & \varphi'(s) > 0 \text{ for } 0 < s < \alpha, \varphi'(s) < 0 \quad \text{for } s > \alpha\ (\alpha > 0); \\ \text{(iii)} & \varphi''(s) \geq 0 \text{ for any } s \geq s_0, \quad \text{for some } s_0 > 0; \\ \text{(iv)} & \varphi^{(j)} \in L^\infty(\mathbb{R}) \quad \text{for any } j \in \mathbb{N}. \end{cases}$$

By $\varphi^{(j)}$ ($j \in \mathbb{N}$) we denote the j-th derivative of the function φ, while the usual notation φ', φ'' is used for the first and second derivatives. Observe that (H_1)-(i) and (ii) imply $\varphi(s) \to 0$ as $s \to \infty$, $0 < \varphi(s) \leq \varphi(\alpha)$ for $s > 0$.

The main feature of φ is that there exists $\alpha > 0$ such that

$$(s - \alpha)\varphi'(s) \leq 0 \quad \text{for any } s > 0$$

(see assumption (H_1)-(ii)). Therefore the first equation in (1.1) is a quasilinear parabolic equation of *forward–backward* type, and problem (1.1) is *ill-posed* whenever the solution U takes values where $\varphi' < 0$.

Motivation for the present study comes from the Perona–Malik equation [PM] in one space dimension

$$u_t = [\varphi(u_x)]_x, \tag{1.2}$$

which also appears in a mathematical model of oceanography [BBDU]. Typical forms of φ in (1.2) are

$$\varphi(s) = \frac{s}{s^2 + \alpha}, \quad \varphi(s) = s \exp\left(-\frac{s}{\alpha}\right) \quad (\alpha > 0). \tag{1.3}$$

In fact, deriving formally equation (1.2) with respect to x and setting $U := u_x$ gives the first equation in (1.1). Since a natural framework to study (1.2) is the space of real functions of bounded variation [BBDU], the above formal argument suggests to study problem (1.1) in the space of bounded *Radon measures on* $\overline{\Omega}$. Let us mention that problem (1.1) also arises (with φ as in (1.3)) in models of aggregating populations [Pa] and (with a cubic-like φ) in the theory of phase transitions [BS, E2, MTT2].

Since problem (1.1) is ill-posed, several regularizations of it have been considered [Sl, Sm]. Here we are concerned with the following regularization:

$$\begin{cases} U_t = [\varphi(U)]_{xx} + \varepsilon[\psi(U)]_{txx} & \text{in } Q \\ \varphi(U) + \varepsilon[\psi(U)]_t = 0 & \text{in } \partial\Omega \times (0, T] \\ U = U_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{1.4}$$

and its limit as the regularization parameter $\varepsilon > 0$ goes to zero. The *increasing* odd function $\psi : \mathbb{R} \to \mathbb{R}$ in (1.4) is related to φ by several assumptions. In fact, the following will be assumed:

$$(H_2)\qquad \begin{cases} \text{(i)} & \psi \in C^\infty(\mathbb{R}),\ \psi' > 0 \text{ in } \mathbb{R},\ \psi \text{ odd}, \\ & \psi(s) \to \gamma \text{ as } s \to \infty \quad \text{for some } \gamma \in (0, \infty); \\ \text{(ii)} & \psi^{(j)} \in L^\infty(\mathbb{R}) \quad \text{for any } j \in \mathbb{N}; \\ \text{(iii)} & \psi''(s) \le 0 \text{ for any } s \ge s_0, \quad \text{for some } s_0 > 0; \\ \text{(iv)} & |\varphi'| \le k_1 \psi' \text{ in } \mathbb{R} \quad \text{for some } k_1 > 0; \\ \text{(v)} & \left| \left(\dfrac{\varphi'}{\psi'} \right)' \right| \le k_2 \psi' \text{ in } \mathbb{R} \quad \text{for some } k_2 > 0; \\ \text{(vi)} & \dfrac{|\varphi''|}{(\psi')^2} \le k_3 \text{ in } \mathbb{R} \quad \text{for some } k_3 > 0. \end{cases}$$

Notations similar to those used in (H_1) for φ are used above for ψ. Observe that (H_2)-(i) and (iii) imply $\psi'(s) \to 0$ as $s \to \infty$. By abuse of notation, we shall also denote by ψ the extension of ψ to $\overline{\mathbb{R}}$ defined by setting $\psi(\infty) := \gamma$.

The term $\varepsilon[\psi(U)]_{txx}$ in the first equation of (1.4) arises as a formal first order approximation to a modified version of (1.1), which takes into account *time delay* effects ([BBDU]; see also [A] and references therein). By analogy with the theory of hyperbolic conservation laws, we call problem (1.4) the *viscous problem* associated with (1.1), and its limit as $\varepsilon \to 0$ the *vanishing viscosity limit*. Observe that the first equation in (1.4) is *degenerate quasiparabolic* [BBDU], for $\psi'(s) \to 0$ as $s \to \infty$. This makes an important difference with respect to the *Sobolev regularization*, which formally corresponds to the choice $\psi(s) = s$ [Sm]. In fact, for a function φ satisfying assumption (H_1) the only bounded invariant domain in $\mathbb{R}_+$ of problem (1.1) is $[0, \alpha]$, where problem (1.1) is well posed. Therefore, to study (1.1) in the general case, unbounded values of U must be considered.

