Rend. Lincei Mat. Appl. 22 (2011), 207–222
DOI 10.4171/RLM/596

Functional Analysis — *Critical points for Sobolev homeomorphisms*, by CARLO SBORDONE and ROBERTA SCHIATTARELLA, communicated on 14 January 2011.

Dedicated to the memory of Professor Giovanni Prodi.

ABSTRACT. — We consider a class of homeomorphisms $f : \Omega \subset \mathbb{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^2$ of the Sobolev space $\mathscr{W}^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^2)$ whose Jacobian may vanish on a set of positive measure but cannot be zero a.e. in Ω. This class is defined by the bi-Sobolev condition

(1) $$f \text{ and } f^{-1} \in \mathscr{W}^{1,1}_{\text{loc}}$$

and reveals useful also in the theory of changes of variables for Sobolev functions.

KEY WORDS: Sobolev mapping, bounded variation, homeomorphism.

AMS 2000 MATHEMATICS SUBJECT CLASSIFICATION: Primary 30C65, Secondary 46E35.

1. INTRODUCTION

Suppose that Ω and Ω' are planar domains and that $f = (u, v) : \Omega \xrightarrow{\text{onto}} \Omega'$ is a Sobolev homeomorphism, that is f is a continuous bijection that belongs to $\mathscr{W}^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^2)$.

For $p \geq 1$, $\mathscr{W}^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^2)$ is the space of mappings $f : \Omega \to \mathbb{R}^2$ whose components u, v belong to the Sobolev space $\mathscr{W}^{1,p}_{\text{loc}}(\Omega)$ of $\mathscr{L}^p_{\text{loc}}$-functions which have locally p-integrable distributional derivatives.

By the Gehring-Lehto Theorem (see [2] Section 3.3) a Sobolev homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is almost everywhere differentiable in the classical sense, moreover the Jacobian determinant

$$J_f = u_x v_y - u_y v_x$$

is locally integrable and satisfies either $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e. (see [17]). We will always suppose $J_f(z) \geq 0$ at each point $z \in \Omega$ of differentiability.

In the following we will denote by C_f the zero set of the Jacobian of f, namely the *Borel* set

(1.1) $$\mathcal{C}_f = \{z \in \Omega : f \text{ is differentiable at } z \text{ and } J_f(z) = 0\}$$

This paper was performed while the first author was a Professor at the "Centro Interdisciplinare Linceo Beniamino Segre" of the Accademia Nazionale dei Lincei, Rome.

and by $\mathcal{C}_u$ and $\mathcal{C}_v$ the critical sets of the components u and v of f:

$$\mathcal{C}_u = \{z \in \Omega : u \text{ is differentiable at } z \text{ and } |\nabla u(z)| = 0\}, \tag{1.2}$$

$$\mathcal{C}_v = \{z \in \Omega : u \text{ is differentiable at } z \text{ and } |\nabla v(z)| = 0\}, \tag{1.3}$$

and we will consider their reciprocal relations and sizes.

Obviously we have $\mathcal{C}_u \cup \mathcal{C}_v \subset \mathcal{C}_f$ and we are legitimate to expect that regularity of the inverse map f^{-1} should guarantee restrictions on the measure of $\mathcal{C}_f$.

For example, the hypothesis $f^{-1} \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega'; \mathbb{R}^2)$ is sufficient to conclude that critical sets have zero measure:

$$|\mathcal{C}_u| = |\mathcal{C}_v| = |\mathcal{C}_f| = 0 \tag{1.4}$$

as we will prove below (Remark 2.7). Here we notice that there is an example ([27], see also [19] Section 6.5.6) of a Sobolev homeomorphism $f : Q_0 \overset{\text{onto}}{\longrightarrow} Q_0$, $Q_0 = (0,1)^2$ satisfying the condition

$$f^{-1} \in \mathscr{W}^{1,p}_{\mathrm{loc}} \backslash \mathscr{W}^{1,2}_{\mathrm{loc}} \quad \text{for any } 1 < p < 2$$

whose critical sets have positive measure:

$$|\mathcal{C}_u| = |\mathcal{C}_v| = |\mathcal{C}_f| > 0.$$

Notice that, however, it is $|f(\mathcal{C}_f \backslash Z)| = 0$ for some zero set $Z \subset \Omega$, according to Sard Lemma (see Lemma 2.5).

Remark 1.1. Let us now point out that, in the category of $\mathscr{W}^{1,p}$-homeomorphisms, i.e. of planar homeomorphisms that belong to $\mathscr{W}^{1,p}_{\mathrm{loc}}$, the case $1 \le p < 2$ is critical respect to the N-property of Lusin, i.e. that a function maps every sets of zero measure to a set of zero measure. As a matter of fact (see [2] Theorem 3.3.7) for $\mathscr{W}^{1,2}$-homeomorphisms we have the N-property.

It is also natural to try to characterize the most general class of Sobolev homeomorphisms that prevent the pathological situation:

$$|\mathcal{C}_f| = |\Omega| \tag{1.5}$$

that i.e. the Jacobian is zero a.e. in Ω.

