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Mechanics — *Stability of ternary reaction-diffusion dynamical systems*, by SALVATORE RIONERO, communicated on 11 February 2011.

Dedicated to the memory of Professor Giovanni Prodi.

ABSTRACT. — In view of the fundamental role played by the ternary reaction-diffusion dynamical systems of P.D.Es for the description of the behaviour of continuous media and other phenomena (biological, chemical, ...), this paper is devoted to their nonlinear L^2-stability. General conditions guaranteeing the (local) nonlinear L^2-stability via the stability of reduced or symmetric ternary linear systems of O.D.Es are found. The results obtained are applied to a triply convective-diffusive fluid mixture saturating a porous layer and to a biological problem.

KEY WORDS: Ternary reaction-diffusion systems, stability, direct method.

MATHEMATICS SUBJECT CLASSIFICATION: 35K57, 37Cxx, 34D20, 34B16.

1. Introduction

The ternary reaction-diffusion dynamical systems of P.D.Es model or take part in the models of many phenomena of the real world. In particular their role is fundamental for the description of the motion of continuous media and the behaviour of many other (biological, chemical, ...) phenomena. In fact the momentum (vectorial) equation is nothing else that a (scalar) ternary reaction-diffusion system of P.D.Es. Further, the convection-diffusion of a triply mixture in a porous layer can be reduced rigorously to the study of such kind of dynamical systems [13]. As concerns other phenomena, many applications of ternary reaction-diffusion systems of P.D.Es can be found, for instance, in [1]–[5]. In the present paper we consider the L^2-stability of such systems aimed to finding conditions guaranteeing the asymptotic stability via the asymptotic stability of linear reduced or symmetric systems of O.D.Es. The plan of the paper is as follows: Section 2 is devoted to preliminaries. Successively, in Section 3, the condition guaranteeing the nonlinear L^2-stability via the stability of a linear reduced ternary system of O.D.Es are found. Section 4 is devoted to obtaining the stability via the stability of a symmetric linear system of O.D.Es. In Section 5, two applications are furnished concerned with: (1) the stability of a triply convective-diffusive fluid mixture saturating a porous layer (subsection 5.1); (2) the stability of a biological model. The paper ends with an appendix (Section 6) in which some details concerned with the Liapunov functional V introduced in Section 3 are recalled.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^q$, $(q = 1, 2, 3)$, be a smooth bounded domain. This paper is concerned with the reaction-diffusion systems

$$\frac{\partial \mathbf{u}}{\partial t} = L\mathbf{u} + \mathbf{F}, \quad \text{in } \Omega \times \mathbb{R}^+, \tag{2.1}$$

with $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{F} = (F_1, F_2, F_3)^T$,

$$L = \begin{pmatrix} a_{11} + \gamma_1 \Delta & a_{12} & a_{13} \\ a_{21} & a_{22} + \gamma_2 \Delta & a_{23} \\ a_{31} & a_{32} & a_{33} + \gamma_3 \Delta \end{pmatrix}; \tag{2.2}$$

$F_i = F_i(u_1, u_2, u_3, \nabla u_1, \nabla u_2, \nabla u_3)$, $(i = 1, 2, 3)$, being (generally) nonlinear and

$$\begin{cases} a_{ij} = \text{const.} \in \mathbb{R}, \quad \gamma_i = \text{const.} > 0, \quad i, j \in \{1, 2, 3\}, \\ u_i : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \to u_i(\mathbf{x}, t) \in \mathbb{R}, \quad \forall i \in \{1, 2, 3\}. \end{cases} \tag{2.3}$$

To (2.1) we append the Robin boundary conditions

$$\beta \mathbf{u} + (1 - \beta) \nabla \mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{2.4}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$,

$$\begin{cases} \beta : \mathbf{x} \in \partial\Omega \to \beta(\mathbf{x}) \in \mathbb{R}, \\ 0 \le \beta \le 1, \quad \forall \mathbf{x} \in \partial\Omega, \end{cases} \tag{2.5}$$

β being a sufficiently regular function not identically zero.

The nonlinear functions $F_i = F_i(u_1, u_2, u_3, \nabla u_1, \nabla u_2, \nabla u_3)$ are assumed to be sufficiently regular and such that

$$(F_i)_{u_1 = u_2 = u_3 = 0} = 0, \quad \forall i \in \{1, 2, 3\}. \tag{2.6}$$

Therefore (2.1)–(2.6) admits the zero solution. To the L^2-stability of this solution is precisely devoted the present paper.

Remark 2.1. *As it is well known, the stability of a non zero solution of a system S can be reduced to the stability of the zero solution of a system S^* easily linked to S.*

We assume that Ω is of class C^p $(p > 2)$ and has the interior cone property. We denote by

- $\langle \cdot, \cdot \rangle$ the scalar product of $L^2(\Omega)$;
- $\langle \cdot, \cdot \rangle_{\partial\Omega}$ the scalar product of $L^2(\partial\Omega)$;
- $\| \cdot \|$ the norm of $L^2(\Omega)$;

- $\|\cdot\|_{\partial\Omega}$ the norm of $L^2(\partial\Omega)$;
- $W^{1,2}(\Omega,\beta)$ the functional space such that

$$W^{1,2}(\Omega,\beta) = \{\varphi \in W^{1,2}(\Omega) \cap W^{1,2}(\partial\Omega), \beta\varphi + (1-\beta)\nabla\varphi\cdot\mathbf{n} = 0, \text{ on } \partial\Omega\}.$$

For $\beta > 0$, $\beta \not\equiv 1$, it follows {cfr. [2], pp. 92–98} that

$$\left\|\sqrt{\frac{\beta}{1-\beta}}\varphi\right\|_{\partial\Omega}^2 + \|\nabla\varphi\|^2 \geq \bar{\alpha}\|\varphi\|^2, \tag{2.7}$$

where $\bar{\alpha} = \bar{\alpha}(\Omega,\beta) = \text{const.} > 0$, is the smallest eigenvalue of the spectral problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & \text{in } \Omega, \\ \beta\varphi + (1-\beta)\nabla\varphi\cdot\mathbf{n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

i.e. the principal eigenvalue of $-\Delta$ in $W^{1,2}(\Omega,\beta)$.