2. Mathematical preliminaries

We denote by $\mathscr{M}(\Omega)$ (respectively, $\mathscr{M}(\mathbb{R})$) the space of Radon measures on Ω (respectively, on $\mathbb{R}$), and by $\mathscr{M}^+(\Omega)$ (respectively, $\mathscr{M}^+(\mathbb{R})$) the cone of positive Radon measures on Ω (respectively, on $\mathbb{R}$). For any $\mu \in \mathscr{M}(\Omega)$ we denote by μ_r and μ_s the density of the absolutely continuous part, respectively the singular part of μ with respect to the Lebesgue measure on Ω. Moreover, we denote by $\langle \cdot, \cdot \rangle_\Omega$ (respectively, $\langle \cdot, \cdot \rangle_{\mathbb{R}}$) the duality map between the space $\mathscr{M}(\Omega)$ (respectively, $\mathscr{M}(\mathbb{R})$) and the space $C_c(\Omega)$ (respectively, $C_c(\mathbb{R})$) of continuous functions with compact support.

By $\mathscr{M}(\overline{\Omega})$ we denote the space of Radon measures $\mu \in \mathscr{M}(\mathbb{R})$ such that $\operatorname{supp} \mu \subseteq \overline{\Omega}$. For any $\mu \in \mathscr{M}(\overline{\Omega})$ we set

$$\|\mu\|_{\mathscr{M}(\overline{\Omega})} := \|\mu\|_{\mathscr{M}(\mathbb{R})}$$

(observe that $|\mu|(\mathbb{R}) = |\mu|(\overline{\Omega}) < \infty$). For any $\mu \in \mathscr{M}(\overline{\Omega})$ and any $\zeta \in C(\overline{\Omega})$ we also define

$$\langle \mu, \zeta \rangle_{\overline{\Omega}} := \langle \mu, \tilde{\zeta} \rangle_{\mathbb{R}},$$

where $\tilde{\zeta} \in C_c(\mathbb{R})$ is any continuous function with compact support such that $\tilde{\zeta} = \zeta$ in $\overline{\Omega}$. Observe that the duality map $\langle \mu, \zeta \rangle_\Omega$ is well defined for any $\zeta \in C_0(\Omega) := \{\zeta \in C(\overline{\Omega}) \mid \zeta = 0 \text{ on } \partial\Omega\}$, and there holds

$$\langle \mu, \zeta \rangle_\Omega = \langle \mu, \zeta \rangle_{\overline{\Omega}}.$$

Similar notations are used for the space of Radon measures on Q, $\overline{Q}$ and $\mathbb{R}^2$.

We denote by $L^\infty((0,T); \mathcal{M}^+(\overline{\Omega}))$ the set of positive Radon measures $U \in \mathcal{M}^+(\overline{Q})$ which satisfy the following property: for almost every $t \in \mathbb{R}$ there exists a measure $U(\cdot, t) \in \mathcal{M}^+(\overline{\Omega})$, $U(\cdot, t) = 0$ if $t \notin [0, T]$, such that

(i) for any $\zeta \in C(\overline{Q})$ the map $t \to \langle U(\cdot, t), \zeta(\cdot, t) \rangle_{\overline{\Omega}}$ is Lebesgue measurable, and

$$\langle U, \zeta \rangle_{\overline{Q}} = \int_0^T \langle U(\cdot, t), \zeta(\cdot, t) \rangle_{\overline{\Omega}} \, dt; \tag{2.5}$$

(ii) there exists a constant $C > 0$ such that

$$\operatorname*{ess\,sup}_{t \in (0,T)} \|U(\cdot, t)\|_{\mathcal{M}(\overline{\Omega})} \le C.$$

Denoting by $U_r \in L^1(Q)$, $U_r \ge 0$ and by $U_s \in \mathcal{M}^+(\overline{Q})$ the density of the absolutely continuous part, respectively the singular part of U with respect to the Lebesgue measure over $\mathbb{R}^2$, equality (2.5) implies for any $\zeta \in C(\overline{Q})$

$$\begin{aligned} \langle U_r, \zeta \rangle_{\overline{Q}} &= \iint_Q U_r \zeta \, dx \, dt, \\ \langle U_s, \zeta \rangle_{\overline{Q}} &= \int_0^T \langle U_s(\cdot, t), \zeta(\cdot, t) \rangle_{\overline{\Omega}} \, dt. \end{aligned} \tag{2.6}$$

