Partial Differential Equations — *Quasi-periodic solutions of nonlinear Schrö-dinger equations on* $\mathbb{T}^d$, by Massimiliano Berti and Philippe Bolle, communicated on 11 March 2011.[1]

To the memory of Giovanni Prodi.

Abstract. — We present recent existence results of quasi-periodic solutions for Schrödinger equations with a *multiplicative* potential on $\mathbb{T}^d$, $d \geq 1$, finitely *differentiable* nonlinearities, and tangential frequencies constrained along a *pre-assigned* direction. The solutions have only Sobolev regularity both in time and space. If the nonlinearity and the potential are in C^∞ then the solutions are in C^∞. The proofs are based on an improved Nash–Moser iterative scheme and a new multiscale inductive analysis for the inverse linearized operators.

Key words: Nonlinear Schrödinger equation, Nash–Moser Theory, KAM for PDE, quasi-periodic solutions, small divisors, infinite dimensional Hamiltonian systems.

Mathematics Subject Classification AMS: 35Q55, 37K55, 37K50.

1. Introduction

The aim of this Note is to present the recent results of [3] concerning the existence of quasi-periodic solutions for d-dimensional Schrödinger equations

$$(1.1) \qquad iu_t - \Delta u + V(x)u = \varepsilon f(\omega t, x, |u|^2)u + \varepsilon g(\omega t, x)$$

with periodic boundary conditions

$$x \in \mathbb{T}^d := (\mathbb{R}/(2\pi\mathbb{Z}))^d,$$

where the *multiplicative* potential V is in $C^q(\mathbb{T}^d; \mathbb{R})$ for some q large enough, $\varepsilon > 0$ is small, the frequency vector $\omega \in \mathbb{R}^\nu$ is colinear to a fixed Diophantine vector $\bar{\omega} \in \mathbb{R}^\nu$, namely

$$(1.2) \quad \omega = \lambda\bar{\omega}, \quad \lambda \in \Lambda := [1/2, 3/2] \subset \mathbb{R}, \quad |\bar{\omega} \cdot l| \geq \frac{\gamma_0}{|l|^{\tau_0}}, \quad \forall l \in \mathbb{Z}^\nu \backslash \{0\},$$

for some $\gamma_0 \in (0,1)$, $\tau_0 > \nu - 1$ (for definiteness $\tau_0 := \nu$), $|l| := \max\{|l_1|,\ldots,|l_\nu|\}$, and the nonlinearity is quasi-periodic in time and only *finitely many times differentiable*, more precisely

$$(1.3) \qquad f \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R}), \quad g \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d; \mathbb{C})$$

for some $q \in \mathbb{N}$ large enough.

The dynamics of the linear Schrödinger equation

$$iu_t - \Delta u + V(x)u = 0$$

is well understood. All its solutions are the linear superpositions of normal mode oscillations

$$(1.4) \quad u(t,x) = \sum_j a_j e^{i\mu_j t} \psi_j(x), \quad a_j \in \mathbb{C}, \quad \text{where } (-\Delta + V(x))\psi_j(x) = \mu_j \psi_j(x),$$

hence periodic, quasi-periodic or almost periodic in time. The eigenvalues $\mu_j \to +\infty$ as $j \to +\infty$ and the corresponding eigenfunctions $\psi_j(x)$ form a Hilbert basis in $L^2(\mathbb{T}^d)$.

- *Question:* do there exist quasi-periodic solutions of (1.1) for positive measure sets of (ε, λ)?

Note that, if $g(\omega t, x) \not\equiv 0$, then $u = 0$ is not a solution of (1.1) for $\varepsilon \neq 0$.

The above question amounts to looking for $(2\pi)^{\nu+d}$-periodic solutions $u(\varphi, x)$ of

$$(1.5) \qquad i\omega \cdot \partial_\varphi u - \Delta u + V(x)u = \varepsilon f(\varphi, x, |u|^2)u + \varepsilon g(\varphi, x).$$

The solutions $u(\varphi, x)$ will be in some Sobolev space

$$H^s := H^s(\mathbb{T}^\nu \times \mathbb{T}^d; \mathbb{C}) \quad \text{with } s \in [s_0, q], \ s_0 > \frac{d+\nu}{2},$$

which is a Banach algebra. The functions in H^s are characterized in Fourier series

$$u(\varphi, x) := \sum_{(l,j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d} u_{l,j} e^{i(l \cdot \varphi + j \cdot x)}$$

by the condition

$$(1.6) \quad \|u\|_s^2 := K_0 \sum_{(l,j) \in \mathbb{Z}^{\nu+d}} |u_{l,j}|^2 \langle l, j \rangle^{2s} < +\infty \quad \text{where } \langle l, j \rangle := \max(1, |l|, |j|).$$

The constant K_0 is fixed (large enough) so that $|u|_{L^\infty} \leq \|u\|_{s_0}$ and the interpolation inequality

$$(1.7) \quad \|u_1 u_2\|_s \leq \frac{1}{2} \|u_1\|_{s_0} \|u_2\|_s + \frac{C(s)}{2} \|u_1\|_s \|u_2\|_{s_0}, \quad \forall s \geq s_0, \, u_1, u_2 \in H^s,$$

holds with $C(s) = 1$, $\forall s \in [s_0, s_1]$ for some $s_1 := s_1(d, \nu)$ (defined along the proof).