In the sequel *we assume that*

(i) (2.1)–(2.5) has the properties of a dynamical system [4] embedded in $W^{1,2}(\Omega,\beta)$ and hence

$$u_i \in W^{1,2}(\Omega,\beta) \tag{2.9}$$

(ii) the functions F_i are such that

$$\begin{aligned} \left\langle \sum_{i=1}^3 |u_i|, \sum_{j=1}^3 |F_j| \right\rangle &\leq k_1(\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2)^{1+\varepsilon_1} \\ &\quad + k_2(\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2)^{\varepsilon_2} \\ &\quad \times (\|\nabla u_1\|^2 + \|\nabla u_2\|^2 + \|\nabla u_3\|^2), \end{aligned} \tag{2.10}$$

with k_i, ε_i, $(i = 1,2)$, non negative constants.

Remark 2.2. *In the applications, it is of interest to know the value of $\bar{\alpha}$ and the influence of the domain size on $\bar{\alpha}$. Having this in mind, we remark that if $\Omega \subset \mathbb{R}$ is given by*

$$0 \leq x \leq l, \quad l = const. > 0, \tag{2.11}$$

and (2.8) *reduces to*

$$\begin{cases} \varphi'' + \lambda\varphi = 0, & in\ \Omega, \\ \varphi = 0, \quad on\ x = 0; & \varphi' = 0 \quad on\ x = l, \end{cases} \tag{2.12}$$

then the sequence $\{\varphi_n\}$ with

$$\varphi_n = \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi}{l}x\right], \quad n = 1,2,\ldots, \tag{2.13}$$

is a complete orthogonal system of eigenfunctions of (2.12) *in* $W^{1,2}(\Omega, \beta)$*, and*

$$\beta(0) = 1, \quad \beta(1) = 0. \tag{2.14}$$

Further the principal eigenvalue $\bar{\alpha}$ *is given by*

$$\bar{\alpha} = \frac{\pi^2}{4l^2}. \tag{2.15}$$

Analogously if Ω *is given by*

$$\Omega = \{(x, y) : 0 \le x \le l_1, 0 \le y \le l_2\}, \tag{2.16}$$

with l_i $(i = 1.2)$*, positive constants, and* (2.8) *reduces to*

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega - \Sigma, \\ \nabla\varphi \cdot \mathbf{n} = 0 & \text{on } \Sigma, \\ \Sigma = \{(x, y) : x = l_1, y \in [0, l_2]\}, \end{cases} \tag{2.17}$$

then the eigenfunctions are given by

$$\varphi_{n,m} = \sin\left[(n-1)\frac{\pi}{l_1}x\right]\sin\frac{m\pi}{l_2}y, \tag{2.18}$$

with $(n, m \in \mathbb{N}^+)$, β *is defined by*

$$\begin{cases} \beta(0, y) = 1 & \text{for } y \in [0, l_2], \\ \beta(x, 0) = \beta(x, l_2) = 1 & \text{for } x \in [0, l_1], \\ \beta(l, y) = 0 & \text{for } y \in [0, l_2], \end{cases} \tag{2.19}$$

and the principal eigenvalue is given by

$$\bar{\alpha} = \left(\frac{1}{4l_1^2} + \frac{1}{l_2^2}\right)\pi^2 = \frac{(l_2^2 + 4l_1^2)\pi^2}{4\Omega_*^2}, \tag{2.20}$$

with $\Omega_* =$ *measure of* $\Omega = l_1 l_2$.

Obviously in the case (2.12) *the diffusion through* $x = l$ *is not allowed, while in the case* (2.17) *the diffusion is not allowed through* Σ.

Setting

$$b_{11} = a_{11} - \bar{\alpha}\gamma_1, \quad b_{22} = a_{22} - \bar{\alpha}\gamma_2, \quad b_{33} = a_{33} - \bar{\alpha}\gamma_3, \tag{2.21}$$

our aim, in the guideline of [6]–[10], is precisely *to find conditions on* a_{ij}*, with* $i \neq j$*, able to reduce the stability of the zero solution of* (2.1)–(2.6) *to the stability* *of the zero solution of the linear system of O.D.Es*

$$\frac{d\mathbf{u}}{dt} = \mathscr{L}\mathbf{u}, \tag{2.22}$$

with either

$$\mathscr{L} = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & a_{23} \\ 0 & a_{32} & b_{33} \end{pmatrix} \tag{2.23}$$

or—when $a_{ij}a_{ji} > 0$, $(i, j = 1, 2, 3)$—

$$\mathscr{L} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{2.24}$$

b_{11}, b_{22}, b_{33} *being given by* (2.21) *and*

$$b_{ij} = b_{ji} = (\operatorname{sign} a_{ij})\sqrt{a_{ij}a_{ji}}. \tag{2.25}$$

We end this section by recalling the conditions governing the stability of the zero solution of (2.22)–(2.23).

Theorem 2.1. *The zero solution of* (2.22)–(2.23) *is asymptotically stable if and only if*

$$b_{11} < 0, \quad I = b_{22} + b_{33} < 0, \quad A = b_{22}b_{33} - a_{23}a_{32} > 0, \tag{2.26}$$

and is only stable either when $\{b_{11} = I = 0, A > 0\}$ *or* $\{b_{11} = 0, I < 0, A > 0\}$ *or* $\{b_{11} < 0, I = 0, A > 0\}$.

Proof. The proof is well-know and easily found.

3. Reduction to the stability of the zero solution of (2.22)–(2.23)

For the sake of simplicity and concreteness, we will confine ourselves to the case of Ω given by (2.11) with $l = 1$ and the boundary conditions $(2.12)_2$. Setting

$$\begin{cases} f_1 = (a_{12}u_2 + a_{13}u_3), \quad f_2 = a_{21}u_1, \quad f_3 = a_{31}u_1, \\ g_i = \gamma_i(\Delta u_i + (1-\varepsilon)\bar{\alpha}_i u_i), \quad F_i^* = F_i + f_i + g_i, \quad (i = 1, 2, 3), \\ \mathbf{F}^* = (F_1^*, F_2^*, F_3^*)^T, \quad b_{11}^* = a_{11} - (1-\varepsilon)\bar{\alpha}\gamma_1, \quad b_{22}^* = a_{22} - (1-\varepsilon)\bar{\alpha}\gamma_2, \\ b_{33}^* = a_{33} - (1-\varepsilon)\bar{\alpha}a_{33}, \quad \mathscr{L}^* = \begin{pmatrix} b_{11}^* & 0 & 0 \\ 0 & b_{22}^* & a_{23} \\ 0 & a_{32} & b_{33}^* \end{pmatrix}, \end{cases} \tag{3.1}$$

(2.1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \mathscr{L}^* \mathbf{u} + \mathbf{F}^*. \tag{3.2}$$