Actually we will see that the assumption $f^{-1} \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega'; \mathbb{R}^2)$ guarantees the strict inequality

$$|\mathcal{C}_f| < |\Omega|. \tag{1.6}$$

(see Theorem 1.4).

In [14], for any $1 \le p < 2$ an amazing example is given of a Sobolev homeomorphism $f_0 : Q_0 \overset{\text{onto}}{\longrightarrow} Q_0$, $Q_0 = (0,1)^2$ such that $f_0 \in \mathscr{W}^{1,p}(Q_0; \mathbb{R}^2)$ and

$$\mathcal{C}_{f_0} = Q_0 \quad \text{a.e.};$$

that is

$$J_{f_0}(z) = 0 \quad \text{a.e. } z \in Q_0. \tag{1.7}$$

We wish to notice here that for such a homeomorphism the following negative conditions necessarily hold

$$f_0 \notin \mathscr{W}^{1,2}(Q_0; \mathbb{R}^2), \tag{1.8}$$

$$f_0^{-1} \notin \mathscr{W}^{1,1}(Q_0; \mathbb{R}^2). \tag{1.9}$$

Proof of (1.8). Since the area formula for the Sobolev homeomorphism f_0 holds up to a zero set $Z_0 \subset Q_0$, then

$$|Z_0| = 0 = \int_{Q_0 \backslash Z_0} J_{f_0}(z)\, dz = |f_0(Q_0 \backslash Z_0)| \tag{1.10}$$

(see [12]), moreover, being $f_0(Z_0) \cup f_0(Q_0 \backslash Z_0) = Q_0$, by (1.10) we deduce

$$|f_0(Z_0)| = |Q_0| = 1.$$

This means that f_0 sends a zero set into a set of full measure, it does not satisfy the N-condition, hence f_0 does not belong to $\mathscr{W}^{1,2}(Q_0; \mathbb{R}^2)$ (see Remark 1.1). $\square$

Proof of (1.9). The fact that f_0^{-1} cannot belong to $\mathscr{W}^{1,1}(Q_0; \mathbb{R}^2)$ derives from Theorem 1.3 and the fact ([14] Section 7) that there exists a Cantor set C_1 in $(0,1)^2$ of positive measure such that

$$|Df_0(z)| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in C_1,$$

hence f_0^{-1} cannot belong to $\mathscr{W}^{1,1}$. $\square$

To give precise statements we need a definition from [18].

Definition 1.1. The homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a bi-Sobolev map if f and f^{-1} are Sobolev homeomorphisms.

A sufficient condition that a Sobolev homeomorphism is a bi-Sobolev map is contained in the following

Theorem 1.2 ([15]). *Let $f : \Omega \xrightarrow{onto} \Omega'$ be a Sobolev homeomorphism satisfying the condition*

$$|\mathcal{C}_f| = 0 \quad a.e. \tag{1.11}$$

then, f is a bi-Sobolev map and

$$\int_\Omega |Df|\, dz = \int_{\Omega'} |Df^{-1}|\, dw.$$

We emphasize that condition (1.11) is not necessary for f to be a bi-Sobolev map. It can happen that bi-Sobolev maps have positive sets of critical points (see the mentioned example of [27]).

Notice also that bi-Sobolev maps escape the pathological equality (1.5) (see Theorem 1.4).

A first interesting property of a bi-Sobolev map $f = (u, v)$ is shown in the following (see [18]).

THEOREM 1.3. *If* $f : \Omega \xrightarrow{onto} \Omega'$ *is a bi-Sobolev map, then*

$$\mathcal{C}_u = \mathcal{C}_v = \mathcal{C}_f \quad a.e. \tag{1.12}$$

Here, for $A, B \subset \mathbb{R}^2$, by $A = B$ a.e. we mean that $|(A \backslash B) \cup (B \backslash A)| = 0$.

Let us consider the critical sets

$$\mathcal{Z}_x = \{z \in \Omega : |f_x(z)| = 0\}, \tag{1.13}$$

$$\mathcal{Z}_y = \{z \in \Omega : |f_y(z)| = 0\}. \tag{1.14}$$

In Section 5 we will give a simple direct proof of the following result which parallels Theorem 1.3:

THEOREM 1.4. *If* $f : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ *is a bi-Sobolev map, then*

$$\mathcal{Z}_x = \mathcal{Z}_y = \mathcal{C}_f \quad a.e. \tag{1.15}$$

Moreover

$$|\mathcal{Z}_x| = |\mathcal{Z}_y| = |\mathcal{C}_f| < |\Omega|. \tag{1.16}$$

A corresponding result holds also for f^{-1}.

We notice here that if we further weaken the regularity assumptions on the inverse, our previous results may fail.