3. The regularized problem

Concerning the initial data U_0, we shall always assume the following:

$$(H_3) \quad \begin{cases} \text{(i)} \;\; U_0 \in \mathcal{M}^+(\overline{\Omega}); \\ \text{(ii)} \;\; \text{there exists a family } \{U_{0\kappa}\} \subseteq C_c^\infty(\Omega), \quad U_{0\kappa} \ge 0, \\ \qquad \|U_{0\kappa}\|_{L^1(\Omega)} \le \|U_0\|_{\mathcal{M}(\overline{\Omega})} \quad \text{for any } \kappa > 0, \text{ such that as } k \to 0: \\ \qquad \text{(a)} \;\; \int_\Omega U_{0\kappa} \zeta \, dx \to \langle U_0, \zeta \rangle_{\overline{\Omega}} \quad \text{for any } \zeta \in C(\overline{\Omega}), \\ \qquad \text{(b)} \;\; \psi(U_{0\kappa}) \rightharpoonup \psi(U_{0r}) \quad \text{in } H_0^1(\Omega), \\ \qquad \text{(c)} \;\; \kappa U_{0\kappa} \rightharpoonup 0 \quad \text{in } H_0^1(\Omega) \end{cases}$$

Let us make the following definition.

Definition 3.1. By a solution of problem (1.4) we mean any $U^\varepsilon \in \mathcal{M}^+(\overline{Q})$ such that:

(i) $U^\varepsilon \in L^\infty((0,T);\mathcal{M}^+(\overline{\Omega}))$;

(ii) $\varphi(U_r^\varepsilon)$, $\psi(U_r^\varepsilon) \in L^\infty((0,T);H_0^1(\Omega))$, and $[\psi(U_r^\varepsilon)]_t \in L^2((0,T);H_0^1(\Omega))$; moreover,

$$\psi(U_r^\varepsilon)(x,0) = \psi(U_{0r})(x) \quad \text{for any } x \in \overline{\Omega}; \tag{3.1}$$

(iii) there holds

$$\operatorname{supp} U_s^\varepsilon \subseteq \mathscr{S} := \{(x,t) \in \overline{Q} \mid \psi(U_r^\varepsilon)(x,t) = \gamma\}; \tag{3.2}$$

(iv) there holds

$$\begin{aligned}\iint_Q U_r^\varepsilon \zeta_t \, dx\, dt + \int_0^T \langle U_s^\varepsilon(\cdot,t), \zeta_t(\cdot,t)\rangle_\Omega \, dt \\ = \iint_Q \{[\varphi(U_r^\varepsilon)]_x \zeta_x + \varepsilon[\psi(U_r^\varepsilon)]_{tx}\zeta_x\}\, dx\, dt - \langle U_0, \zeta(\cdot,0)\rangle_\Omega\end{aligned} \tag{3.3}$$

for any $\zeta \in C^1([0,T];H_0^1(\Omega))$, $\zeta(\cdot,T) = 0$ in $\overline{\Omega}$.

Remark 3.1. In the above Definition 3.1, as always in the following, we identify $\psi(U_r^\varepsilon) \in L^\infty(\Omega)$ with its *continuous representative* $w \in C(\overline{Q})$, $\psi(U_r^\varepsilon) \equiv w$, which exists by Definition 3.1-(ii). Therefore the set $\mathscr{S}$ defined in (3.2) is closed.

Similarly, since $\varphi(U_r^\varepsilon) \equiv \varphi(\psi^{-1}(\psi(U_r^\varepsilon)))$ in Q, by assumptions (H_2)-(iv) and (v) there holds $[\varphi(U_r^\varepsilon)]_t \in L^2((0,T);H_0^1(\Omega))$. Together with Definition 3.1-(ii), this implies $\varphi(U_r^\varepsilon) \in C(\overline{Q})$. Since $\varphi(s) \to 0$ as $s \to \infty$, equality (3.5) below implies $\varphi(U_r^\varepsilon) = 0$ on the set $\mathscr{S}$.

The following result shows that for any $\varepsilon > 0$ there exists a unique measure-valued function U^ε, which solves problem (1.4) in the sense of Definition 3.1. The proof makes use of a family of approximating problems, defined by a regularization of ψ and U_0, and of uniform a priori estimates of their solutions.

Theorem 3.1. *Let assumptions* (H_1)–(H_3) *be satisfied. Then there exists a unique solution* U^ε *of problem* (1.4). *Moreover,*

(i) *for almost every* $t \in (0,T)$ *there holds*

$$\|U_r^\varepsilon(\cdot,t)\|_{L^1(\Omega)} + \|U_s^\varepsilon(\cdot,t)\|_{\mathscr{M}(\overline{\Omega})} \le \|U_0\|_{\mathscr{M}(\overline{\Omega})}; \tag{3.4}$$

(ii) $U_r^\varepsilon \in H^1(Q_0)$ *for any open subset* $Q_0 \subseteq Q$ *such that* $\operatorname{dist}(\overline{Q}_0, \mathscr{S}) > 0$. *Moreover,* $U_r^\varepsilon \in C(\overline{Q}\backslash\mathscr{S})$ *and*

$$\lim_{dist((x,t),\mathscr{S})\to 0} U_r^\varepsilon(x,t) = \infty. \tag{3.5}$$

Since $U_r^\varepsilon \in L^1(Q)$, by (3.5) it is reasonable to expect that the set $\mathscr{S}$ defined in (3.2) has zero Lebesgue measure. On this subject the following holds.