The above question can be regarded as a bifurcation problem for equation (1.5) from the trivial solution $(u, \varepsilon) = (0, 0)$. The main difficulty is that the unperturbed linear operator

$$i\omega \cdot \partial_\varphi - \Delta + V(x)$$

possesses arbitrarily small eigenvalues,

$$-\omega \cdot l + \mu_j,$$

called the "small divisors". As a consequence, its inverse operator, if any, is unbounded and the standard implicit function theorem can not be applied.

The main strategies which have been developed to overcome the small divisors difficulty are KAM (Kolmogorov–Arnold–Moser) theory and Newton–Nash–Moser Implicit function theorems.

1.1. Some literature

The first existence results of quasi-periodic solutions of Hamiltonian PDEs have been proved via KAM theory by Kuksin [17] and Wayne [19] for one dimensional, analytic, nonlinear perturbations of linear wave and Schrödinger equations. These pioneering results were limited to Dirichlet boundary conditions because the eigenvalues of ∂_{xx} had to be simple. In this case one can impose the "second order Melnikov" non-resonance conditions to solve the homological equations, which are linear PDEs with constant coefficients, at each KAM step. Already for periodic boundary conditions, where two consecutive eigenvalues are possibly equal, these non-resonance conditions are violated.

Then, another more direct bifurcation approach has been proposed by Craig and Wayne [12], who introduced the Lyapunov–Schmidt decomposition method for PDEs and solved the small divisors problem, for periodic solutions, with an analytic Newton iterative scheme. The advantage of this approach is to require only the "first order Melnikov" non-resonance conditions, which are essentially the minimal assumptions. On the other hand, the main difficulty of this strategy lies in the inversion of the linearized operators obtained at each step of the iteration, and in achieving suitable estimates for their inverses in high (analytic) norms. Indeed these operators come from linear PDEs with *non-constant* coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues.

For solving this problem, Craig and Wayne developed a coupling technique whose key properties are:

(i) "separations" between the singular sites, namely the Fourier indices of the small divisors,
(ii) "localization" of the eigenfunctions of $-\partial_{xx} + V(x)$ with respect to the exponentials.

Roughly speaking, property (ii) means that, if we expand the eigenfunctions of $-\partial_{xx} + V(x)$ with respect to the exponentials, the Fourier coefficients converge to zero, at a speed which increases with the regularity of $V(x)$. It implies that the matrix which represents, in the eigenfunction basis, the multiplication operator for an analytic (resp. Sobolev) function has an exponentially (resp. polynomially) fast decay off the diagonal. Then the "separation properties" (i) imply a very "weak interaction" between the singular sites.

Property (ii) holds in dimension 1, but, for $x \in \mathbb{T}^d$, $d \geq 2$, some counterexamples are known, see [16].

The "separation properties" (i) are quite different for periodic or quasi-periodic solutions. In the first case the singular sites are "separated at infinity", namely the distance between distinct singular sites increases when the Fourier indices tend to infinity. On the contrary, this property never holds for quasi-periodic solutions, neither for finite dimensional systems. For example, in the ODE case where the small divisors are $\omega \cdot l$, $l \in \mathbb{Z}^\nu$, if the frequency vector $\omega \in \mathbb{R}^\nu$ is diophantine, then the singular sites l where $|\omega \cdot l| \leq \rho$ are "uniformly distributed" in a neighborhood of the hyperplane $\omega \cdot l = 0$, with nearby indices at distance $O(\rho^{-\alpha})$ for some $\alpha > 0$.

Nevertheless Bourgain extended in [6] the approach of Craig–Wayne via a multiscale inductive argument, proving the existence of quasi-periodic solutions with Gevrey regularity of 1-dimensional wave and Schrödinger equations with polynomial nonlinearities.

At present, the theory for 1-dimensional semilinear PDEs has been sufficiently understood, but much work remains for PDE in higher space dimensions, due to the more complex spectral analysis of $-\Delta + V(x)$. The main difficulties for PDEs in higher dimensions are:

1. the eigenvalues μ_j of $-\Delta + V(x)$ appear in clusters of unbounded sizes,
2. the eigenfunctions $\psi_j(x)$ are, in general, "not localized" with respect to the exponentials.