Introducing the scalings μ_i (to be chosen suitably), and setting

$$u_i = \mu_i U_i, \quad b^*_{ij} = a_{ij}\frac{\mu_j}{\mu_i}, \quad i \neq j, \tag{3.3}$$

(3.2) becomes

$$\frac{\partial \mathbf{U}}{\partial t} = \tilde{L}\mathbf{U} + \tilde{\mathbf{F}}, \tag{3.4}$$

with

$$\begin{cases} \mathbf{U} = (U_1, U_2, U_3)^T, \quad \tilde{L} = \begin{pmatrix} b^*_{11} & 0 & 0 \\ 0 & b^*_{22} & b^*_{23} \\ 0 & b^*_{32} & b^*_{33} \end{pmatrix}, \\ \tilde{\mathbf{F}} = \left(\frac{1}{\mu_1}F^*_1, \frac{1}{\mu_2}F^*_2, \frac{1}{\mu_3}F^*_3\right)^T, \quad \frac{1}{\mu_i}F^*_i = \frac{1}{\mu_i}(F_i + f_i + g_i), \\ \frac{1}{\mu_1}f_1 = \left(a_{12}\frac{\mu_2}{\mu_1}U_2 + a_{13}\frac{\mu_3}{\mu_1}U_3\right), \quad \frac{1}{\mu_2}f_2 = a_{21}\frac{\mu_1}{\mu_2}U_1, \\ \frac{1}{\mu_3}f_3 = a_{31}\frac{\mu_1}{\mu_3}U_1, \quad \frac{1}{\mu_i}g_i = \gamma_i(\Delta U_i + (1-\varepsilon)\bar{\alpha}U_i). \end{cases} \tag{3.5}$$

Lemma 3.1. *Let* Ω *be given by* (2.11) *with* $l = 1$ *and* $(2.12)_2$ *hold. Then* $\forall \varphi \in W^{1,2}(\Omega, \beta)$, *the inequalities*

$$\begin{cases} \|\varphi\|^2 \le \dfrac{4}{\pi^2}\|\nabla\varphi\|^2, \quad \|\varphi^2\|^2 \le \dfrac{16}{\pi^4}\|\nabla\varphi\|^4, \\ \langle \varphi, \Delta\varphi \rangle \le -(1-\varepsilon)\|\nabla\varphi\|^2 - \varepsilon\dfrac{\pi^2}{4}\|\varphi\|^2, \end{cases} \tag{3.6}$$

with $\varepsilon = const. \in [0, 1]$, *hold.*

Proof. $(3.6)_1$ is the well-known Poincarè inequality in the case at hand {cfr. [4], p. 338}. The Sobolev inequality $(3.6)_2$ is easily obtained since, by virtue of $(3.6)_1$, it follows that

$$\frac{1}{2}\varphi^2(x) = \int_0^x \varphi\varphi'\,dx \le \|\varphi\| \cdot \|\nabla\varphi\| \le \frac{2}{\pi}\|\nabla\varphi\|^2,$$

and hence squaring and integrating on $[0,1]$, $(3.6)_2$ is immediately obtained. Finally, in view of $(2.12)_2$–$(2.12)_3$, one obtains

$$\langle\varphi,\Delta\varphi\rangle = \langle\varphi,\nabla\varphi\cdot\mathbf{n}\rangle_{\partial\Omega} - \|\nabla\varphi\|^2 = -(1-\varepsilon)\|\nabla\varphi\|^2 - \varepsilon\frac{\pi^2}{4}\|\varphi\|^2.$$

Setting

$$\text{(3.7)}\quad \begin{cases} A^* = b_{22}^*b_{33}^* - b_{23}^*b_{32}^* = b_{22}^*b_{33}^* - a_{23}a_{32}, \quad A_1 = A^* + (b_{32}^*)^2 + (b_{33}^*)^2, \\ A_2 = A^* + (b_{22}^*)^2 + (b_{23}^*)^2, \quad A_3 = b_{22}^*b_{32}^* + b_{23}^*b_{33}^*, \\ \Phi_1 = \langle A_1U_2 - A_3U_3, \gamma_2(\Delta U_2 + \bar{\alpha}U_2)\rangle \\ \qquad + \langle A_2U_3 - A_3U_2, \gamma_3(\Delta U_3 + \bar{\alpha}U_3)\rangle, \end{cases}$$

the following Lemma holds

Lemma 3.2. *Let*

$$\text{(3.8)}\qquad A^* > 0, \quad (\gamma_2+\gamma_3)A_3 \le 2\sqrt{A_1A_2\gamma_2\gamma_3}.$$

Then

$$\text{(3.9)}\qquad \Phi_1 \le 0.$$

Proof. Assuming for simplicity $\gamma_2 \le \gamma_3$, by virtue of (3.8), the following cases are possible

$$\text{(3.10)}\qquad (\gamma_2+\gamma_3)|A_3| = 2\sqrt{A_1A_2\gamma_2\gamma_3},$$

$$\text{(3.11)}\qquad \gamma_2 < \gamma_3, \quad (\gamma_2+\gamma_3)|A_3| < 2\gamma_2\sqrt{A_1A_2},$$

$$\text{(3.12)}\qquad \gamma_2 < \gamma_3, \quad 2\gamma_2\sqrt{A_1A_2} \le (\gamma_2+\gamma_3)|A_3| \le 2\sqrt{A_1A_2\gamma_2\gamma_3}.$$

In view of the boundary data it turns out that

$$\text{(3.13)}\quad \Phi_1 = \begin{cases} \gamma_2A_1(-\|\nabla U_2\|^2 + \bar{\alpha}\|U_2\|^2) + \gamma_3A_2(-\|\nabla U_3\|^2 + \bar{\alpha}\|U_3\|^2) \\ +(\gamma_2+\gamma_3)A_3\langle\nabla U_2,\nabla U_3\rangle - (\gamma_2+\gamma_3)A_3\bar{\alpha}\langle U_2,U_3\rangle. \end{cases}$$

In the case (3.10) it turns out that

$$\text{(3.14)}\qquad (\gamma_2+\gamma_3)A_3 = \pm\sqrt{A_1A_2\gamma_2\gamma_3},$$

which, by virtue of $(3.6)_1$, implies

$$\text{(3.15)}\quad \Phi_1 = -[\|\nabla(\sqrt{\gamma_2A_1}U_2 \pm \sqrt{\gamma_3A_2}U_3)\|^2 - \bar{\alpha}\|\sqrt{\gamma_2A_1}U_2 \pm \sqrt{\gamma_3A_2}U_3\|^2].$$