THEOREM 3.2. *Let assumptions* (H_1)–(H_3) *be satisfied. Let* U^ε *be the solution of problem* (1.4) *given by Theorem* 3.1. *Then the set* $\mathscr{S}$ *defined by* (3.2)*:*

(i) *has zero Lebesgue measure;*
(ii) *has a strictly positive distance from* $\partial\Omega \times [0, T] \subseteq \partial Q$.

Set

$$V_r^\varepsilon := \varphi(U_r^\varepsilon) + \varepsilon[\psi(U_r^\varepsilon)]_t \quad (\varepsilon > 0). \tag{3.6}$$

It can be proven that $V_r^\varepsilon \in L^\infty(Q) \cap L^2((0, T); H_0^1(\Omega))$ [ST1, Lemma 4.6]; moreover, there exists a constant $C > 0$ (which does not depend on ε) such that

$$\|V_r^\varepsilon\|_{L^\infty(Q)} + \|V_r^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C \tag{3.7}$$

for any $\varepsilon > 0$ [ST1, Lemmata 4.6 and 4.7]. Therefore equality (3.3) reads

$$\begin{aligned} \iint_Q U_r^\varepsilon \zeta_t \, dx\, dt + \int_0^T \langle U_s^\varepsilon(\cdot, t), \zeta_t(\cdot, t)\rangle_\Omega \, dt \\ = \iint_Q V_{rx}^\varepsilon \zeta_x \, dx\, dt - \langle U_0, \zeta(\cdot, 0)\rangle_\Omega. \end{aligned} \tag{3.8}$$

As a consequence of Theorems 3.1–3.2, we can prove that the density U_r^ε satisfies the first equation of problem (1.4) in a suitable weak sense, out of a set of arbitrarily small Lebesgue measure. In fact, the following holds.

THEOREM 3.3. *Let assumptions* (H_1)–(H_3) *be satisfied. Let* U^ε *be the solution of problem* (1.4) *given by Theorem* 3.1, $\mathscr{S}$ *the set defined in* (3.2) *and* $A \subseteq Q$ *any open set such that* $\operatorname{dist}(\bar{A}, \mathscr{S}) > 0$. *Then:*

(i) U_{rt}^ε, $V_{rxx}^\varepsilon \in L^2(A)$, *and*

$$U_{rt}^\varepsilon = V_{rxx}^\varepsilon \quad \text{in } L^2(A); \tag{3.9}$$

(ii) *for almost every* $t \in (0, T)$ *there holds*

$$\operatorname{supp} U_s^\varepsilon(\cdot, t) \subseteq \{x \in \overline{\Omega} \mid \psi(U_r^\varepsilon)(x, t) = \gamma\} \subseteq \{x \in \Omega \mid V_r^\varepsilon(x, t) = 0\} \tag{3.10}$$

where the set $\{x \in \overline{\Omega} \mid \psi(U_r^\varepsilon)(x, t) = \gamma\}$ *has zero Lebesgue measure.*

Further we prove that the couple $(U_r^\varepsilon, V_r^\varepsilon)$ satisfies in a weak sense a family of infinitely many inequalities, which we call *viscous entropy inequalities* by analogy with the case of hyperbolic conservation laws. Similar inequalities are known to hold for the Sobolev regularization, both for a cubic-like φ [NP] and for a φ of Perona–Malik type [Sm], and play an important role when studying the vanishing viscosity limit. With respect to [NP], [Sm] we prove an improved version of these inequalities, which holds for almost every $t \in (0, T)$ (in this connection, see [ST]).

Define for any $g \in C^1(\mathbb{R})$

$$G(z) := \int_0^z g(\varphi(s))\,ds \quad (z \in \mathbb{R}). \tag{3.11}$$

Then the following holds.

THEOREM 3.4. *Let U^ε be the solution of problem* (1.4) *given by Theorem* 3.1. *Let $g \in C^1([0, \varphi(\alpha)])$, $g' \geq 0$, $g(0) = 0$. Then $G(U_r^\varepsilon) \in C(\overline{Q})$, and for any $t_1, t_2 \in [0, T]$, $t_1 < t_2$, there holds*

$$\begin{aligned} &\int_\Omega G(U_r^\varepsilon)(x, t_2)\zeta(x, t_2)\,dx - \int_\Omega G(U_r^\varepsilon)(x, t_1)\zeta(x, t_1)\,dx \\ &\qquad \leq \int_{t_1}^{t_2} \int_\Omega [G(U_r^\varepsilon)\zeta_t - g(V_r^\varepsilon)V_{rx}^\varepsilon\zeta_x - g'(V_r^\varepsilon)(V_{rx}^\varepsilon)^2\zeta]\,dx\,dt \end{aligned} \tag{3.12}$$

for any $\zeta \in C^1([0, T]; H_0^1(\Omega))$, $\zeta \geq 0$.