Problem 2 has been often bypassed considering pseudo-differential PDEs substituting the multiplicative potential $V(x)$ with a "convolution potential"

$$V * (e^{ij \cdot x}) := m_j e^{ij \cdot x}, \quad m_j \in \mathbb{R}, \ j \in \mathbb{Z}^d,$$

which is diagonal on the exponentials. The scalars m_j are called the "Fourier multipliers".

Concerning problem 1, since the approach of Craig–Wayne and Bourgain requires only the first order Melnikov non-resonance conditions it works well, in

principle, in the case of multiple eigenvalues. Actually, the first existence results of periodic solutions for NLW and NLS on $\mathbb{T}^d$, $d \geq 2$, has been established by Bourgain in [7], [10]. The nonlinearities are polynomials and the solutions are Gevrey regular. Here the singular sites form huge clusters (not only points as in $d = 1$) but are still "separated at infinity".

Recently these results were extended in [4]–[5] to prove the existence of periodic solutions, with only Sobolev regularity, for NLS and NLW in any dimension and with finitely differentiable nonlinearities. Actually [4], [5] deal with PDEs defined not only on tori, but on any compact Zoll manifold, Lie group and homogeneous space. For PDEs on Lie groups only weak properties of "localization" (ii) of the eigenfunctions hold.

Regarding quasi-periodic solutions, Bourgain [10]–[11] was the first to prove their existence for PDEs in higher dimension, actually for nonlinear Schrödinger and wave equations with Fourier multipliers and polynomial nonlinearities on $\mathbb{T}^d$ with $d \geq 2$. The Fourier multipliers, in number equal to the tangential frequencies of the quasi-periodic solution, play the role of "external parameters".

The techniques used in [11]—sub-harmonic function theory, semi-algebraic sets, Cartan theorem—mainly concern fine properties of rational and analytic functions.

We also remark that, in the last years, the KAM approach has been extended by Eliasson–Kuksin [15] for nonlinear Schrödinger equations on $\mathbb{T}^d$ with a convolution potential and analytic nonlinearities. The potential plays the role of "external parameters". The quasi-periodic solutions are C^∞ in space. An advantage of the KAM approach is to provide also a stability result, made available by the fact that the linearized equations on the perturbed invariant tori are reducible to constant coefficients.

1.2. Main result

The main result proved in [3] concerning the existence of quasi-periodic solutions of NLS is:

THEOREM 1.1 [3]. *Assume* (1.2). *There is* $s := s(d, \nu)$, $q := q(d, \nu) \in \mathbb{N}$, *such that:* $\forall V \in C^q$ *satisfying*

$$(1.8) \qquad -\Delta + V(x) \geq \beta_0 I, \quad \beta_0 > 0,$$

$\forall f, g \in C^q$, *there exist* $\varepsilon_0 > 0$, *a map*

$$u \in C^1([0, \varepsilon_0] \times \Lambda; H^s) \quad \text{with } u(0, \lambda) = 0,$$

and a Cantor like set $\mathscr{C}_\infty \subset [0, \varepsilon_0] \times \Lambda$ *of asymptotically full Lebesgue measure, i.e.*

$$(1.9) \qquad |\mathscr{C}_\infty|/\varepsilon_0 \to 1 \quad \text{as } \varepsilon_0 \to 0,$$

such that, $\forall (\varepsilon, \lambda) \in \mathscr{C}_\infty$, $u(\varepsilon, \lambda)$ *is a solution of* (1.5) *with* $\omega = \lambda \bar{\omega}$.

Moreover, if V, f, g *are of class* C^∞ *then* $u(\varepsilon, \lambda) \in C^\infty(\mathbb{T}^d \times \mathbb{T}^\nu, \mathbb{C})$.

The main improvements of this result with respect to the previous literature are that we prove the existence of quasi-periodic solutions for nonlinear Schrödinger equations on $\mathbb{T}^d$, $d \geq 1$, with:

1. finitely *differentiable* nonlinearities, see (1.3),
2. a *multiplicative* (finitely differentiable) potential $V(x)$, see (1.8),
3. a *pre-assigned* (Diophantine) direction of the tangential frequencies, see (1.2).

1. Theorem 1.1 confirms the natural conjecture about the persistence of quasi-periodic solutions for Hamiltonian PDEs into a setting of finitely many derivatives (as in the classical KAM theory), stated for example by Bourgain [9], page 97. The nonlinearities in Theorem 1.1, as well as the potential, are sufficiently many times differentiable, depending on the dimension and the number of the frequencies. Of course we cannot expect the existence of quasi-periodic solutions of the Schrödinger equation under too weak regularity assumptions on the non-linearities. Actually, for finite dimensional Hamiltonian systems, it has been rigorously proved that, if the vector field is not sufficiently smooth, then all the invariant tori could be destroyed and only discontinuous Aubry-Mather invariant sets survive. We have not tried to estimate the minimal smoothness exponents.