When (3.11) holds, then exists a positive number $\bar{\gamma} < \gamma_2$ such that setting $\{\eta_2 = \gamma_2 - \bar{\gamma}, \eta_3 = \gamma_3 - \bar{\gamma}\}$, it follows that

$$(\gamma_2 + \gamma_3)|A_3| = 2\bar{\gamma}\sqrt{A_1 A_2}, \tag{3.16}$$

$$\Phi = \begin{cases} -A_1\eta_2(\|\nabla U_2\|^2 - \bar{\alpha}\|U_2\|^2) - A_2\eta_3(\|\nabla U_3\|^2 - \bar{\alpha}\|U_3\|^2) \\ -\bar{\gamma}\|\nabla\sqrt{A_1}U_2 \pm \sqrt{A_2}U_3\|^2 - \bar{\alpha}\|\sqrt{A_1}U_2 \pm \sqrt{a_2}U_3\|^2. \end{cases} \tag{3.17}$$

Finally in the case (3.12), there exists a positive constant $\bar{\gamma} < \gamma_3$ such that, setting $\eta = \gamma_3 - \bar{\gamma} > 0$, (3.10) continues to hold with $\bar{\gamma}$ in place of γ_3 and, analogously, Φ_1 is given by the right-hand side of (3.15) with $\bar{\gamma}$ in place of γ_3, plus the negative term $-\eta A_2(\|\nabla U_3\|^2 - \bar{\alpha}\|U_3\|^2)$.

Remark 3.1. *We remark that*

(i) *in the case* $\gamma_2 = \gamma_3$, (3.9) *holds for any positive value of the ratio* $\dfrac{\mu_2}{\mu_3}$ *appearing in* b_{23}^* *and* b_{32}^*,

(ii) *when*

$$b_{22}^* b_{33}^* a_{23} a_{32} < 0, \tag{3.18}$$

choosing

$$\frac{\mu_2}{\mu_3} = \left|\frac{b_{33}^* a_{23}}{b_{22}^* a_{32}}\right|^{1/2} \tag{3.19}$$

it follows that (3.9) *holds for any* $\gamma_2 > 0$ *and* $\gamma_3 > 0$.

The proof of (i) *can be easily reached by the reader and, anyway, found in* [7]. *In the case* (3.18), *by virtue of* (3.19), *it follows that* $A_3 = 0$ *and hence* (3.8) *holds.*

Lemma 3.3. *Let*

$$b_{11}^* < 0, \quad I^* = b_{11}^* + b_{22}^* < 0, \quad A^* > 0, \tag{3.20}$$

and either

$$a_{12} = a_{13} = 0, \tag{3.21}$$

or

$$a_{21} = a_{31} = 0, \tag{3.22}$$

or

$$a_{13} = a_{31} = 0, \quad a_{12}a_{21} \le 0, \quad b_{22}^* a_{23} a_{32} b_{33}^* < 0, \tag{3.23}$$

or

$$a_{12} = a_{21} = 0, \quad a_{13}a_{31} \le 0, \quad b_{22}^* a_{23} a_{32} b_{33}^* < 0, \tag{3.24}$$

or

$$\sup(b_{11}^*, b_{22}^*, b_{33}^*) < 0, \quad a_{23} = a_{32} = 0, \quad a_{ij}a_{ji} \le 0, \quad i \ne j. \tag{3.25}$$

Then, setting

$$\Phi^* = \begin{cases} \left[\left(A_1 a_{21}\dfrac{\mu_1}{\mu_2} - A_3 a_{31}\dfrac{\mu_1}{\mu_3}\right) + a_{12}\dfrac{\mu_2}{\mu_1}\right]\langle U_1, U_2\rangle + \\ \left[\left(A_2 a_{31}\dfrac{\mu_1}{\mu_3} - A_3 a_{21}\dfrac{\mu_1}{\mu_2}\right) + a_{13}\dfrac{\mu_3}{\mu_1}\right]\langle U_1, U_3\rangle, \end{cases} \tag{3.26}$$

in each of the cases (3.21)–(3.25), *exist suitable values for* μ_1, μ_2, μ_3 *such that it turns out that*

$$\Phi^* \le \frac{1}{2}[|b_{11}|\,\|U_1\|^2 + I^*A^*(\|U_2\|^2 + \|U_3\|^2)]. \tag{3.27}$$

Proof. In fact in the case (3.21), choosing $\mu_2 = \mu_3 = 1$ and setting

$$m = \sup(|A_1a_{21} - A_3a_{31}|, |A_2a_{31} - A_3a_{21}|), \tag{3.28}$$

it follows that

$$\begin{aligned}\Phi^* &\le m\mu_1(\langle |U_1|, |U_2| + |U_3|\rangle) \le m\mu_1(\|U_1\|(\|U_2\| + \|U_3\|)) \\ &\le \frac{m^2\mu_1^2}{|I^*A^*|}\|U_1\|^2 + \frac{1}{2}|I^*A^*|(\|U_2\|^2 + \|U_3\|^2),\end{aligned}$$

and hence

$$\mu_1^2 = \frac{|b_{11}I^*A^*|}{2m^2} \Rightarrow (3.27). \tag{3.29}$$

A completely analogous procedure can be used in the case (3.22). In the cases (3.23)–(3.24), for $\dfrac{\mu_2}{\mu_3}$ given by (3.19) one obtains $A_3 = 0$ and in the case (3.23), choosing $\mu_2 = 1$, it follows that

$$\Phi^* = \left(A_1a_{21}\mu_1 + \frac{a_{12}}{\mu_1}\right)\langle U_1, U_2\rangle, \tag{3.30}$$

with

$$A_1 = A^* + \frac{a_{32}^2}{\mu_3^2} + (b_{33}^*)^2, \tag{3.31}$$

independent of μ_1. Therefore—when $a_{12}a_{21} < 0$—choosing

$$\mu_1 = \left|\frac{a_{12}}{A_1a_{21}}\right|^{1/2}, \tag{3.32}$$

one obtains $\Phi^* = 0$. If $a_{12}a_{21} = 0$, with (for example) $a_{12} = 0$, (3.30) reduces to

$$\Phi^* = A_1 a_{21}\mu_1\langle U_1, U_2\rangle \le \frac{1}{2}|I^*A^*|\,\|U_2\|^2 + \frac{1}{2}\frac{(A_1a_{21})^2}{|I^*A^*|}\mu_1^2\|U_1\|^2, \tag{3.33}$$

and (3.27) is reached choosing

$$\mu_1^2 = \left|\frac{b_{11}^* I^* A^*}{A_1 a_{21}}\right|. \tag{3.34}$$

Obviously an analogous procedure can be used in the case (3.24).