A major consequence of the above family of inequalities is that the singular measure $U_s^\varepsilon(\cdot, t)$ is *nondecreasing in time*. This is the content of the following

THEOREM 3.5. *Let assumptions (H_1)–(H_3) be satisfied. Let U^ε be the solution of problem* (1.4) *given by Theorem* 3.1. *Then for any $\eta \in H_0^1(\Omega)$, $\eta \geq 0$ there holds*

$$\langle U_{0s}, \eta \rangle_\Omega \leq \langle U_s^\varepsilon(\cdot, t), \eta \rangle_\Omega \tag{3.13}$$

for almost every $t \in (0, T)$, and also

$$\langle U_s^\varepsilon(\cdot, t_1), \eta \rangle_\Omega \leq \langle U_s^\varepsilon(\cdot, t_2), \eta \rangle_\Omega \tag{3.14}$$

for almost every $t_1 \leq t_2$, $t_1, t_2 \in (0, T)$.

By the above result, if the singular measure $U_s^\varepsilon(\cdot, t)$ exists at some time $\bar{t} \geq 0$, it also exists at any later time. It is natural to wonder whether the singular measure U_s^ε exists at all. As shown in [BBDU], $U_s^\varepsilon(\cdot, t)$ can arise at some time $t = \bar{t} > 0$ even if the initial data U_0 are regular. On the other hand, it can be proven that for a class of smooth initial data and for a suitable choice of ψ the singular measure is always absent [ST1, Theorem 2.8]. Expectedly, this depends on the order of degeneracy of ψ (namely, on the rate of growth of ψ') at infinity.

4. THE VANISHING VISCOSITY LIMIT

Consider the sets

$$\begin{aligned} S_1 &:= \{(u, \varphi(u)) \mid u \in [0, \alpha]\} \equiv \{(s_1(v), v) \mid v \in [0, \varphi(\alpha)]\}, \\ S_2 &:= \{(u, \varphi(u)) \mid u \in [\alpha, \infty)\} \equiv \{(s_2(v), v) \mid v \in (0, \varphi(\alpha)]\}. \end{aligned} \tag{4.15}$$

Following [Pl1], we always assume in the sequel:

$$(S) \qquad \begin{cases} \text{The functions } s_1', s_2' \text{ are linearly independent} \\ \text{on any open subset of the interval } (0, \varphi(\alpha)). \end{cases}$$

Our purpose is to study the behaviour and the limiting points as $\varepsilon \to 0$ of the families $\{U_r^\varepsilon\}$, $\{U_s^\varepsilon\}$, $\{V_r^\varepsilon\}$ and $\{\varphi(U_r^\varepsilon)\}$ considered above. The main question is whether describing the limiting points (in some topology) of the family $\{U^\varepsilon\}$ of its solutions enables us to define in some suitable sense measure-valued solutions of the original ill-posed problem (1.1).

To this aim, observe that by inequalities (3.4) and (3.7) there exist a sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, $U \in \mathscr{M}^+(\bar{Q})$, $\mu_1 \in \mathscr{M}^+(\bar{Q})$, $V_r \in L^\infty(Q) \cap L^2((0,T); H_0^1(\Omega))$ such that

$$\langle U^{\varepsilon_k}, f \rangle_{\bar{Q}} \to \langle U, f \rangle_{\bar{Q}} \tag{4.16}$$

$$\langle U_s^{\varepsilon_k}, f \rangle_{\bar{Q}} \to \langle \mu_1, f \rangle_{\bar{Q}} \tag{4.17}$$

for any $f \in C(\bar{Q})$, and

$$V_r^{\varepsilon_k} \overset{*}{\rightharpoonup} V_r \quad \text{in } L^\infty(Q), \tag{4.18}$$

$$V_r^{\varepsilon_k} \rightharpoonup V_r \quad \text{in } L^2((0,T); H_0^1(\Omega)). \tag{4.19}$$

Since $U^\varepsilon, U_s^\varepsilon \in L^\infty((0,T); \mathscr{M}^+(\bar{\Omega}))$, it follows easily from (3.4), (4.16), (4.17) that $U, \mu_1 \in L^\infty((0,T); \mathscr{M}^+(\bar{\Omega}))$ as well, and there holds

$$\langle U^{\varepsilon_k}(\cdot,t), \rho \rangle_\Omega \to \langle U(\cdot,t), \rho \rangle_\Omega \tag{4.20}$$

for any function $\rho \in C_0(\Omega)$ and almost every $t \in (0,T)$ (see [ST1, Propositions 4.2 and 4.3]).

Letting $\varepsilon_k \to 0$ in equality (3.8) (written with $\varepsilon = \varepsilon_k$), by (3.4), (4.19), (4.20) we obtain the following result.

THEOREM 4.1. *Let $U \in L^\infty((0,T); \mathscr{M}^+(\bar{\Omega}))$ be the limiting measure in* (4.16) *and $V_r \in L^\infty(Q) \cap L^2((0,T); H_0^1(\Omega))$ the limiting function in* (4.18). *Then*

$$\int_0^T \langle U(\cdot,t), \zeta_t(\cdot,t) \rangle_\Omega \, dt = \iint_Q V_{rx} \zeta_x \, dx \, dt - \langle U_0, \zeta(\cdot,0) \rangle_\Omega \tag{4.21}$$

for any $\zeta \in C^1([0,T]; H_0^1(\Omega))$, $\zeta(\cdot,T) = 0$ in Ω.