2. Theorem 1.1 is the first existence result of quasi-periodic solutions with a multiplicative potential $V(x)$ on $\mathbb{T}^d$, $d \geq 2$. We do not exploit properties of "localizations" of the eigenfunctions of $-\Delta + V(x)$ with respect to the exponentials, that actually might *not* be true, see [16]. Along the multiscale analysis we use the exponential basis which diagonalizes $-\Delta + m$ where m is the average of $V(x)$, and not the eigenfunctions of $-\Delta + V(x)$. In [10] Bourgain considered also analytic periodic potentials of the special form $V_1(x_1) + \cdots + V_d(x_d)$ to ensure localization properties of the eigenfunctions, leaving open the natural problem for a general multiplicative potential $V(x)$.

In the setting of Theorem 1.1, the potential $V(x)$ is fixed: we do *not* extract parameters from V, since the role of the parameters is played by $\omega = \lambda \bar{\omega}$. The positivity condition (1.8) is used for the measure estimates. We note that, for autonomous NLS it is always verified after a gauge-transformation $u \mapsto e^{-i\sigma t} u$ for σ large enough.

3. For finite dimensional systems, the existence of quasi-periodic solutions with tangential frequencies constrained along a fixed direction has been proved by Eliasson [13] (with KAM theory) and Bourgain [8] (with a multiscale approach). The main difficulty relies in satisfying the Melnikov non-resonance conditions, required at each step of the iterative process, using only *one* parameter. Bourgain raised in [8] the question if a similar result holds true also for infinite dimensional Hamiltonian systems. This has been recently proved in [1] for 1-dimensional PDEs, verifying the second order Melnikov non-resonance conditions of KAM theory. Theorem 1.1 (and its method of proof) answers positively to Bourgain's conjecture also for PDEs in higher space dimension. The non-resonance conditions to be fulfilled are of first order Melnikov type.

Finally, we note that Theorem 1.1 is stated for quasi-periodically forced NLS but the small divisors difficulty for autonomous NLS is the same.

The proof of Theorem 1.1 is based on a Nash–Moser iterative scheme and a multiscale analysis of the linearized operators as in [11]. However, our approach presents many novelties with respect to that of Bourgain [11], concerning:

1. the iterative scheme,
2. the multiscale proof of the Green's functions polynomial decay estimates.

We outline in section 2 the main ideas of the proof of Theorem 1.1. All the techniques employed are elementary and based on abstract arguments valid for many PDEs. Only the "separation properties" of the "singular" sites will change, of course, for different PDEs.

2. Ideas of the proof

Vector NLS. We prove Theorem 1.1 finding solutions of the "vector" NLS equation

$$(2.1) \quad \begin{cases} \mathrm{i}\omega \cdot \partial_\varphi u^+ - \Delta u^+ + V(x)u^+ = \varepsilon f(\varphi, x, u^- u^+)u^+ + \varepsilon g(\varphi, x) \\ -\mathrm{i}\omega \cdot \partial_\varphi u^- - \Delta u^- + V(x)u^- = \varepsilon f(\varphi, x, u^- u^+)u^- + \varepsilon \bar{g}(\varphi, x) \end{cases}$$

where

$$\mathbf{u} := (u^+, u^-) \in \mathbf{H}^s := H^s \times H^s$$

(the second equation is obtained by formal complex conjugation of the first one). In the system (2.1) the variables u^+, u^- are independent. However (2.1) reduces to the scalar NLS equation (1.1) in the set

$$\mathscr{U} := \{\mathbf{u} := (u^+, u^-) : \overline{u^+} = u^-\}$$

in which u^- is the complex conjugate of u^+ (and viceversa). In (2.1) we choose, for example, the following smooth extension of $f(\varphi, x, \cdot)$ to $\mathbb{C}$,

$$f(\varphi, x, z) := (1 - \mathrm{i})f(\varphi, x, \mathrm{Re}(z)) + \mathrm{i}f(\varphi, x, \mathrm{Re}(z) + \mathrm{Im}(z)), \quad z \in \mathbb{C}.$$

Note that with this choice the differential of $f(\varphi, x, \cdot)$ at $s \in \mathbb{R}$ is $\mathbb{C}$-linear.