EXAMPLE 1.5. Let $c : (0, 1) \to (0, 1)$ be the usual Cantor ternary function and define $k(t) = t + c(t)$ and $h = k^{-1} : (0, 2) \to (0, 1)$. Then the mapping $f = (u, v)$ defined by

$$f(x, y) = (h(x), y) \quad \text{for } (x, y) \in (0, 2) \times (0, 1),$$

is a Sobolev homeomorphism whose inverse is in BV (see Section 2 for definition of mapping of bounded variation) but fails to belong to $\mathscr{W}^{1,1}_{\mathrm{loc}}$. One can check that $|\mathcal{Z}_x| = 1$, $|\mathcal{Z}_y| = 0$.

After a Section of Preliminaries, in Section 3 we study the composition of Sobolev functions with bi-Sobolev maps. In Section 4 various distortions quotients are related each other. In Section 5 an elementary proof of Theorem 1.4 is given.

2. Preliminaries

2.1. Lipschitz, Sobolev and BV mappings. Let Ω be a bounded domain in $\mathbb{R}^n$. We say that the mapping $f : \Omega \to \mathbb{R}^n$ is Lipschitz if there exists a constant $K > 0$ such that

$$|f(z) - f(z')| \leq K|z - z'| \tag{2.1}$$

for every $z, z' \in \Omega$. Further, $f : \Omega \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^n$ is said to be bi-Lipschitz if

$$\frac{|z - z'|}{K} \leq |f(z) - f(z')| \leq K|z - z'| \tag{2.2}$$

for every $z, z' \in \Omega$ and a $K \geq 1$. For $1 \leq p \leq \infty$, we say that $f \in \mathscr{L}^p(\Omega; \mathbb{R}^n)$ belongs to the Sobolev space $\mathscr{W}^{1,p}(\Omega; \mathbb{R}^n)$ if the distributional derivatives of the coordinate functions of f belong to $\mathscr{L}^p(\Omega; \mathbb{R}^n)$. Further, $f \in \mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ if $f \in \mathscr{W}^{1,p}(\tilde{\Omega}; \mathbb{R}^n)$ for each open $\tilde{\Omega} \subset\subset \Omega$.

The function $h : \Omega \to \mathbb{R}^n$ is said to be a representative of $g : \Omega \to \mathbb{R}^n$ if $h = g$ a.e..

It is well known (see [1] Section 3.11) that a mapping $f \in \mathscr{L}^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ is in $\mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ if and only if there is a representative which is an absolutely continuous function on almost all lines parallel to coordinate axes and the variation on these lines is integrable.

Recall that for a function $\varphi : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ the total variation of φ is

$$V(\varphi, (a, b)) = \sup\left\{ \sum_{i=1}^{k} |\varphi(a_i) - \varphi(b_i)| : (a_i, b_i) \text{ are disjoint intervals in } (a, b) \right\}. \tag{2.3}$$

A real function $u \in \mathscr{L}^1(\Omega)$ is of bounded variation, $u \in \mathrm{BV}(\Omega)$ if the distribution partial derivatives of u are measures with finite total variation in Ω: there are Radon (signed) measures $\mu_1, \dots, \mu_n$ in Ω such that, for $i = 1, \dots, n$ the total variations verify $|\mu_i|(\Omega) < \infty$ and

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dz = -\int_\Omega \varphi \, d\mu_i$$

for all $\varphi \in C^1_0(\Omega)$. The gradient of u, ∇u, is then a vector-valued measure with finite total variation

$$|\nabla u|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div} \underline{v} \, dz : \underline{v} = (v_1, \dots, v_n) \in C^1_0(\Omega; \mathbb{R}^n), |\underline{v}| \leq 1 \text{ a.e.} \right\}. \tag{2.4}$$

If $u \in \mathscr{W}^{1,1}(\Omega)$, then $|\nabla u|(\Omega) = \displaystyle\int_\Omega |\nabla u| \, dz$ for all this see [1].

Further, we say that the mapping $f \in \mathscr{L}^1(\Omega; \mathbb{R}^n)$ belongs to $\mathrm{BV}(\Omega; \mathbb{R}^n)$ if the coordinate functions of f belong to $\mathrm{BV}(\Omega)$. Finally $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ requires that $f \in \mathrm{BV}(\tilde{\Omega}; \mathbb{R}^n)$ for every subdomain $\tilde{\Omega} \subset \Omega$.

For $1 \le p \le \infty$, $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^n$ domains, the homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a $\mathscr{W}^{1,p}$-homeomorphism if $f \in \mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. For $p = 1$ we simply say that f is a Sobolev homeomorphism. For $p = \infty$ we also say that f is a Lipschitz homeomorphism.

