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Mathematical Logic — *Involutions on Zilber fields*, by VINCENZO MANTOVA, presented by U. Zannier, communicated on 15 April 2011.

To the memory of Giovanni Prodi.

ABSTRACT. — In this paper, we briefly outline the definition of Zilber field, which is a structure analogue to the complex field with the exponential function. An open conjecture, including Schanuel's Conjecture, is whether the complex field is itself one of these structure.

In view of this conjecture, a natural question raised by Zilber, Kirby, Macintyre and others is whether they have an automorphism of order two akin to complex conjugation.

We announce, without proof, the positive answer: for cardinality up to the continuum there exists an involution of the field commuting with the exponential function. Moreover, in the case of cardinality of the continuum, the automorphism can be taken such that its fixed field is exactly $\mathbb{R}$, and the kernel of the exponential function is $2\pi i\mathbb{Z}$.

KEY WORDS: pseudoexponentiation, conjugation, involution, Zilber fields, real closed fields.

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1. INTRODUCTION

In [6] Zilber defined a new algebraic structure, commonly called "Zilber field", which is an analogue of the complex field equipped with the exponential function, but with good model-theoretic properties.

In more details, a Zilber field is a pair (K, E), where K is an (uncountable) algebraically closed field of characteristic 0 and E is a function from K to K^* such that the fundamental relation

$$E(x + y) = E(x) \cdot E(y)$$

holds, and several other axioms are satisfied. These axioms are either some known facts about the classical exp which we want to be true for E, or deep conjectures, among which, most notably, Schanuel's Conjecture. A function E satisfying such properties is generally called "pseudoexponentiation". We will give a full account of the axioms in Section 2.

The remarkable result by Zilber is that, in an appropriate infinitary language, the sentence Ψ expressing the axioms is *uncountably categorical*. In particular, there is just one model of cardinality $2^{\aleph_0}$, up to isomorphism. In Zilber's view, categorical structures should correspond to natural mathematical objects, so he conjectured that the model of cardinality $2^{\aleph_0}$ is just the classical field $(\mathbb{C}, \exp)$. If true, it would imply Schanuel's Conjecture as well.

Motivated by this idea, a natural question has been raised by Macintyre, Kirby, Zilber and others: can we find a 'pseudoconjugation' on (K, E) analogous to the complex conjugation? In other words, is there a field automorphism of order two which commutes with E? Its fixed field would be a real closed field, and it would constitute an example of a 'Zilber's real exponentiation', together with a 'Zilber's cosine'.

Here we announce the positive answer for the Zilber fields of cardinality up to $2^{\aleph_0}$; the proof itself, with all the details, will be the subject of a subsequent paper.

THEOREM 1. *Let (K, E) be a Zilber field. If $|K| \leq 2^{\aleph_0}$, then there exists an involution on (K, E), i.e., a field automorphism $\sigma : K \to K$ of order two such that $\sigma \circ E = E \circ \sigma$.*

We deduce this theorem from a quite stronger statement, which in some sense is the reverse: we show that given σ, we can construct an appropriate function E such that (K, E) is a Zilber field. On the other hand, by categoricity of Zilber's axioms, we obtain that the existence of σ transfers to all Zilber fields of the same cardinality.

At the present moment, our method works only for some special σ's, and this is the reason for the restriction on the cardinality. The full statement is the following:

THEOREM 2. *Let K be an uncountable algebraically closed field of characteristic 0, and $\sigma : K \to K$ a field automorphism of order two. If the order topology on K^σ is second-countable, then there is a function $E : K \to K^*$ such that (K, E) is a Zilber field, and $\sigma \circ E = E \circ \sigma$.*

If we take $K = \mathbb{C}$ and σ as the complex conjugation, then $\mathbb{C}^\sigma = \mathbb{R}$ is second-countable, and we obtain that there is a pseudoexponentiation E on $\mathbb{C}$ commuting with σ. In particular, by categoricity, the Zilber field of cardinality $2^{\aleph_0}$ has an involution. Since Zilber fields form an abstract elementary class, we can use the downward Löwenheim-Skolem property to recover the same statement for smaller fields, and deduce the full statement of Theorem 1.