Linearized equations. We look for solutions of the vector NLS equation (2.1) in $\mathbf{H}^s \cap \mathscr{U}$ by a Nash–Moser iterative scheme. The main step concerns the invertibility of (any finite dimensional restriction of) the family of linearized operators at any $\mathbf{u} \in \mathbf{H}^s \cap \mathscr{U}$, namely

$$(2.2) \quad \mathscr{L}(\mathbf{u}) := L_\omega - \varepsilon T_1$$

where

$$(2.3) \qquad L_\omega := \begin{pmatrix} i\omega \cdot \partial_\varphi - \Delta + V(x) & 0 \\ 0 & -i\omega \cdot \partial_\varphi - \Delta + V(x) \end{pmatrix},$$

$$T_1 := \begin{pmatrix} p(\varphi, x) & q(\varphi, x) \\ \bar{q}(\varphi, x) & p(\varphi, x) \end{pmatrix},$$

and

$$(2.4) \qquad p(\varphi, x) := f(\varphi, x, |u^+|^2) + f'(\varphi, x, |u^+|^2)|u^+|^2,$$

$$q(\varphi, x) := f'(\varphi, x, |u^+|^2)(u^+)^2,$$

with f' denoting the derivative of f with respect to s. The functions p, q depend also on ε, λ through $\mathbf{u}$. Note that $p(\varphi, x)$ is real valued and so the operator $\mathscr{L}(\mathbf{u})$ is *symmetric* in $\mathbf{H}^0$, i.e.

$$(\mathscr{L}(\mathbf{u})h, k) = (h, \mathscr{L}(\mathbf{u})k)$$

for all h, k in the domain of $\mathscr{L}(\mathbf{u})$. As a consequence, the eigenvalues of all its finite dimensional restrictions vary smoothly with respect to one dimensional parameter.

The operator $\mathscr{L}(\mathbf{u})$ in (2.2) can also be written as

$$\mathscr{L}(\mathbf{u}) = D_\omega + T, \quad T := T_2 - \varepsilon T_1,$$

where D_ω is the constant coefficient operator

$$D_\omega := \begin{pmatrix} i\omega \cdot \partial_\varphi - \Delta + m & 0 \\ 0 & -i\omega \cdot \partial_\varphi - \Delta + m \end{pmatrix}, \quad T_2 := \begin{pmatrix} V_0(x) & 0 \\ 0 & V_0(x) \end{pmatrix},$$

m is the average of $V(x)$ and $V_0(x) := V(x) - m$ has zero mean value.

In the exponential basis $\mathscr{L}(\mathbf{u})$ is represented by the infinite dimensional self-adjoint matrix

$$(2.5) \qquad A(\varepsilon, \lambda) := D_\omega + T$$

of 2×2 complex matrices, where

$$(2.6) \qquad D_\omega := \mathrm{diag}_{i \in \mathbb{Z}^b} \begin{pmatrix} -\omega \cdot l + \|j\|^2 + m & 0 \\ 0 & \omega \cdot l + \|j\|^2 + m \end{pmatrix},$$

$$i := (l, j) \in \mathbb{Z}^b := \mathbb{Z}^\nu \times \mathbb{Z}^d,$$

with $\|j\|^2 := j_1^2 + \cdots + j_d^2$, and

$$(2.7) \qquad T := (T_i^{i'})_{i \in \mathbb{Z}^b, i' \in \mathbb{Z}^b}, \quad T_i^{i'} := -\varepsilon(T_1)_i^{i'} + (T_2)_i^{i'},$$

$$(T_1)_i^{i'} = \begin{pmatrix} p_{i-i'} & q_{i-i'} \\ (\bar{q})_{i-i'} & p_{i-i'} \end{pmatrix}, \quad (T_2)_i^{i'} = \begin{pmatrix} (V_0)_{j-j'} & 0 \\ 0 & (V_0)_{j-j'} \end{pmatrix},$$

where p_i, q_i, $(V_0)_j$ denote the Fourier coefficients of $p(\varphi, x)$, $q(\varphi, x)$, $V_0(x)$.

Note that the matrix T is *Töplitz*, namely $T_i^{i'}$ depends only on the difference of the indices $i - i'$. Moreover, since the functions p, q in (2.4), as well as the potential V, are in H^s, then $T_i^{i'} \to 0$ as $|i - i'| \to \infty$ at a polynomial rate.

We introduce the one-parameter family of infinite dimensional matrices

$$A(\varepsilon, \lambda, \theta) := A(\varepsilon, \lambda) + \theta Y := D_\omega + T + \theta Y \quad \text{where} \quad Y := \text{diag}_{i \in \mathbb{Z}^b} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The reason for adding θY is the crucial covariance property (2.8) below.

The core of the proof of Theorem 1.1 is a polynomial off-diagonal decay for the inverse of the $(2N + 1)^b$-dimensional sub-matrices of $A(\varepsilon, \lambda, \theta)$ centered at (l_0, j_0) denoted by

$$A_{N, l_0, j_0}(\varepsilon, \lambda, \theta) := A_{|l - l_0| \leq N, |j - j_0| \leq N}(\varepsilon, \lambda, \theta).$$

If $l_0 = 0$ we use the simpler notation

$$A_{N, j_0}(\varepsilon, \lambda, \theta) := A_{N, 0, j_0}(\varepsilon, \lambda, \theta).$$

If also $j_0 = 0$, we write $A_N(\varepsilon, \lambda, \theta) := A_{N,0}(\varepsilon, \lambda, \theta)$, and, for $\theta = 0$, we denote $A_{N, j_0}(\varepsilon, \lambda) := A_{N, j_0}(\varepsilon, \lambda, 0)$.