Let us investigate in more detail the structure of the singular term U. Consider first the case $U_s^\varepsilon = 0$ for any $\varepsilon > 0$ (for instance, this is the case if $U_0 \in H_0^1(\Omega)$ and ψ "grows slowly at infinity"; see [ST1, Theorem 2.8]). Since the sequence $\{U_r^{\varepsilon_k}\} = \{U^{\varepsilon_\kappa}\}$ is uniformly bounded in $L^1(Q)$ by inequality (3.4), we can consider the associated sequence $\{\tau^{\varepsilon_k}\}$ of *Young measures* (e.g., see [GMS, V]). Let τ denote the *narrow limit* of the sequence $\{\tau^{\varepsilon_k}\}$ and $\nu_{(x,t)}$ its *disintegration*, defined for almost every $(x,t) \in Q$. By the Prohorov Theorem (e.g., see [V]) we have the following result.

PROPOSITION 4.2. *Let $U^\varepsilon \in L^\infty((0,T);\mathscr{M}^+(\overline{\Omega}))$ be the unique solution of problem* (1.4), *and τ^ε the Young measure over $Q \times \mathbb{R}$ associated to the density U_r^ε of its absolutely continuous part* ($\varepsilon > 0$). *Then there exist a subsequence of the sequence $\{\varepsilon_k\}$ in* (4.16) (*denoted again by $\{\varepsilon_k\}$*) *and a Young measure τ on $Q \times \mathbb{R}$ such that:*

(i) *$\tau^{\varepsilon_k} \to \tau$ narrowly in $Q \times \mathbb{R}$;*
(ii) *for any $f \in C(\mathbb{R})$ such that the sequence $\{f(U_r^{\varepsilon_k})\}$ is bounded in $L^1(Q)$ and equi-integrable there holds*

$$f(U_r^{\varepsilon_k}) \rightharpoonup f_* \quad \text{in } L^1(Q), \tag{4.22}$$

where

$$f_*(x,t) := \int_{[0,\infty)} f(\xi)\, d\nu_{(x,t)}(\xi) \quad ((x,t) \in Q). \tag{4.23}$$

In general, the sequence $\{U_r^{\varepsilon_k}\}$ need not be equi-integrable in the cylinder Q, thus the above result cannot be applied with $f(\xi) = \xi$. However, we can associate to $\{U_r^{\varepsilon_k}\}$ an equi-integrable subsequence "by removing sets of small measure" and using the so-called *biting convergence* [GMS, V]. This leads to the following

PROPOSITION 4.3. *Let the assumptions of Proposition* 4.2 *be satisfied. Then there exist a subsequence $\{U_r^{\varepsilon_j}\} \equiv \{U_r^{\varepsilon_{k_j}}\} \subseteq \{U_r^{\varepsilon_k}\}$ and a sequence $\{A_j\}$ of measurable sets, with $A_{j+1} \subseteq A_j \subseteq Q$ for any $j \in \mathbb{N}$ and Lebesgue measure $|A_j| \to 0$ as $j \to \infty$, such that:*

(i) *the sequence $\{U_r^{\varepsilon_j}\chi_{Q\setminus A_j}\}$ is equi-integrable and*

$$U_r^{\varepsilon_j}\chi_{Q\setminus A_j} \rightharpoonup W := \int_{[0,\infty)} \xi\, d\nu(\xi) \quad \text{in } L^1(Q) \tag{4.24}$$

(*χ_E denoting the characteristic function of a set $E \subseteq Q$*);
(ii) *there exists a measure $\mu_2 \in L^\infty((0,T);\mathscr{M}^+(\overline{\Omega}))$ such that*

$$\langle U_r^{\varepsilon_j}\chi_{A_j}, f\rangle_{\overline{Q}} \to \langle \mu_2, f\rangle_{\overline{Q}} \tag{4.25}$$

for any $f \in C(\overline{Q})$.

By the above results, there holds

$$\begin{aligned}\int_0^T \langle U(\cdot,t),\zeta(\cdot,t)\rangle_{\overline{\Omega}}\, dt &= \lim_{j\to\infty} \langle U^{\varepsilon_j}(\cdot,t),\zeta\rangle_{\overline{Q}} \\ &= \lim_{j\to\infty} \iint_Q U_r^{\varepsilon_j}\zeta\, dx\, dt = \iint_Q W\zeta\, dx\, dt + \langle \mu_2,\zeta\rangle_{\overline{Q}} \\ &= \int_0^T \left\{ \int_\Omega W(x,t)\zeta(x,t)\, dx + \langle \mu_2(\cdot,t),\zeta(\cdot,t)\rangle_{\overline{\Omega}} \right\} dt\end{aligned}$$

for any $\zeta \in C(\overline{Q})$. Choosing in the above equality $\zeta(x,t) = \rho(x)h(t)$, with $\rho \in C(\overline{\Omega})$ and $h \in C([0,T])$, we obtain immediately

$$U = W + \mu_2 \quad \text{in } L^\infty((0,T);\mathcal{M}^+(\overline{\Omega})). \tag{4.26}$$

The above equality shows that the limiting quantity U in (4.16) is in general measure-valued, even if $U^\varepsilon = U_r^\varepsilon \in L^1(Q)$ for any $\varepsilon > 0$. Let us mention that when φ is cubic-like and $\psi(s) = s$ a uniform L^∞-estimate of the family $\{U^\varepsilon\}$ holds, which implies its equi-integrability (see [NP, Pl1]). Therefore, the appearance of the measure μ_2 in (4.26) is related to the behaviour of φ at infinity (see assumption (H_1)-(i)).