Finally, in the case (3.25), being

$$I^* = b_{22}^* + b_{33}^* < 0, \quad A^* = b_{22}^* b_{33}^* > 0, \tag{3.35}$$

one obtains $b_{22}^* < 0$, $b_{33}^* < 0$. Further by virtue of

$$A_1 = A^* + (b_{33}^*)^2, \quad A_2 = A^* + (b_{11}^*)^2, \quad A_3 = 0, \tag{3.36}$$

for $\mu_1 = 1$ it follows that

$$\Phi^* = \left(\frac{A_1a_{21}}{\mu_2} + a_{12}\mu_2\right)\langle U_1, U_2\rangle + \left(\frac{A_2a_{31}}{\mu_3} + a_{13}\mu_3\right)\langle U_1, U_3\rangle, \tag{3.37}$$

with A_1 and A_3 independent of μ_2 and μ_3. Therefore choosing

$$\mu_2 = \left|\frac{A_1a_{21}}{a_{12}}\right|^{1/2}, \quad \mu_3 = \left|\frac{A_2a_{31}}{a_{13}}\right|^{1/2} \tag{3.38}$$

when

$$a_{21}a_{12} < 0, \quad a_{31}a_{13} < 0, \tag{3.39}$$

it follows that $\Phi^* = 0$. If, for instance $\{a_{12} = 0, a_{13}a_{31} < 0\}$ then, for μ_3 given by $(3.38)_2$, one obtains

$$\Phi^* = \frac{A_1a_{21}}{\mu_2}\langle U_1, U_2\rangle,$$

and it is very easy to choose μ_2 in such a way that (3.18) hold. If $\{a_{12} = 0, a_{31} = 0\}$ then (3.29) reduces to

$$\Phi^* = \frac{A_1a_{21}}{\mu_2}\langle U_1, U_2\rangle + a_{13}\mu_3\langle U_1, U_3\rangle,$$

and one can choose μ_2 and μ_3 in such a way that (3.27) holds.

THEOREM 3.1. *Let the assumptions of Lemma* 3.3 *hold and let* $\bar\mu_1, \bar\mu_2, \bar\mu_3$ *be scalings guaranteeing the applicability of Lemma* 3.3. *If* (3.8) *holds with* $\mu_i = \bar\mu_i$

$(i = 1, 2, 3)$, *then the zero solution of* (2.1)–(2.4) *is asymptotically stable with respect to the* $L^2(\Omega)$*-norm.*

PROOF. For the sake of simplicity we assume $A_3 = 0$, as happens in the case (3.25) for any μ_1, μ_2, μ_3 and in the cases (3.23), (3.24) for $\frac{\mu_2}{\mu_3}$ given by (3.19). In that cases there are no restrictions imposed by (3.8) on γ_2 and γ_3.

Introducing the functional

$$W = \frac{1}{2}\|U_1\|^2 + V, \tag{3.40}$$

with

$$V = \frac{1}{2}[A^*(\|U_2\|^2 + \|U_3\|^2) + \|b_{22}^* U_3 - b_{32}^* U_2\|^2 + \|b_{23}^* U_3 - b_{33}^* U_2\|^2], \tag{3.41}$$

it follows that {cfr Appendix} the temporal derivative of W along the solutions of (3.4) is given by

$$\dot{W} = b_{11}^*\|U_1\|^2 + I^* A^*(\|U_2\|^2 + \|U_3\|^2) + \Phi^* + \Phi, \tag{3.42}$$

with

$$\Phi = \gamma_1(1 - \varepsilon)\langle U_1, \Delta U_1 + \bar{\alpha} U_1\rangle + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4, \tag{3.43}$$

Φ_1 being given by $(3.7)_5$ and

$$\begin{cases} \Phi_2 = \varepsilon[\gamma_2\langle A_1 U_2, \Delta U_2\rangle + \gamma_3\langle A_2 U_3, \Delta U_3\rangle], \\ \Phi_3 = \dfrac{1}{\mu_1}\langle U_1, F_1\rangle + \dfrac{1}{\mu_2}\langle A_1 U_2, F_2\rangle + \dfrac{1}{\mu_3}\langle A_2 U_3, F_3\rangle, \\ \Phi_4 = \varepsilon \displaystyle\sum_{i=1}^{3} \gamma_i\langle U_i, \Delta U_i\rangle. \end{cases} \tag{3.44}$$

In view of Lemmas 3.1–3.3, it follows that

$$\Phi \le -\varepsilon\gamma^*(\|\nabla U_1\|^2 + \|\nabla U_2\|^2 + \|\nabla U_3\|^2) + \Phi_3, \tag{3.45}$$

with $\gamma^* = \inf(\gamma_1, \gamma_2, \gamma_3)$. On the other hand, by virtue of (2.10), it follows that exist two positive constants m_1 and m_2 such that

$$\begin{aligned} \Phi_3 \le\ & m_1(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{1+\varepsilon_1} + m_2(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\varepsilon_2} \\ & \cdot (\|\nabla U_1\|^2 + \|\nabla U_2\|^2 + \|\nabla U_3\|^2). \end{aligned} \tag{3.46}$$

Therefore (3.42)–(3.46) imply

$$\dot{W} \le \frac{1}{2}[b_{11}^* \|U_1\|^2 + I^* A^* (\|U_2\|^2 + \|U_3\|^2)] + m_1(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{1+\varepsilon_1} - [\varepsilon\gamma^* - m_2(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2)^{\varepsilon_2}] \cdot (\|\nabla U_1\|^2 + \|\nabla U_2\|^2 + \|\nabla U_3\|^2). \tag{3.47}$$

In view of (3.42), it turns out that

$$p(\|U_2\|^2 + \|U_3\|^2) \le V \le q(\|U_2\|^2 + \|U_3\|^2), \tag{3.48}$$

with

$$p = \frac{1}{2}A^*, \quad q = \frac{1}{2}A^* + [(b_{22}^*)^2 + (b_{23}^*)^2 + (b_{32}^*)^2 + (b_{33}^*)^2]. \tag{3.49}$$