Since the matrix T is Töplitz, the following *covariance* property holds:

(2.8) $$A_{N, l_0, j_0}(\varepsilon, \lambda, \theta) = A_{N, j_0}(\varepsilon, \lambda, \theta + \lambda \overline{\omega} \cdot l_0).$$

Matrices with off-diagonal decay. In the space of matrices

$$\mathscr{M}_C^B := \{ M = (M_k^{k'})_{k' \in B, k \in C}, M_k^{k'} \in \mathbb{C} \},$$

where B, C are finite subsets of $\mathbb{Z}^b \times \{0, 1\}$ (the indices 0, 1 are introduced to distinguish the $\pm$ sign in matrices like (2.6)), we consider the s-norm

$$|M|_s^2 := K_0 \sum_{n \in \mathbb{Z}^b} [M(n)]^2 \langle n \rangle^{2s} \quad \text{where} \quad \langle n \rangle := \max(1, |n|),$$

$$[M(n)] := \begin{cases} \max_{i - i' = n, i \in \bar{C}, i' \in \bar{B}} |M_i^{i'}| & \text{if } n \in \bar{C} - \bar{B} \\ 0 & \text{if } n \notin \bar{C} - \bar{B} \end{cases}$$

with $\bar{B} := \text{proj}_{\mathbb{Z}^b} B$, $\bar{C} := \text{proj}_{\mathbb{Z}^b} C$, and $K_0 > 0$ is introduced in (1.6).

The s-norm is designed to estimate the off-diagonal decay of matrices like T in (2.7): if $p, q, V \in H^s$ then

$$|T_1|_s \leq K \|(q, p)\|_s, \quad |T_2|_s \leq K \|V\|_s.$$

The set of (square) matrices with finite s-norm form an algebra. Hence products and powers of matrices with finite s-norm will exhibit the same off-diagonal decay. We refer to section 3 of [3] for more details.

Improved Nash–Moser iteration. We construct inductively better and better approximate solutions

$$\mathbf{u}_n \in H_n := \left\{ \mathbf{u} = (u^+, u^-) \in \mathbf{H}^s : \mathbf{u} = \sum_{|(l,j)| \leq N_n} \mathbf{u}_{l,j} e^{\mathrm{i}(l\cdot\varphi + j\cdot x)}, \mathbf{u}_{l,j} \in \mathbb{C}^2 \right\}$$

of the NLS equation (2.1), solving, by a Nash–Moser iterative scheme, the "truncated" equations

$$(P_n) \qquad\qquad P_n(L_\omega \mathbf{u} - \varepsilon(f(\mathbf{u}) + g)) = 0, \quad \mathbf{u} \in H_n,$$

where $P_n : \mathbf{H}^s \to H_n$ denote the orthogonal projectors onto H_n and $N_n := N_0^{2^n}$, see Theorem 7.1 in [3].

The main step is to prove that the finite dimensional matrices

$$\mathscr{L}_n := \mathscr{L}_n(\mathbf{u}_{n-1}) := P_n \mathscr{L}(\mathbf{u}_{n-1})_{|H_n}$$

are invertible for "most" parameters $(\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$ and satisfy

$$(2.9) \qquad\qquad |\mathscr{L}_n^{-1}|_s = O(N_n^{\tau' + \delta s}), \quad \delta \in (0, 1), \ \tau' > 0, \ \forall s > 0.$$

The bound (2.9) implies the interpolation estimates

$$\|\mathscr{L}_n^{-1} h\|_s \leq C(s)(N_n^{\tau' + \delta s} \|h\|_{s_0} + N_n^{\tau' + \delta s_0} \|h\|_s), \quad \forall s \geq s_0,$$

which are sufficient for the Nash–Moser convergence, see section 7 in [3]. Note that the exponent $\tau' + \delta s$ in (2.9) grows with s, unlike the usual Nash–Moser theory where the "tame" exponents are s-independent. Actually the conditions (2.9) are *optimal* for the convergence, as a famous counter-example of Lojasiewicz–Zehnder [18] shows: if $\delta = 1$ the Nash–Moser iterative scheme does not converge.