In Section 3 we will sketch, without proofs, an explicit construction of such a function E. The verification that the resulting structure (K, E) is a Zilber field is not immediate, and it will be the subject of a subsequent paper. As hinted above, we can use the construction on $\mathbb{C}$ with the complex conjugation, obtaining the following special case.

THEOREM 3. *There is a function $E : \mathbb{C} \to \mathbb{C}^*$ such that $(\mathbb{C}, E)$ is a Zilber field, $E(\bar{z}) = \overline{E(z)}$ for all $z \in \mathbb{C}$, and $E\left(2\pi i \frac{p}{q}\right) = e^{2\pi i (p/q)}$ for all $p \in \mathbb{Z}$, $q \in \mathbb{Z}^*$.*

This answers specifically to the question posed in [3, 3.10], and moreover it says that there is a σ whose fixed field is $\mathbb{R}$ itself. However, our construction produces a function E on $\mathbb{C}$ which certainly does not coincide with exp, although a priori there could be an automorphism of the complex field $\mathbb{C}$ which brings E to exp (see the comments in Section 4).

2. Zilber's axioms

Here is a list of the axioms defining Zilber fields. As we anticipated in the introduction, some of the axioms are just properties of $(\mathbb{C}, \exp)$, while the others are conjecturally true on $(\mathbb{C}, \exp)$ but still unknown.

2.1. Trivial properties of $(\mathbb{C}, \exp)$. The first 'trivial' axioms of (K, E) are easy to state:

(ACF) K is an algebraically closed field of characteristic 0;
(E) E is a homomorphism $E : (K, +) \to (K^*, \cdot)$;
(LOG) E is surjective;
(STD) the kernel is a cyclic group, i.e., $\ker E = \omega\mathbb{Z}$ for some $\omega \in K^*$.

These axioms can be stated in just one $\mathscr{L}_{\omega_1,\omega}$ sentence: an infinite conjunction for the first ones, and an infinite disjunction for the last one.

2.2. Conjecturally true axioms. The following two axioms are not known to be true on the structure $(\mathbb{C}, \exp)$.

(SC) Schanuel's Condition: for any $x_1, \dots, x_n \in K$ linearly independent over $\mathbb{Q}$,

$$\mathrm{tr.deg.}_{\mathbb{Q}}(x_1, \dots, x_n, E(x_1), \dots, E(x_n)) \geq n.$$

One can interpret Schanuel's Condition as stating that there are not too many algebraic relations among the values of the exponential, except for the trivial ones forced by the $\mathbb{Q}$-linear dependence of $x_1, \dots, x_n$ (which induce monomial relations between the values $E(x_1), \dots, E(x_n)$). Indeed, one can see that Schanuel's Condition implies that the smallest subfield closed under E is always isomorphic to the same structure, the "free E-field over $\emptyset$" [4, Theorem 4].

A consequence of (SC) is that the function $\delta(\bar{x}) := \mathrm{tr.deg.}_{\mathbb{Q}}(\bar{x}, E(\bar{x})) - \mathrm{lin.d.}_{\mathbb{Q}}(\bar{x})$ is always non-negative. As such, δ is a suitable predimension for Hrushovski amalgamation, and it is exactly how it is used in Zilber's proof of categoricity.

Under axiom (STD), the field $\mathbb{Q}$ is definable, so (SC) is a countable conjunction of first order formulae.

(SEC) Strong Exponential-algebraic Closure: for any 'free rotund'[1] irreducible variety $V \subset K^n \times (K^*)^n$ (see below), and any finite tuple $\bar{a} \in K^{<\omega}$ with V defined over $\bar{a}$, there is an $\bar{x} \in K^n$ such that

$$(\bar{x}, E(\bar{x})) \in V,$$

and $\mathrm{tr.deg.}_{\bar{a}}(\bar{x}, E(\bar{x})) = \dim V$.

[1] In the original paper by Zilber, this is called "free ex-normal". Here we prefer to use the notation proposed in [1], in order to distinguish it from the classical concept of "normal variety" in algebraic geometry.