2.2. Differentiability of $\mathscr{W}^{1,1}$-homeomorphisms. Let $f : \Omega \subset \mathbb{R}^n \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^n$ be a homeomorphism. We decompose ([11]) the domain Ω of f as follows

$$\Omega = \mathcal{R}_f \cup \mathcal{C}_f \cup \mathcal{E}_f \tag{2.5}$$

where

$$\mathcal{R}_f = \{z \in \Omega : f \text{ is differentiable at } z \text{ and } J_f(z) \neq 0\} \tag{2.6}$$

is the set (possible empty) of regular points of f, and

$$\mathcal{C}_f = \{z \in \Omega : f \text{ is differentiable at } z \text{ and } J_f(z) = 0\} \tag{2.7}$$

$$\mathcal{E}_f = \{z \in \Omega : f \text{ is not differentiable at } z\}. \tag{2.8}$$

Differentiability is understood in the classical sense. Since f is a homeomorphism, those are Borel sets ([31]). Moreover

$$f(\mathcal{R}_f) = \mathcal{R}_{f^{-1}}$$

and for all $z \in \mathcal{R}_f$:

$$Df^{-1}(f(z)) = (Df(z))^{-1}. \tag{2.9}$$

A Sobolev homeomorphism f is known to be differentiable a.e. in Ω (see forthcoming Theorem 2.2). For such a map $|\mathcal{E}_f|$ vanishes and either $J_f(z) \ge 0$ or $J_f(z) \le 0$ a.e.. Moreover, Df is a Borel function and is the differential also in the sense of distributions.

An important property of Lipschitz functions is their a.e. differentiability. The prototype of such kind of results is the following theorem due to Rademacher and Stepanov (see [29] p. 311 and also [1], Theorem 2.14):

Theorem 2.1. *Any function $u \in \mathscr{W}^{1,\infty}(\Omega)$ is differentiable a.e.*

Let us list some other classical a.e. differentiability theorems for Sobolev functions.

It is well known that if a function u belongs to $\mathscr{W}^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $p > n$, then u is differentiable a.e. ([4] for $n = 2$ and [3] for general n).

In 1981 Stein [30] proved that if $u \in \mathscr{W}^{1,1}(\Omega)$ and $|\nabla u|$ belongs to the Lorentz space $\mathscr{L}^{n,1}(\Omega)$ then u is differentiable a.e. (for Lorentz space see [22]). Better result are available for Sobolev homeomorphisms $f \in \mathscr{W}^{1,1}(\Omega; \mathbb{R}^n)$. First of all, in the planar case, the following result of Gehring-Lehto is largely satisfactory ([23] Theorem III.3.1).

Theorem 2.2 (Gehring-Lehto). *Let* $f : \Omega \subset \mathbb{R}^2 \overset{onto}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ *be a Sobolev homeomorphism. Then* f *is differentiable a.e.*

For general n weaker results hold true. In [31] Vaisala proved that to get a.e. differentiability of f it suffices to assume that $|Df|$ is p-integrable for some $p > n - 1$, whereas $|Df| \in \mathscr{L}^{n-1}$ is not sufficient when $n > 2$.

Recently J. Onninen showed that it is sufficient to assume $|Df| \in \mathscr{L}^{n-1,1}(\Omega)$ ([25]) which, for $n = 2$ reduces to Gehring-Lehto Theorem.

Remark 2.3. In [7] and in [13] a n-dimensional version of (2.9) is proved. Since bi-Sobolev maps $f \in \mathscr{W}^{1,1}(\Omega; \mathbb{R}^n)$ are not necessarily a.e. differentiable for $n > 2$, in those papers the Authors use the notion of approximate differentiability and prove formula (2.9) for all z belonging to a Borel set $A \subset \{z \in \Omega : f$ is approximately differentiable at z and $J_f(z) > 0\} = \tilde{\mathcal{R}}_f$ which is of full measure in $\tilde{\mathcal{R}}_f$, such that $f(A) \subset \{w \in \Omega' : f^{-1}$ is approximately differentiable at w and $J_{f^{-1}}(w) > 0\} = \tilde{\mathcal{R}}_{f^{-1}}$ which is of full measure in $\tilde{\mathcal{R}}_{f^{-1}}$ as well.

2.3. Area Formula. A continuous mapping $f : \Omega \to \mathbb{R}^2$ is said to satisfy the N-condition of Lusin if $|f(E)| = 0$ for every $E \subset \Omega$ such that $|E| = 0$. For homeomorphisms this is equivalent to say that $f(E)$ is measurable for any $E \subset \Omega$ measurable.

A $\mathscr{W}^{1,2}$-homeomorphism satisfies the N-condition, according to the following result ([23] p. 150).

Proposition 2.4. *Let* $g : \Omega \subset \mathbb{R}^2 \overset{onto}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ *be a homeomorphism belonging to* $\mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$. *Then* g *verifies the N condition.*

On the other hand there exists a homeomorphism that does not satisfy the condition N and $|Df|$ belongs to $\mathscr{L}^p$ for each $1 < p < 2$; see the examples by Ponomarev [26], [27].

In a recent paper [21] it is shown that a sharp regularity assumption to rule out the failure of the N-condition for planar homeomorphism is that

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega |Df|^{2-\varepsilon}\, dz = 0. \tag{2.10}$$

See [20] and [9] for the introduction and the study of condition (2.10).