Hence one obtains

$$\dot{W} \le -\frac{1}{2}\left[|b_{11}^*|\,\|U_1\|^2 + \frac{|I^*A^*|}{q}(\|U_2\|^2 + \|U_3\|^2)\right] + m_1\Big(\|U_1\|^2 + \frac{1}{p}V\Big)^{1+\varepsilon_1} - \left[\varepsilon\gamma^* - m_2\Big(\|U_1\|^2 + \frac{1}{p}V\Big)^{\varepsilon_2}\right](\|\nabla U_1\|^2 + \|\nabla U_2\|^2 + \|\nabla U_3\|^2), \tag{3.50}$$

i.e.

$$\dot{W} \le -(\delta - \delta_1 W^{\varepsilon_1})W - (\delta_2 - \delta_3 W^{\varepsilon_2})(\|\nabla U_1\|^2 + \|\nabla U_2\|^2 + \|\nabla U_3\|^2), \tag{3.51}$$

with

$$\begin{cases} \delta = \dfrac{1}{2}\inf\Big(|b_{11}|, \dfrac{|I^*A^*|}{q}\Big), \\ \delta_1 = m_1\Big(1 + \dfrac{1}{p}\Big)^{1+\varepsilon_1}, \quad \delta_2 = \varepsilon\gamma^*, \quad \delta_3 = m_2\Big(1 + \dfrac{1}{p}\Big)^{\varepsilon_2}. \end{cases} \tag{3.52}$$

Therefore—by recursive argument—it follows that

$$W_0 < \inf\left[\Big(\frac{\delta}{\delta_1}\Big)^{1/\varepsilon_1}, \Big(\frac{\delta_2}{\delta_3}\Big)^{1/\varepsilon_2}\right], \tag{3.53}$$

implies

$$W \le W_0 \exp[-(\delta - \delta_1 W_0^{\varepsilon_1})t]. \tag{3.54}$$

Remark 3.2. *We remark that*

(1) *the assumptions* (3.20) *are implied by*

$$b_{11} < 0, \quad I < 0, \quad A > 0, \tag{3.55}$$

and—as put in evidence at the end of Section 2—(3.55) *are the conditions necessary and sufficient for the asymptotic stability of the zero solution of the reduced linear system of O.D.Es* (2.22)–(2.23)*;*

In fact—for continuity reasons—when (3.55) *hold—exist positive* $\varepsilon \in]0,1[$ *such that* (3.20) *hold too.*

(2) *When* $\mathbf{F}$ *depends only on* $\nabla\mathbf{u}$ *and depends linearly on it, then Theorem* 3.1*—in the case* $A_3 = 0$*—guarantees the global stability. In fact then one easily finds that* (3.46) *holds with* $\varepsilon_1 = \varepsilon_2 = 0$.

(3) *When* $\mathbf{F}$ *depends on* $\mathbf{u}$ *quadratically, then* $(3.6)_2$ *is useful in evaluating* $\langle 1, u_i^3 \rangle$, $(i = 1, 2, 3)$.

(4) *It is easily verified* {*cfr.* [9] *Theorem* 3} *that when the assumptions of Lemma* 3.3 *hold, then the Routh-Hurwitz conditions* [11]*, guaranteeing the stability of the matrix* $\mathscr{L}^*$*, are verified.*

4. Reduction to the stability of the zero solution of (2.22) with $\mathscr{L}$ symmetrized according to (2.24)–(2.25)

Lemma 4.1. *Let*

$$a_{ij}a_{ji} > 0, \quad i \neq j, \tag{4.1}$$

and

$$a_{12}a_{23}a_{31} = a_{13}a_{21}a_{32}. \tag{4.2}$$

Then, choosing the scalings μ_i *in such a way that*

$$\frac{\mu_j}{\mu_i} = \sqrt{\frac{a_{ij}}{a_{ji}}}, \quad \forall i \neq j, \tag{4.3}$$

system (2.1) *can be reduced to*

$$\frac{\partial \mathbf{U}}{\partial t} = \tilde{L}\mathbf{U} + \tilde{\mathbf{F}}, \tag{4.4}$$

with

$$\begin{cases} \tilde{L} = \begin{pmatrix} b_{11}^* & b_{12} & b_{13} \\ b_{21} & b_{22}^* & b_{23} \\ b_{31} & b_{32} & b_{33}^* \end{pmatrix}, \quad \tilde{\mathbf{F}} = (\tilde{F}_1, \tilde{F}_2\tilde{F}_2)^T, \\ \tilde{F}_i = \dfrac{1}{\mu_i}F_i + g_i, \quad \mathbf{U} = \left(\dfrac{u_1}{\mu_1}, \dfrac{u_2}{\mu_2}, \dfrac{u_3}{\mu_3}\right)^T \end{cases}$$

where $(b_{11}^*, b_{22}^*, b_{33}^*)$ *and* (g_1, g_2, g_3) *are given in* (3.1) *and* b_{ij} *are given by* (2.25).

PROOF. Obviously one has only to check the consistency of (4.3). This consistency is guaranteed by (4.2). In fact, in view of

$$\frac{\mu_1}{\mu_2}\cdot\frac{\mu_2}{\mu_3} = \sqrt{\frac{a_{21}}{a_{12}}\cdot\frac{a_{32}}{a_{23}}},$$

and taking into account that (4.2) implies $a_{21}a_{32} = \dfrac{a_{12}a_{23}a_{31}}{a_{13}}$, it turns out that

$$\frac{\mu_1}{\mu_2}\cdot\frac{\mu_2}{\mu_3} = \sqrt{\frac{a_{11}}{a_{13}}} = \frac{\mu_1}{\mu_3}.$$

Analogously the consistency of $\dfrac{\mu_1}{\mu_3}\cdot\dfrac{\mu_3}{\mu_2} = \dfrac{\mu_1}{\mu_2}$ and of $\dfrac{\mu_2}{\mu_1}\cdot\dfrac{\mu_1}{\mu_3} = \dfrac{\mu_2}{\mu_3}$ is immediately verified.

REMARK 4.1. *Let us notice that, obviously,* (4.2), *holds when both the right and left sides are zero. This is the case, for instance, of* $a_{12} = a_{21} = 0$. *Then if* $a_{13}a_{31} > 0$, $a_{23}a_{32} > 0$, *the symmetrization is obtained by choosing* $\mu_1 = \mu_3\sqrt{\dfrac{a_{13}}{a_{31}}}$, $\mu_2 = \mu_3\sqrt{\dfrac{a_{23}}{a_{32}}}$ *with any constant value of* μ_3.