Analogously to other situations where we introduce the existential closure, e.g. for differential fields, we ask that each non-over-determined system of equations involving one iteration of the exponential has a solution. A system of equations can be seen as an absolutely irreducible affine variety containing all the solutions, which are the points of the form $(\bar{x}, E(\bar{x}))$, and (SC) imposes severe restrictions to the shape of this variety.

Indeed, a point of the form $(\bar{x}, E(\bar{x}))$ must have high transcendence degree, and the same must be true for any $\mathbb{Q}$-linear combination of the element of $\bar{x}$; this implies that V has large dimension, and the same is true for the varieties obtained composing the coordinate functions of V.

If we write the action of $\mathbb{Z}$ on the group $K \times K^*$ as $[m] : (x, w) \mapsto (mx, w^m)$, and for each $M \in \mathscr{M}_{k,n}(\mathbb{Z})$ the induced map $[M] : K^n \times (K^*)^n \to K^k \times (K^*)^k$ as $M : (\bar{x}, \overline{w}) \mapsto (M \cdot \bar{x}, \overline{w}^M)$, we can see that (SC) implies

$$\text{tr.deg.}_{\mathbb{Q}}(M \cdot (\bar{x}, E(\bar{x}))) \geq \operatorname{rank} M,$$

when the coordinates of $\bar{x}$ are $\mathbb{Q}$-linearly independent. If V is a variety containing $(\bar{x}, E(\bar{x}))$ as a generic point (in the geometric sense), an analogous condition on the dimension of V holds. This is the meaning of rotund:

Definition 4. A variety $V \subset K^n \times (K^*)^n$ is *rotund* if for any $k \in \mathbb{N}^*$, and for any matrix $M \in \mathscr{M}_{k,n}(\mathbb{Z})$, we have

$$\dim M \cdot V \geq \operatorname{rank} M.$$

Moreover, we want our system of equations not to be “over-determined”. In our case it translates to the following.

Definition 5. A variety $V \subset K^n \times (K^*)^n$ is *free* if, for any vector $\overline{m} \in \mathbb{Z}^n$, the two projections of $\overline{m} \cdot V \subset K \times K^*$ on the coordinates are not constant.

Note that indeed when the variety is not free, we can eliminate one variable from the system of equations.

The property of being free and rotund for a variety can be expressed with a first-order formula [6, Theorem 3.2]. Under the axioms already stated, the existence of generic points can be also formulated as a countable conjunction of first-order formulas [2, Proposition 5].

2.3. A non-trivial property of $(\mathbb{C}, \exp)$. This last axiom is known to be true on $(\mathbb{C}, \exp)$ [6, Lemma 5.12].

(CCP) Countable Closure Property: for any given free rotund irreducible variety $V \subset K^n \times (K^*)^n$, defined over a finite tuple $\bar{c} \in K^{<\omega}$, such that $\dim V = n$, there are at most countably many points $(\bar{x}, E(\bar{x})) \in V$ such that $\text{tr.deg.}_{\bar{c}}(\bar{x}, E(\bar{x})) = \dim V$.

This condition is the reformulation for Zilber fields of a general property that one can impose on the closure operator of a so called “quasi-minimal excellent class”, as the class of Zilber fields themselves: it says that the closure of a finite set must be at most countable (hence the name). It is the last ingredient needed to get categoricity [5, Theorem 2].

3. The construction

We can prove Theorem 2 by effectively building the function E starting from the knowledge of the involution σ. Let us call R the fixed field K^σ. The field R is necessarily real closed, with $K = R[\sqrt{-1}] = R[i]$, and as such, it is an ordered field equipped with the corresponding topology.

It is easy to see that $\sigma \circ E = E \circ \sigma$ if and only if

(1) $E(R) \subset R_{>0}$;
(2) $E(iR) \subset \mathbb{S}_1(R) = \{z \in K : z \cdot \sigma(z) = 1\}$.

Moreover, R and iR are $\mathbb{Q}$-linearly independent. This means that we can build the function E on R and on iR independently, and we will do so that the restrictions (1), (2) are satisfied.