Let f be a Sobolev homeomorphism i.e. $f \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ and let η be a non-negative Borel measurable function on $\mathbb{R}^2$. Without any additional assumption we have ([6], Theorem 3.1.8)

$$\int_\Omega \eta(f(z)) J_f(z)\, dz \le \int_{\mathbb{R}^2} \eta(w)\, dw. \tag{2.11}$$

Moreover there exists a set $\tilde{\Omega} \subset \Omega$ of full measure such that the area formula holds for f on $\tilde{\Omega}$ ([12]):

$$\int_{\tilde{\Omega}} \eta(f(z)) J_f(z)\, dz = \int_{f(\tilde{\Omega})} \eta(w)\, dw. \tag{2.12}$$

We say that the area formula holds for f on the Borel set B if the equality

$$\int_{B} \eta(f(z)) J_f(z)\, dz = \int_{f(B)} \eta(w)\, dw \tag{2.13}$$

is satisfied for any nonnegative Borel function η on $\mathbb{R}^2$.

Also, the area formula holds on each set on which the Lusin condition N is satisfied. This follows from the area formula for Lipschitz mappings, from the a.e. differentiability of f and a general property of a.e. differentiable planar mappings, namely that Ω can be exhausted up to a set of measure zero by sets the restriction to which of f is Lipschitz. More precisely, we can decompose $\Omega \subset \mathbb{R}^2$ into pairwise disjoint sets

$$\Omega = Z \cup \bigcup_{k=1}^{\infty} \Omega_k \tag{2.14}$$

such that $|Z| = 0$ and $f_{|\Omega_k}$ is Lipschitz ([18]).

From (2.12) we deduce immediately the following general version for $\mathscr{W}^{1,1}$-homeomorphisms of the classical Sard Lemma concerning the image of the set $\mathcal{C}_f$ (defined in (1.1)) of zeros of the Jacobian.

Lemma 2.5 (Sard). *Let $f : \Omega \to \Omega'$ be a $\mathscr{W}^{1,1}$-homeomorphism. Then exists $Z \subset \Omega$ such that $|Z| = 0$ and $|f(\mathcal{C}_f \backslash Z)| = 0$.*

Proof. Let $\tilde{\Omega} \subset \Omega$ be a full measure set such that the area formula (2.12) holds for f on $\tilde{\Omega}$. We may assume that the Borel $\mathcal{C}_f$ has positive measure, otherwise we choose $Z = \mathcal{C}_f$.

Since $|\mathcal{C}_f| > 0$, there exists a zero set Z such that $\mathcal{C}_f \backslash Z \subset \tilde{\Omega}$ and by (2.12) with $\eta(w) = \chi_{f(\mathcal{C}_f \backslash Z)}(w)$, we have:

$$\int_{\tilde{\Omega}} \chi_{f(\mathcal{C}_f \backslash Z)}(f(z)) J_f(z)\, dz = \int_{\mathcal{C}_f \backslash Z} J_f(z)\, dz = |f(\mathcal{C}_f \backslash Z)|.$$

Since $J_f = 0$ on $\mathcal{C}_f \backslash Z$ we deduce that $|f(\mathcal{C}_f \backslash Z)| = 0$. □

For completeness let us prove the following useful

Proposition 2.6. *Let $f : \Omega \subset \mathbb{R}^2 \to \Omega' \subset \mathbb{R}^2$ be a Sobolev homeomorphism. Then f^{-1} verifies the N condition if, and only if, the Jacobian of f satisfies the condition*

$$J_f(z) > 0 \quad a.e.\ z \in \Omega. \tag{2.15}$$

PROOF. Suppose f^{-1} verifies the N-condition. By Lemma 2.5 exists a set Z of measure zero such that $|f(\mathcal{C}_f \setminus Z)| = 0$ and thus $|\mathcal{C}_f| = 0$.

Conversely, suppose (2.15) and assume, by contradiction, that there exists $E' \subset \Omega'$, $|E'| = 0$ such that

$$|f^{-1}(E')| > 0.$$

By (2.11) with $\eta = 1$, $B = f^{-1}(E')$ and (2.15) we have

$$0 < \int_{f^{-1}(E')} J_f \, dz \leq |E'| = 0$$

which is a contradiction. □

REMARK 2.7. Combining Proposition 2.6 with Proposition 2.4 for $g = f^{-1}$, we deduce that if f is a bi-Sobolev map such that $f^{-1} \in \mathscr{W}^{1,2}$ then $|\mathcal{C}_f| = 0$.

3. Composition with Sobolev homeomorphisms

In this section we shall be concerned with the behaviour of a function $\varphi \in \mathscr{W}^{1,1}(\Omega')$ when composed with a Sobolev homeomorphism

$$f : \Omega \subset \mathbb{R}^2 \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^2. \tag{3.1}$$

The major difficulty lies in the fact that, also if we assume that f is a bi-Sobolev map, the map f^{-1} need not satisfy the N-condition. In other words, the image of a null set under f^{-1} may fail to be measurable. This poses serious problems concerning measurability of the composition $\varphi \circ f : \Omega \overset{\text{onto}}{\longrightarrow} \mathbb{R}$ and forces us to assume that f^{-1} satisfies the N-condition. In fact, it is well known ([23] p. 121) that the N-condition on f^{-1} guarantees that $\varphi \circ f$ is measurable for any measurable function $\varphi : \Omega' \overset{\text{onto}}{\longrightarrow} \mathbb{R}$.