Similar arguments can be used when $U_s^\varepsilon \neq 0$. In this general case, taking (4.17) into account we obtain the following result.

THEOREM 4.4. *Let $U \in L^\infty((0,T);\mathcal{M}^+(\overline{\Omega}))$ be the limiting measure in* (4.16). *Let $W \in L^1(Q)$, $W \geq 0$ be the barycenter of the Young disintegration ν which appears in* (4.24), *and $\mu_1, \mu_2 \in L^\infty((0,T);\mathcal{M}^+(\overline{\Omega}))$ the measures in* (4.17) *and* (4.25). *Set*

$$\mu := \mu_1 + \mu_2. \tag{4.27}$$

Then:

(i) *there holds*

$$U = W + \mu \quad \text{in } L^\infty((0,T);\mathcal{M}^+(\overline{\Omega})); \tag{4.28}$$

(ii) *for almost every $t \in (0,T)$ there holds*

$$\operatorname{supp} \mu(\cdot,t) \subseteq \mathcal{T}_t := \{x \in \overline{\Omega} \mid V_r(x,t) = 0\}. \tag{4.29}$$

In the light of the above result, the above measure μ is the sum of two contributions, one coming from the convergence in $\mathcal{M}^+(\overline{Q})$ of the sequence $\{U_s^{\varepsilon_k}\}$ of the singular parts of the solutions of the viscous problem (1.4), the other resulting from the biting convergence of the sequence $\{U_r^{\varepsilon_k}\}$ of the corresponding regular parts.

We can now point out more clearly the relationship between U and V_r. Set

$$Q_{1k} := \{(x,t) \in Q \mid U_r^{\varepsilon_k}(x,t) \leq \alpha\}, \quad Q_{2k} := Q \backslash Q_{1k}, \tag{4.30}$$

where $\{\varepsilon_k\}$, $\varepsilon_k \to 0$ is the sequence in (4.16), and denote by $\chi_{Q_{1k}}, \chi_{Q_{2k}}$ the characteristic functions of these sets. Since the sequences $\{\chi_{Q_{1k}}\}$, $\{\chi_{Q_{2k}}\}$ are uniformly bounded in $L^\infty(Q)$, there exist two subsequences (denoted again $\{\chi_{Q_{1k}}\}$, $\{\chi_{Q_{2k}}\}$ for simplicity) and a function $\lambda \in L^\infty(Q)$, $0 \leq \lambda \leq 1$ such that

$$\chi_{Q_{1k}} \overset{*}{\rightharpoonup} \lambda, \quad \chi_{Q_{2k}} \overset{*}{\rightharpoonup} 1 - \lambda \quad \text{in } L^\infty(Q). \tag{4.31}$$

Then we have the following result, which shows that for almost every $(x,t) \in Q$ the measure $\nu_{(x,t)}$ is *atomic* with support

$$\operatorname{supp} \nu_{(x,t)} = \begin{cases} \{s_1(V_r(x,t)), s_2(V_r(x,t))\} & \text{if } V_r(x,t) > 0, \\ \{0\} & \text{if } V_r(x,t) = 0. \end{cases}$$

THEOREM 4.5. *Let ν be the disintegration of the limiting Young measure τ over $Q \times \mathbb{R}$ given by Proposition* 4.2, *$\lambda \in L^\infty(Q)$, $0 \le \lambda \le 1$ the limiting function in* (4.31), *$V_r \in L^\infty(Q) \cap L^2((0,T); H_0^1(\Omega))$ the limiting function in* (4.18)–(4.19), *and s_1, s_2 the functions in* (4.15). *Then for almost every $(x,t) \in Q$*

$$\nu_{(x,t)} = \begin{cases} \lambda(x,t)\delta(\cdot - s_1(V_r(x,t))) \\ \qquad + [1-\lambda(x,t)]\delta(\cdot - s_2(V_r(x,t))) & \text{if } V_r(x,t) > 0, \\ \delta(\cdot - 0) & \text{if } V_r(x,t) = 0 \end{cases} \tag{4.32}$$

with $\lambda = 1$ almost everywhere in the set $\mathcal{T} := \{(x,t) \in \bar{Q} \mid V_r(x,t) = 0\}$.