For the sake of simplicity, let us assume that $K = \mathbb{C}$ and that σ is the complex conjugation; in this case, $R = \mathbb{R}$, and we can use the standard notation $\Re$, $\Im$ for real and imaginary parts, and $|\cdot|$ for the modulus. We will denote by Θ the 'phase', i.e., the quantity $\Theta(z) = z/|z|$.

We proceed by back-and-forth, using transfinite induction: at each step, we take an element in $\mathbb{C}$ or in $\mathbb{C}^*$ and we define E so that the chosen element appears in the domain or in the image; at the end, the function E will be surjective and defined everywhere. Moreover, at each step we also take a free rotund irreducible variety, and we define E on some new points such in a way that the variety gets solutions.

We carefully do this construction, so that the domain of E is always a $\mathbb{Q}$-vector space, and consequently its image is a divisible group. This is crucial to make sure that E is well defined.

In our basic step we take a transcendental element $\omega \in i\mathbb{R}$, and we define $E\left(\frac{p}{q}\omega\right) := \zeta_q^p$, where (ζ_q) is a coherent system of roots of unity. As ω is arbitrary, we can choose in particular $\omega = 2\pi i$.

After that, we take an enumeration $\{\alpha_j\}_{j<2^{\aleph_0}}$ of $\mathbb{R}$, an enumeration $\{\beta_j\}_{j<2^{\aleph_0}}$ of the multiplicative group $\mathbb{C}^*$, and an enumeration $\{V_j\}_{j<2^{\aleph_0}}$ of all free "perfectly" rotund irreducible varieties (a particular subset of all rotund varieties; see below). We proceed by transfinite induction.

Suppose that we are at the stage $j \le 2^{\aleph_0}$, and that the function E has already been defined on some elements of $\mathbb{C}$. We extend it as follows:

(1) If E is already defined on α_j, we proceed to the next step; otherwise, we choose an element β in $\mathbb{R}_{>0}$ transcendental over the current domain, the current image of E, and α_j, and we define $E\left(\frac{p}{q}\alpha_j\right) := \beta^{p/q}$ for some coherent choice of positive roots of β. We repeat the same for $i\alpha_j$, taking β in $\mathbb{S}_1(\mathbb{R})$ and setting $E\left(i\frac{p}{q}\alpha_j\right) := \beta^{p/q}$.

(2) If β_j is already in the image of E, we proceed to the next step; otherwise, we choose an element α in $\mathbb{R}$ transcendental over the current domain of E, the current image of E, and $|\beta_j|$, and we define $E\left(\frac{p}{q}\alpha\right) := |\beta_j|^{p/q}$. We repeat the same for $\Theta(\beta_j)$, taking another α in $\mathbb{R}$ and setting $E(i\frac{p}{q}\alpha) := \Theta(\beta_j)^{p/q}$.

(3) This is more complicated. Given V_j, let $\mathscr{V}$ be the family of all the irreducible components of the varieties $\frac{m}{n} \cdot V_j$, i.e., all the irreducible varieties W such that $(m\,\mathrm{Id}) \cdot W = (n\,\mathrm{Id}) \cdot V_j$, for some $m, n \in \mathbb{N}^*$.

If for all $W \in \mathscr{V}$ we can find countably many algebraically independent generic points of the form $(\bar{x}, E(\bar{x}))$ in W, such that they are *dense* in the order topology over W, we proceed to the next step.

Otherwise, we take a countable dense set of such points on each W, with the restriction that they must be "real generic" (see below) over the current domain of E, the current image of E, and the field of definition of W. If $(\bar{x} + i\bar{y}, \bar{\rho}\bar{\theta})$ is one of such points, we define $E\left(\frac{p}{q}x_k\right) := \rho_k^{p/q}$, $E\left(i\frac{p}{q}y_k\right) := \theta_k^{p/q}$.

(4) On limit ordinals, we define E taking the union of the previously defined functions, as usual.