We will give sharp conditions under which $\varphi \circ f \in \mathscr{W}^{1,1}_{\text{loc}}(\Omega)$. The point is that we will admit that Jacobian J_f may vanish on some positive set, while on the contrary, the classical results ([23] p. 151) require that f and f^{-1} belong to $\mathscr{W}^{1,2}$, hence f and f^{-1} carry zero sets into zero sets and, equivalently, $J_f \neq 0$ a.e. and $J_{f^{-1}} \neq 0$ a.e..

Let us precisely state a result from [23].

THEOREM 3.1. *Let $f : \Omega \subset \mathbb{R}^2 \overset{onto}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ be a homeomorphism which satisfies*

$$|Df| \in \mathscr{L}^2(\Omega), \quad |Df^{-1}| \in \mathscr{L}^2(\Omega'). \tag{3.2}$$

If $\varphi \in \mathscr{W}^{1,1}(\Omega')$ satisfies $|\nabla \varphi| \in \mathscr{L}^2(\Omega')$ then

$$\varphi \circ f \in \mathscr{W}^{1,1}(\Omega). \tag{3.3}$$

Notice that (3.2) implies that f and f^{-1} verify the N-condition. See also ([33]).

There are other recent results about the Sobolev regularity of compositions. For example in [15] assumption (3.2) has been relaxed into

$$|Df| \in \mathscr{L}^1(\Omega), \quad |Df^{-1}| \in \mathscr{L}^2(\Omega'). \tag{3.4}$$

and in [14] this last condition was proved to be sharp to obtain (3.3), in the sense that for $\Omega = \Omega' = (-1,1)^2$, for any $0 < \epsilon < 1$ there exists a bi-Sobolev homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ such $|Df^{-1}| \in \mathscr{L}^2(\Omega')$ and there exists $\varphi \in \mathscr{W}^{1,2-\epsilon}(\Omega')$ such that $\varphi \circ f \notin \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega)$. In [10] the following result has been proved

Theorem 3.2. *Let $f : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ be a homeomorphism which satisfies*

$$|Df| \in \mathscr{L}^1_{\mathrm{loc}}(\Omega), \quad |Df^{-1}| \in \mathscr{L}^2 \log^\alpha \mathscr{L}_{\mathrm{loc}}(\Omega') \quad (\alpha \in \mathbb{R}). \tag{3.5}$$

If $\varphi \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega')$ satisfies $|\nabla\varphi| \in \mathscr{L}^2 \log^{-\alpha} \mathscr{L}_{\mathrm{loc}}(\Omega')$, then (3.3) *holds true.*

4. The distortion quotients of planar homeomorphisms

Let $f : \Omega \subset \mathbb{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^2$ be a homeomorphism which is a.e. differentiable together with its inverse. Let us define the distortion quotient

$$K_f(z) = \frac{|Df(z)|^2}{J_f(z)} \tag{4.1}$$

whenever z belongs to the regular set $\mathcal{R}_f$ of f (see (2.6)). If $|Df(z)|^2 = 0$, we may set $K_f(z) = 1$. However, where $|Df(z)|^2 > 0$ but $J_f(z) = 0$ there is no meaningful definition for $K_f(z)$. Assuming that the set of all such degenerate points has zero measure, that is, assuming

$$\{z \in \Omega : J_f(z) = 0\} = \{z \in \Omega : |Df(z)|^2 = 0\} \quad \text{a.e.}, \tag{4.2}$$

then $K_f : \Omega \to [1, \infty[$ is a Borel map which in [11] has been defined precisely at every $z \in \Omega$ as follows:

$$K_f(z) = \begin{cases} \dfrac{|Df(z)|^2}{J_f(z)}, & \text{for all } z \in \mathcal{R}_f \\ 1 & \text{for all } z \in \Omega \backslash \mathcal{R}_f. \end{cases} \tag{4.3}$$

Hereafter the undetermined ratio $\frac{0}{0}$ is understood to be equal to 1 for $z \in \mathcal{C}_f$ (see (2.7)).

Let us notice that bi-Sobolev maps automatically verify (4.2) ([18]). In [11] for bi-Sobolev maps the formula

$$K_{f^{-1}}(w) = K_f(f^{-1}(w)) \tag{4.4}$$

was established at every point $w \in \Omega'$. The main difficulty in establishing (4.4) for a bi-Sobolev map f is that f need not satisfy Lusin condition N; this poses some problem for the definition and the measurability of $K_f \circ f^{-1}$ in (4.4). To overcome this difficulty the precise definition (4.4) of K_f was proposed, so that it is defined at every point, and not only a.e.. Notice that analogous formula holds for K_f when f is bi-Sobolev

$$K_f(z) = K_{f^{-1}}(f(z)) \quad \text{for all } z \in \Omega. \tag{4.5}$$

Using the same technique, in the following we will give similar formulas, for various distortion quotients.