L^2*-bounds.* The first step is to show that, for "most" parameters $(\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$, the eigenvalues of the restricted linearized operators $\mathscr{L}_n := P_n \mathscr{L}(\mathbf{u}_{n-1})_{|H_n}$ are in modulus bounded from below by $O(N_n^{-\tau})$ and so the L^2-norm of the inverse satisfies

$$(2.10) \qquad\qquad \|\mathscr{L}_n^{-1}\|_0 = O(N_n^\tau).$$

The proof is based on an eigenvalue variation argument. Dividing $\mathscr{L}_n$ by λ, and setting $\xi := 1/\lambda$, we observe that the derivative with respect to ξ satisfies

$$\partial_\xi(\xi \mathscr{L}_n) = P_n \begin{pmatrix} -\Delta + V(x) & 0 \\ 0 & -\Delta + V(x) \end{pmatrix}_{|H_n} + O(\varepsilon\|T_1\|_0 + \varepsilon\|\partial_\lambda T_1\|_0) \overset{(1.8)}{\geq} \frac{\beta_0}{2},$$

for ε small, i.e. it is positive definite. So, the eigenvalues $\mu_{l,j}(\xi,\varepsilon)$ (which depend C^1-smoothly on ξ for fixed ε) of the self-adjoint matrix $\xi\mathscr{L}_n$ satisfy

$$\partial_\xi \mu_{l,j}(\xi,\varepsilon) \geq \frac{\beta_0}{2}, \quad \forall |(l,j)| \leq N_n,$$

which easily implies (2.10) except in a set of measure $O(\varepsilon_0 N_n^{-\tau+d+v})$, see Lemma 6.7 in [3].

REMARK 2.1. *The L^2-estimate* (2.10) *alone implies only that*

$$|\mathscr{L}_n^{-1}|_s \leq N_n^{s+d+v} \|\mathscr{L}_n^{-1}\|_0 = O(N_n^{s+d+v+\tau}), \quad \forall s > 0,$$

which has the form (2.9) *with* $\delta = 1$.

In order to prove the sublinear decay (2.9) for the Green functions we have to exploit (mild) "separation properties" of the small divisors: not all the eigenvalues of $\mathscr{L}_n$ are $O(N_n^{-\tau})$ small. We have to worry only about the SINGULAR sites (l,j) such that

(2.11) $$|\pm\omega \cdot l + |j|^2 + m| \leq \Theta,$$

where $\Theta \geq 1$ is a fixed constant, depending, in particular, on V.

Multiscale Step. The bounds (2.9) follow by an inductive application of a "multiscale argument".

A matrix $A \in \mathscr{M}_E^E$, $E \subset \mathbb{Z}^b \times \{0,1\}$, with $\mathrm{diam}(E) \leq N$ is called N-good if

$$|A^{-1}|_s \leq N^{\tau'+\delta s}, \quad \forall s \in [s_0, s_1],$$

for some $s_1 := s_1(d,v)$ large. Otherwise we say that A is N-bad.

The aim of the multiscale step is to deduce that a matrix $A \in \mathscr{M}_E^E$ with

$$\mathrm{diam}(E) \leq N' = N^\chi \quad \text{with } \chi \gg 1,$$

is N'-good, knowing

- (H1) (Off-diagonal decay) $|A - \mathrm{Diag}(A)|_{s_1} \leq \Upsilon$ where $\mathrm{Diag}(A) := (\delta_{kk'} A_k^{k'})_{k,k' \in E}$.

Condition (H1) means that A is "polynomially localized" close to the diagonal. For the matrix A in (2.5) the constant $\Upsilon = O(\|V\|_{s_1} + \varepsilon\|(p,q)\|_{s_1})$ and Θ, defined in (2.11), must be $\Theta \gg \Upsilon$.

- (H2) (L^2-bound) $\|A^{-1}\|_0 \leq (N')^\tau$.

Condition (H2) is usually verified with an exponent $\tau \geq d + v$ large, imposing lower bounds on the modulus of the eigenvalues of A.

In order to prove an off-diagonal decay for A^{-1}, we need assumptions concerning the N-dimensional submatrices centered along the diagonal of A. We define an index $k \in E$ to be

1. REGULAR for A if $|A_k^k| \geq \Theta$. Otherwise, k is SINGULAR.
2. (A, N)-REGULAR if there is $F \subset E$ such that $\mathrm{diam}(F) \leq 4N$, $\mathrm{d}(k, E \backslash F) \geq N$ and A_F^F is N-good.
3. (A, N)-GOOD if it is regular for A or (A, N)-regular. Otherwise we say that k is (A, N)-BAD.

We suppose that

- (H3) (Separation properties) There is a partition of the (A, N)-bad sites $B = \bigcup_\alpha \Omega_\alpha$ with

$$(2.12) \qquad \mathrm{diam}(\Omega_\alpha) \leq N^{C_1}, \quad \mathrm{d}(\Omega_\alpha, \Omega_\beta) \geq N^2, \quad \forall \alpha \neq \beta,$$

for some $C_1 := C_1(d, \nu) \geq 2$.