LEMMA 4.2. *Let the assumptions of Lemma* 4.1 *hold. Then* (2.24)*—with* $b_{i,j}$, $(i \neq j)$, *given by* (2.25)*—has the same characteristic values* (*invariants*) *of*

$$\mathscr{L}_1 = \begin{pmatrix} b_{11} & a_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & a_{32} & b_{33} \end{pmatrix}, \tag{4.5}$$

and hence, in particular, the following identities hold

$$det\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = det\begin{pmatrix} b_{11} & a_{12} \\ a_{21} & b_{22} \end{pmatrix}, \tag{4.6}$$

$$det\,\mathscr{L} = det\,\mathscr{L}_1. \tag{4.7}$$

PROOF. The proof is immediate.

LEMMA 4.3. *Let*

$$Q = \sum_{i,j}^{1-3} \alpha_{ij}\xi_i\xi_j, \tag{4.8}$$

be a definite symmetric quadratic form. Then, setting

$$D_0 = 1, \quad D_1 = \alpha_{11}, \quad D_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \quad D_3 = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}, \tag{4.9}$$

it turns out that

$$Q = \frac{X_1^2}{D_0 D_1} + \frac{X_2^2}{D_1 D_2} + \frac{X_3^2}{D_2 D_3}, \tag{4.10}$$

where X_1, X_2, X_3 are the Jacobi's variables given by

$$X_1 = A_1, \quad X_2 = \begin{vmatrix} \alpha_{11} & A_1 \\ \alpha_{21} & A_2 \end{vmatrix}, \quad X_3 = \begin{vmatrix} \alpha_{11} & \alpha_{12} & A_1 \\ \alpha_{21} & \alpha_{22} & A_2 \\ \alpha_{31} & \alpha_{32} & A_3 \end{vmatrix}, \tag{4.11}$$

with

$$A_i = \sum_{j=1}^{3} \alpha_{ij}\xi_j. \tag{4.12}$$

PROOF. The proof can be found in {[12], p. 302}.

LEMMA 4.4. *Let* (4.8) *be a definite symmetric quadratic form and let* (X_1, X_2, X_3) *be the Jacobi's variables associated to* (ξ_1, ξ_2, ξ_3). *Then exist two positive constants p and q such that*

$$p^{-1}\sum_{i=1}^{3}\xi_i^2 \le \sum_{i=1}^{3} X_i^2 \le q\sum_{i=1}^{3}\xi_i^2. \tag{4.13}$$

PROOF. In fact it easily follows that

$$\begin{cases} D_1\xi_1 + \alpha_{12}\xi_2 + \alpha_{13}\xi_3 = X_1, \\ D_2\xi_2 + \delta\xi_3 = X_3, \\ D_3\xi_3 = X_3, \end{cases} \tag{4.14}$$

with

$$\delta = \alpha_{11}\alpha_{23} - \alpha_{21}\alpha_{13}, \tag{4.15}$$

and hence

$$\begin{cases} \xi_1 = \dfrac{1}{D_1}\left[X_1 - \dfrac{\alpha_{12}}{D_2}X_2 + \dfrac{1}{D_3}(\alpha_{12}\delta - \alpha_{13})X_3\right], \\ \xi_2 = \dfrac{1}{D_2}\left(X_2 - \dfrac{\delta}{D_3}X_3\right), \\ \xi_3 = \dfrac{1}{D_3}X_3. \end{cases} \tag{4.16}$$

By virtue of the Cauchy inequality, (4.14) and (4.16) imply respectively

$$\text{(4.17)} \qquad \begin{cases} X_1^2 \le 2(D_1^2 + \alpha_{12}^2 + \alpha_{13}^2)(\xi_1^2 + \xi_2^2 + \xi_3^2), \\ X_2^2 \le 2(D_2^2 + \delta^2)(\xi_2^2 + \xi_3^2), \\ X_3^2 \le D_3^2 \xi_3^2, \end{cases}$$

$$\text{(4.18)} \quad \begin{cases} \xi_1^2 \le \dfrac{2}{D_1^2 D_2^2 D_3^2}[(D_2^2 D_3^2 + \alpha_{12}^2 D_3^2) + D_2^2(\alpha_{12}\delta - \alpha_{13})^2](X_1^2 + X_2^2 + X_3^2), \\ \xi_2^2 \le \dfrac{1}{D_2^2 D_3^2}(D_3^2 + \delta^2)(X_2^2 + X_3^2), \\ \xi_3^2 = \dfrac{1}{D_3^2} X_3, \end{cases}$$

and hence (4.13) immediately follows with

$$\text{(4.19)} \qquad \begin{cases} p = \sup\left\{ \dfrac{2}{D_1^2 D_2^2 D_3^2}[D_2^2 D_3^2 + \alpha_{12}^2 D_3^2 + D_2^2(\alpha_{12}\delta - \alpha_{13})^2], \right. \\ \qquad\qquad \left. \dfrac{1}{D_2^2 D_3^2}(D_3^2 + \delta^2), \dfrac{1}{D_3^2} \right\}, \\ q = 2\sup\left\{ D_1^2 + \alpha_{12}^2 + \alpha_{13}^2, D_2^2 + \delta^2, \dfrac{1}{2} D_3^2 \right\}. \end{cases}$$

THEOREM 4.1. *Let the assumptions of Lemma* 4.1 *hold together with*

$$\text{(4.20)} \qquad b_{11} < 0, \quad \begin{vmatrix} b_{11} & a_{12} \\ a_{21} & b_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & a_{32} & b_{22} \end{vmatrix} < 0,$$

and (2.10). *Then the zero solution of* (4.4) *is asymptotically stable with respect to the* L^2*-norm.*

PROOF. When (4.20) hold, then exists a positive constant $\varepsilon \in]0, 1[$ such that

$$\text{(4.21)} \qquad \begin{aligned} &\tilde{D}_0 = 1, \quad \tilde{D}_1 = b_{11}^* < 0, \quad \tilde{D}_2 = \begin{vmatrix} b_{11}^* & a_{12} \\ a_{21} & b_{22}^* \end{vmatrix} > 0, \\ &\tilde{D}_3 = \begin{vmatrix} b_{11}^* & a_{12} & a_{13} \\ a_{21} & b_{22}^* & a_{23} \\ a_{31} & a_{32} & b_{33}^* \end{vmatrix} < 0. \end{aligned}$$