For homeomorphisms which are a.e. differentiable with their inverse and have nonnegative Jacobians, under suitable assumptions, it is possible to introduce different distortions quotients. Namely, if $f = (u, v)$ satisfies the condition

$$\{z : J_f(z) = 0\} = \{z : |\nabla u(z)| = 0\} \quad \text{a.e.} \tag{4.6}$$

then we are allowed to define the Borel function

$$K_f^{(1)}(z) := \begin{cases} \dfrac{|\nabla u(z)|^2}{J_f(z)} & \text{for all } z \in \mathcal{R}_f, \\ 1 & \text{otherwise.} \end{cases} \tag{4.7}$$

Similarly, if $f = (u, v)$ satisfies the condition

$$\{z : J_f(z) = 0\} = \{z : |\nabla v(z)| = 0\} \quad \text{a.e.} \tag{4.8}$$

then the Borel function

$$K_f^{(2)}(z) := \begin{cases} \dfrac{|\nabla v(z)|^2}{J_f(z)} & \text{for all } z \in \mathcal{R}_f, \\ 1 & \text{otherwise.} \end{cases} \tag{4.9}$$

is well defined.

On the other hand, if $f = (u, v)$ satisfies the condition

$$\{z : J_f(z) = 0\} = \{z : |f_x(z)| = 0\} \quad \text{a.e.} \tag{4.10}$$

then we can define the Borel function

$$H_f^{(1)}(z) := \begin{cases} \dfrac{|f_x(z)|^2}{J_f(z)} & \text{for all } z \in \mathcal{R}_f, \\ 1 & \text{otherwise.} \end{cases} \tag{4.11}$$

Finally, for f satisfying

$$\{z : J_f(z) = 0\} = \{z : |f_y(z)| = 0\} \quad \text{a.e.} \tag{4.12}$$

we define

$$H_f^{(2)}(z) := \begin{cases} \dfrac{|f_y(z)|^2}{J_f(z)} & \text{for all } z \in \mathcal{R}_f, \\ 1 & \text{otherwise.} \end{cases} \tag{4.13}$$

We would like to point out here that various interaction between such quotients of the type (4.4) and (4.5) via composition with f and f^{-1} hold true.

As a prototype of our results, let us prove the following

THEOREM 4.1. *Let $f : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ be a homeomorphism and suppose that $f = (u, v)$, $f^{-1} = (x, y)$ are differentiable a.e. and satisfy conditions* (4.10) *and*

$$\{w \in \Omega' : J_{f^{-1}}(w) = 0\} = \{w \in \Omega' : |\nabla y(w)| = 0\} \quad a.e. \tag{4.14}$$

respectively. Moreover, assume $J_f(z) \geq 0$, $J_{f^{-1}}(z) \geq 0$ a.e. Then,

$$H_f^{(1)}(z) = K_{f^{-1}}^{(2)}(f(z)) \quad \textit{for all } z \in \Omega. \tag{4.15}$$

where

$$K_{f^{-1}}^{(2)}(w) := \begin{cases} \dfrac{|\nabla y(w)|^2}{J_{f^{-1}}(w)} & \textit{if } J_{f^{-1}}(w) > 0, \\ 1 & \textit{otherwise.} \end{cases} \tag{4.16}$$

PROOF. We will now use the elementary formulas for the differential of the inverse

$$Df^{-1}(f(z)) = (Df(z))^{-1} \quad \text{for every } z \in \mathcal{R}_f \tag{4.17}$$

which in two dimension reads as

$$\begin{pmatrix} x_u(f(z)) & x_v(f(z)) \\ y_u(f(z)) & y_v(f(z)) \end{pmatrix} = \begin{pmatrix} \dfrac{v_y(z)}{J_f(z)} & \dfrac{-u_y(z)}{J_f(z)} \\ \dfrac{-v_x(z)}{J_f(z)} & \dfrac{u_x(z)}{J_f(z)} \end{pmatrix}$$

hence

$$y_u(f(z))^2 + y_v(f(z))^2 = \frac{u_x(z)^2 + v_x(z)^2}{J_f(z)^2}. \tag{4.18}$$

Clearly the image of regular points of f are regular points of f^{-1}:

$$f(\mathcal{R}_f) = \mathcal{R}_{f^{-1}}$$

and then, if $z \in \mathcal{R}_f$ by (4.18)

$$H_f^{(1)}(z) = K_{f^{-1}}^{(2)}(f(z))$$

because $f(z) \in \mathcal{R}_{f^{-1}}$ and

$$J_{f^{-1}}(f(z)) = \frac{1}{J_f(z)} > 0.$$

On the other hand, if $z \in \Omega \setminus \mathcal{R}_f$, we have by (4.11)

$$H_f^{(1)}(z) = 1$$

and also

$$K_{f^{-1}}^{(2)}(f(z)) = 1$$

because $f(z) \in \Omega' \setminus \mathcal{R}_{f^{-1}}$. $\square$

REMARK 4.2. Similarly we can prove

$$H_f^{(2)}(z) = K_{f^{-1}}^{(1)}(f(z)) \quad \text{for all } z \in \Omega \tag{4.19}$$

under the assumptions (4.12) and

$$\{w \in \Omega' : J_{f^{-1}}(w) = 0\} = \{w \in \Omega' : |\nabla x(w)| = 0\} \quad \text{a.e.} \tag{4.20}$$

and assuming f and f^{-1} differentiable a.e. and with non negative Jacobians.