The goal of the multiscale proposition is to deduce that A is N'-good, from (H1)–(H2)–(H3), with suitable relations between the constants χ, C_1, δ, s_1, see Proposition 4.1 in [3] for a precise statement. Roughly, the main conditions on the exponents are $C_1 < \delta\chi$ and $2s_1 \gg \chi\tau$. The first means that the size N^{C_1} of any bad clusters Ω_α is small with respect to the size $N' := N^\chi$ of the matrix. The second means that s_1 is large enough to "separate" the resonance effects of two nearby bad clusters Ω_α, Ω_β.

The proof of Proposition 4.1-[3] is based on "resolvent identity" arguments.

Separation properties. We apply the previous multiscale step to the matrix $A_{N_{n+1}}(\varepsilon, \lambda)$. The key property to verify is (H3). It is sufficient to prove the "separation properties" (2.12) for the N_n-BAD sites of $A(\varepsilon, \lambda)$, namely the indices (l_0, j_0) which are singular and for which there exists a site (l, j), with $|(l, j) - (l_0, j_0)| \leq N$, such that $A_{N_n, l, j}(\varepsilon, \lambda)$ is N_n-bad.

Such separation properties are obtained for all the parameters (ε, λ) which are N_n-good, namely such that

$$(2.13) \qquad \forall j_0 \in \mathbb{Z}^d, \quad B_{N_n}(j_0; \varepsilon, \lambda) := \{\theta \in \mathbb{R} : A_{N_n, j_0}(\varepsilon, \lambda, \theta) \text{ is } N_n\text{-bad}\}$$
$$\subset \bigcup_{q=1, \ldots, N_n^{2d+\nu+4}} I_q \quad \text{where } I_q \text{ are disjoint}$$
$$\text{intervals with } |I_q| \leq N_n^{-\tau}.$$

We first use the covariance property (2.8) and the "complexity" information (2.13) to bound the number of "bad" time-Fourier components (this idea goes back to [14]). Indeed

$$A_{N_n, l_0, j_0}(\varepsilon, \lambda) \text{ is } N_n\text{-bad} \iff A_{N_n, j_0}(\varepsilon, \lambda, \omega \cdot l_0) \text{ is } N_n\text{-bad} \iff \omega \cdot l_0 \in B_{N_n}(j_0; \varepsilon, \lambda).$$

Then, using that ω is Diophantine, the complexity bound (2.13) implies that, for each fixed j_0, there are at most $CN_n^{3d+2v+4}$ sites (l_0, j_0), $|l_0| \leq N_{n+1}$, which are N_n-bad, see Corollary 5.1 in [3].

Next, we prove that a N_n^2-"chain" of singular sites, i.e. a sequence of integers

$$k_1, k_2, \ldots, k_L \text{ satisfying (2.11) with } |k_{i+1} - k_i| \leq N_n^2,$$

which are also N_n-bad, has a "length" L bounded by

$$L \leq N_n^{C(d,v)},$$

see Lemma 5.2 in [3]. The proof uses ideas similar to [11]. This implies a partition of the $(A_{N_{n+1}}(\varepsilon, \lambda), N_n)$-bad sites as in (2.12) at order N_n, see Proposition 5.1 in [3]

Measure and "complexity" estimates. In order to conclude the inductive proof we have to verify that "most" parameters (ε, λ) are N_n-good. For this, we do not invoke sub-harmonicity and semi-algebraic set theory as in [11].

We prove first that, except a set of measure $O(\varepsilon_0 N_n^{-1})$, all parameters $(\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$ are N_n-good in a weak sense, namely

$$(2.14) \qquad \forall j_0 \in \mathbb{Z}^d, \quad B_{N_n}^0(j_0; \varepsilon, \lambda) := \{\theta \in \mathbb{R} : \|A_{N_n, j_0}^{-1}(\varepsilon, \lambda, \theta)\|_0 > N_n^\tau\}$$

$$\subset \bigcup_{q=1, \ldots, N_n^{2d+v+4}} I_q, \quad I_q \text{ interval}, |I_q| \leq N_n^{-\tau}.$$

The proof is again based on simple eigenvalue variation arguments, using that $-\Delta + V(x)$ is positive definite, and performing the measure estimates in the set of variables $\xi := 1/\lambda$, $\eta := \theta/\lambda$. In this way we prove that, except a set of parameters $(\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$ of measure $O(\varepsilon_0 N_n^{-1})$, the set $B_{N_n}^0(j_0; \varepsilon, \lambda)$ in (2.14) (of "strongly" bad θ) has a small measure $O(N_n^{-\tau+2d+v+4})$. This and the Lipschitz dependence of the eigenvalues with respect to parameters imply also the complexity bound (2.14), see section 6 in [3].

Finally, the multiscale Proposition step, and the fact that the separation properties of the N_n-bad sites of $A(\varepsilon, \lambda, \theta)$ hold uniformly in $\theta \in \mathbb{R}$, imply inductively that most of the parameters (ε, λ) are actually N_n-good (in the strong sense), concluding the inductive argument, see Lemma 7.6 in [3].

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