As a consequence of (4.24) and (4.32) we obtain the following equality, which together with (4.28) makes the relationship between U and V_r clear:

$$W(x,t) = \begin{cases} \lambda(x,t)s_1(V_r(x,t)) + [1-\lambda(x,t)]s_2(V_r(x,t)) & \text{if } V_r(x,t) > 0, \\ 0 & \text{if } V_r(x,t) = 0 \end{cases}$$

for almost every $(x,t) \in Q$, with $\lambda \in L^\infty(Q)$, $0 \le \lambda \le 1$, $\lambda = 1$ almost everywhere in the set $\mathcal{T}$. This can be interpreted by saying that, when $V_r(x,t) > 0$, the function W takes the *fraction* $\lambda(x,t)$ of its value at (x,t) on the "stable branch" s_1 of the graph of φ, and the fraction $1 - \lambda(x,t)$ on the "unstable branch" s_2 (see [MTT1, Pl1, Pl2, Pl3] for an analogous result when φ is cubic-like, and [Sm] for the case of Sobolev regularization).

Let us state another result, which is closely related to Theorem 4.5. Since the family $\{\varphi(U_r^\varepsilon)\}$ is uniformly bounded in $L^\infty(Q)$, we can consider the associated family $\{\theta^\varepsilon\}$ of Young measures. As in Proposition 4.2, there exist a sequence $\{\theta^{\varepsilon_k}\}$ and a Young measure θ over $Q \times \mathbb{R}$ such that

$$\theta^{\varepsilon_k} \to \theta \quad \text{narrowly in } Q \times \mathbb{R}. \tag{4.33}$$

Let $\sigma \equiv \sigma_{(x,t)}$ denote the disintegration of the Young measure θ, with support $\operatorname{supp} \sigma_{(x,t)} \subseteq [0, \varphi(\alpha)]$ for almost every $(x,t) \in Q$ (see assumption (H_1)-(ii)). Then we have the following result.

THEOREM 4.6. *Let $V_r \in L^\infty(Q) \cap L^2((0,T); H_0^1(\Omega))$ be the limiting function in* (4.18)–(4.19), *and let σ be the disintegration of the limiting Young measure θ over $Q \times \mathbb{R}$ which appears in* (4.33). *Then for almost every $(x,t) \in Q$*

$$\sigma_{(x,t)} = \delta(\cdot - V_r(x,t)). \tag{4.34}$$

Finally, we establish a limiting version of the viscous entropy inequalities stated in Theorem 3.4 above. Set

$$G_*(x,t) := \begin{cases} \lambda(x,t)G(s_1(V_r(x,t))) \\ \qquad + [1-\lambda(x,t)]G(s_2(V_r(x,t))) & \text{if } V_r(x,t) > 0, \\ 0 & \text{if } V_r(x,t) = 0, \end{cases} \tag{4.35}$$

where G is the function defined in (3.11).

Theorem 4.7. *Let G be the function* (3.11) *with $g \in C^1([0,\varphi(\alpha)])$, $g(0) = 0$, $g' \geq 0$. Then for almost every $t_1, t_2 \in (0,T)$, $t_1 \leq t_2$, for any g as above and any $\zeta \in C^1([0,T]; H_0^1(\Omega))$, $\zeta \geq 0$ there holds*

$$\int_\Omega G_*(x,t_2)\zeta(x,t_2)\,dx - \int_\Omega G_*(x,t_1)\zeta(x,t_1)\,dx \leq \int_{t_1}^{t_2}\int_\Omega \{G_*\zeta_t - g(V_r)V_{rx}\zeta_x - g'(V_r)V_{rx}^2\zeta\}(x,t)\,dx\,dt. \tag{4.36}$$

Inequalities (4.36) are referred to as *entropy inequalities*. In particular, relying on them we can prove the following result, which is the analogous of Theorem 3.5.

Theorem 4.8. *Let $\mu \in L^\infty((0,T); \mathscr{M}^+(\overline{\Omega}))$ be the measure defined by* (4.27). *Then for any $\rho \in H_0^1(\Omega)$, $\rho \geq 0$ there holds:*

(i) *for almost every $t \in (0,T)$*

$$\langle U_{0s}, \rho\rangle_\Omega \leq \langle \mu(\cdot,t), \rho\rangle_\Omega; \tag{4.37}$$

(ii) *for almost every $t_1, t_2 \in (0,T)$, $t_1 < t_2$,*

$$\langle \mu(\cdot,t_1), \rho\rangle_\Omega \leq \langle \mu(\cdot,t_2), \rho\rangle_\Omega. \tag{4.38}$$

The above inequality points out a remarkable *nondecreasing property* of the map $t \to \mu(\cdot,t)$ $(t \in (0,T))$, which implies that singularities can appear and spread as time progresses.

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F. Smarrazzo
Dipartimento di Matematica "G. Castelnuovo"
Università di Roma "La Sapienza"
P.le A. Moro 5
I-00185 Roma, Italy
smarrazz@mat.uniroma1.it

A. Tesei
Dipartimento di Matematica "G. Castelnuovo"
Università di Roma "La Sapienza"
P.le A. Moro 5
I-00185 Roma, Italy
tesei@mat.uniroma1.it