REMARK 4.3. The following assumptions of Theorem 4.1

$$f \text{ and } f^{-1} \text{ differentiable a.e.} \tag{4.21}$$

$$J_f \geq 0,\ J_{f^{-1}} \geq 0 \quad \text{a.e.} \tag{4.22}$$

are certainly satisfied if f is a BV-homeomorphism (see [16], [5])

REMARK 4.4. Using Theorem 4.1 and Remark 4.2 we can recover formulas (4.4) and (4.5) for bi-Sobolev maps.

In fact, obviously we have

$$\{z \in \Omega : |Df(z)| = 0\} = \{z \in \Omega : |\nabla u(z)| = 0\} \cap \{z \in \Omega : |\nabla v(z)| = 0\}$$

then K_f is well defined if we assume

$$\{z \in \Omega : J_f(z) = 0\} \subset \{z \in \Omega : |Df(z)| = 0\} \tag{4.23}$$

hence also $K_f^{(1)}$ and $K_f^{(2)}$ are well defined under such a condition. Since we have

$$\{w \in \Omega' : |Df^{-1}(w)| = 0\} = \{w \in \Omega' : |\nabla x(w)| = 0\} \cap \{w \in \Omega' : |\nabla y(w)| = 0\}$$

also $K_{f^{-1}}$, $H_f^{(1)}$ and $H_f^{(2)}$ are well defined for f a bi-Sobolev map.

Moreover, (4.2) implies similar condition on f^{-1} and we check that

$$K_f(z) = K_f^{(1)}(z) + K_f^{(2)}(z) = H_f^{(1)}(z) + H_f^{(2)}(z) \quad \text{for all } z \in \Omega$$

and similarly for the inverse

$$K_{f^{-1}}(w) = K^{(1)}_{f^{-1}}(w) + K^{(2)}_{f^{-1}}(w) = H^{(1)}_{f^{-1}}(w) + H^{(2)}_{f^{-1}}(w) \quad \text{for all } w \in \Omega'.$$

5. Proof of Theorem 1.4

We first prove that $\mathcal{C}_f = \mathcal{Z}_y$. Obviously we have $\mathcal{Z}_y \subseteq \mathcal{C}_f$.

By contradiction we suppose that there exists a set $A \subset \Omega$ with positive Lebesgue measure such that f is differentiable in A, $J_f(z) = 0$ and $|f_y| > 0$ on A. Since, analogously to (2.14), we can decompose A up to a set of measure zero into countably many pieces where f is Lipschitz and one of them must have positive measure, we can also assume that f is Lipschitz on A.

Using area formula (2.13), we get

$$0 = \int_A J_f(z)\,dz = |f(A)|.$$

We denote, by

$$p_2 : (x_1, x_2) \in \mathbb{R}^2 \to (x_1, 0) \in \mathbb{R}^2$$

the orthogonal projection and by

$$p^{(2)} : (x_1, x_2) \in \mathbb{R}^2 \to x_2 \in \mathbb{R}$$

the second coordinate function.

Applying the area formula to the differentiable function $f(x,\cdot) : y \in p^{(2)}(A) \to f(x, y) \in \Omega'$ we get:

$$\begin{aligned} 0 < \int_{A \cap p_2^{-1}(\{(x,0)\})} |f_y(x, y)| &= \int_{\mathbb{R}^2} N(f, A \cap p_2^{-1}(\{(x,0)\}), \sigma)\,d\mathscr{H}^1(\sigma) \\ &= \mathscr{H}^1(f(A \cap p_2^{-1}(\{(x,0)\}))). \end{aligned}$$

Since $|f(A)| = 0$ we get that $\mathscr{H}^1(f(A \cap p_2^{-1}(\{(x,0)\}))) = 0$ for almost every $(x,0) \in \mathbb{R}^2$ and this is a contradiction.

The other equality $\mathcal{C}_f = \mathcal{Z}_x$ a.e., is completely analogous.

To prove (1.16) notice that, if f satisfies $|\mathcal{C}_f| = |\Omega|$, then $|f_x| = 0$, $|f_y| = 0$ a.e. in Ω and hence $|Df| = 0$ a.e. The contradiction follows from the ACL condition for f, i.e. that f is absolutely continuous on almost all lines parallel to coordinate axes.

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Received 9 December 2010,
and in revised form 12 January 2011.

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