By "perfectly rotund" varieties we mean the rotund varieties which in some sense are 'irreducible components' of other rotund varieties. They correspond to simple extensions in Hrushovski's amalgamation. The actual definition is almost the same as [1, Definition 6.3], where it corresponds to simple algebraicity, but we drop the assumption "$\dim V = n$", taking algebraicity out of the picture.

By "real generic point" of a variety V over some set A, we mean a point $(\bar{x} + i\bar{y}, \bar{\rho}\bar{\theta}) \in V$, where $\bar{x}, \bar{y}$ are the real and imaginary parts, and $\bar{\rho}, \bar{\theta}$ are the moduli and the phases, such that not only $\mathrm{tr.deg.}_A(\bar{x} + i\bar{y}, \bar{\rho}\bar{\theta}) = \dim V$, as in the usual definition of generic, but also $\mathrm{tr.deg.}_{\Re(A), \Im(A)}(\bar{x}, \bar{y}, \bar{\rho}, \bar{\theta}) = 2 \dim V$.

This construction clearly yields a function E such that $\sigma \circ E = E \circ \sigma$, thanks to requirements (1) and (2). Moreover, the domain of E is the whole set $\mathbb{C}$, and the image is $\mathbb{C}^*$, and it is easy to verify that the kernel is exactly $\omega\mathbb{Z}$. Hence, the trivial axioms are verified.

It is also easy to see that (SEC) holds in the resulting (K, E), thanks to step (3). Axiom (SC) is true at the basic step, and it is easy to see that it is preserved by steps (1), (2) and (4). However, verifying that step (3) preserves (SC) is more difficult, as it involves a careful study of rotund varieties. It is even more difficult to verify (CCP) in the final structure.

Indeed, axiom (CCP) is the reason why we have to take a *dense* set of points in step (3). Suppose that (CCP) is not true. Let j be the least ordinal such that at the stage j there is a rotund variety $X \subset K^n \times (K^*)^n$, with $\dim X = n$, containing uncountably many generic points of the form $(\bar{x}, E(\bar{x}))$ (note that it can be $j = 2^{\aleph_0}$). One can see that for this to happen, there must be a sequence (V_l) of uncountably many rotund varieties, such that we have added solutions to all of them before the stage j, and when adding new solutions, they induce new solutions on X too. In this situation, one can prove that there is a sequence of open sets (U_l) in X such that the solutions in V_l induce solutions in U_l, and vice versa (strictly speaking, it happens between the sets $\bigcup_{m,n \in \mathbb{Z}} \frac{m}{n} \cdot V_l$ and $\bigcup_{m,n \in \mathbb{Z}} \frac{m}{n} \cdot U_l$).

However, since the topology on X is second-countable, one can see that the contributions of the V_l's to the solutions of X actually come all from a countable subsequence of (V_l). This implies that X must have gained uncountably many solutions at an earlier stage than j, a contradiction.

4. Comments

As stated in the introduction, our construction is able to produce a function E on $\mathbb{C}$ which commutes with the complex conjugation. However, our function E cannot happen to be exp itself, as the set of generic points of the form $(\bar{x}, E(\bar{x}))$ on a free rotund variety $V \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$, with $\dim V = n$, is dense, while Zilber proved that for exp the corresponding set is discrete [6, Lemma 5.12]. Taking the opposite point of view, under Zilber's conjecture that $E \cong \exp$, it means that we have picked a class of involutions which does not contain complex conjugation.

Moreover, this approach isn't able to guarantee that $E_{\restriction\mathbb{R}}$ is monotone. Note that if $E_{\restriction\mathbb{R}}$ were monotone, there would be a real number $c \in \mathbb{R}$ such that $E_{\restriction\mathbb{R}}(x) = \exp_{\restriction\mathbb{R}}(c \cdot x)$, and this would imply Schanuel's Conjecture for the function $\exp_{\restriction\mathbb{R}}(c \cdot x)$.

The question if there is an involution σ on (K, E) such that $E_{\restriction K^\sigma}$ is also a monotone function remains open, and it seems to be much more difficult. Work is in progress about finding a way to adapt the above construction, but most probably we have to renounce to some axioms, such as (CCP).

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