On the other hand, setting

$$\text{(4.22)} \qquad E = \frac{1}{2}(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2),$$

in view of (4.4), one obtains

$$\dot{E} = \int_\Omega P\, d\Omega + \langle \mathbf{U}, \tilde{\mathbf{F}} \rangle, \tag{4.23}$$

with

$$P = b_{11}^* U_1^2 + b_{22}^* U_2^2 + b_{33}^* U_3^2 + \sum_{i \neq j}^{1-3} b_{ij} U_i U_j, \tag{4.24}$$

and b_{ij} given by (2.25). Then, by virtue of Lemmas 4.2–4.3, it follows that

$$\int_\Omega P\, d\Omega \leq \frac{\|X_1^2\|^2}{\tilde{D}_0 \tilde{D}_1} + \frac{\|X_2^2\|^2}{\tilde{D}_1 \tilde{D}_2} + \frac{\|X_3^2\|^2}{\tilde{D}_2 \tilde{D}_3} \leq -m \sum_{i=1}^{3} \|X_i\|^2, \tag{4.25}$$

with

$$m = \inf\left(\left| \frac{1}{\tilde{D}_1} \right|, \frac{1}{|\tilde{D}_1 \tilde{D}_2|}, \frac{1}{|\tilde{D}_2 \tilde{D}_3|} \right), \tag{4.26}$$

and hence, in view of (4.13), one obtains

$$\begin{cases} \displaystyle\int_\Omega P\, d\Omega \leq -\frac{m}{p}(\|U_1\|^2 + \|U_2\|^2 + \|U_3\|^2) = -dE, \\ d = \dfrac{m}{p}, \text{ with } p \text{ given by } (4.19)_1 \text{ with } \tilde{D}_i \text{ at the place of} D_i. \end{cases} \tag{4.27}$$

Since—in view of (2.10)—one obtains

$$\langle \mathbf{U}, \mathbf{F} \rangle < -\varepsilon\gamma^* (\|\nabla U_1\|^2 + \|\nabla U_2\|^2 + \|\nabla U_3\|^2) + d_1 E^{1+\varepsilon_1} + d_2 E_2^{\varepsilon_2} \sum_{i=1}^{3} \|\nabla U_i\|^2, \tag{4.28}$$

with d_1, d_2 positive constants, then by virtue of (4.23) it turns out that

$$\dot{E} \leq -(d - d_1 E^{\varepsilon_1}) E - (\varepsilon\gamma^* - d_2 E^{\varepsilon_2}) \sum_{i=1}^{3} \|\nabla U_i\|^2, \tag{4.29}$$

and hence the asymptotic stability under the local condition

$$E_0 < \inf\left[\left(\frac{d}{d_1} \right)^{1/\varepsilon_1}, \left(\frac{\varepsilon\gamma^*}{d_2} \right)^{1/\varepsilon_2} \right]. \tag{4.30}$$

5. Applications

We furnish here, in the Subsection 5.1, an application of the results obtained in Section 3 to a triply convective diffusive fluid mixture saturating a porous layer and, in the Subsection 5.2, an application of the results of Section 2 to a model concerned with the dispersal of epidemics.

5.1. *Triply convective-diffusive fluid mixture saturating a porous horizontal layer in the Darcy-Oberbeck-Boussinesq scheme*

The equations governing the perturbations to the conduction solution in the porous horizontal layer $\{z = 0, z = 1\}$ are found to be [13]

$$(5.1) \qquad \begin{cases} \nabla p = -\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2) \cdot \mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta_t + \mathbf{u} \cdot \nabla\theta = HR\mathbf{u} \cdot \mathbf{k} + \Delta\theta, \\ P_1(\Phi_{1t} + \mathbf{u} \cdot \nabla\Phi_1) = H_1R_1\mathbf{u} \cdot \mathbf{k} + \Delta\Phi_1, \\ P_2(\Phi_{2t} + \mathbf{u} \cdot \nabla\Phi_2) = H_2R_2\mathbf{u} \cdot \mathbf{k} + \Delta\Phi_2, \end{cases}$$

under the boundary conditions

$$(5.2) \qquad (\mathbf{u} \cdot \mathbf{i})_z = (\mathbf{u} \cdot \mathbf{j})_z = \mathbf{u} \cdot \mathbf{k} = \theta = \Phi_1 = \Phi_2 = 0, \quad \text{on } z = 0, 1.$$

In (5.1)–(5.2) $(\mathbf{u}, p, \theta, \Phi_1, \Phi_2)$ are respectively the perturbation to the velocity $\mathbf{v}$, pressure π, temperature T and concentrations C_α, $(\alpha = 1, 2)$, to different chemical species S_1 and S_2 dissolved in the porous fluid layer. $Oxyz$ is a Cartesian frame of reference with fundamental unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, with $\mathbf{k}$ pointed vertically upward. R is the temperature Rayleigh number, while R_α and P_α are the Rayleigh and Prandtl numbers respectively of S_α, $(\alpha = 1, 2)$. The boundary equations (5.2) are obtained via the boundary data on the temperature T, concentration C_i, $(i = 1, 2)$, and velocity $\mathbf{v}$ given by

$$(5.3) \qquad \begin{aligned} &T(0) = T_1, \quad T(1) = T_2; \quad C_{\alpha l}(0) = C_{\alpha u}, \quad \alpha = 1, 2; \\ &\mathbf{v} \cdot \mathbf{k} = 0, \quad \text{on } z = 0, 1, \end{aligned}$$

and H, H_1, H_2 are given by

$$(5.4) \qquad \begin{aligned} &H = \operatorname{sign}(\delta T), \quad H_\alpha = \operatorname{sign}(\delta C_\alpha), \\ &\delta T = T_1 - T_2, \quad \delta C_\alpha = C_{\alpha l} - C_{\alpha u}, \quad \alpha = 1, 2. \end{aligned}$$

The perturbations $(\mathbf{u}, \theta, \Phi_1, \Phi_2)$ are assumed—as usual—periodic with periods $\dfrac{2\pi}{a_x}$ and $\dfrac{2\pi}{a_y}$ in the x and y directions (with $a_x > 0$, $a_y > 0$) and belonging to $L^2(\Omega)$, $\forall t \geq 0$, Ω being the periodicity cell,

$$(5.5) \qquad \Omega = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0, 1].$$

As it is well known the set of functions $\{\sin(n\pi z)\}_{n\in\mathbb{N}}$ is a complete orthogonal set for $L^2[0,1]$. Then for any function $f \in \{\omega = \mathbf{u}\cdot\mathbf{k}, \theta, \Phi_1, \Phi_2\}$ it follows that

$$f = \sum_{n=1}^{\infty} f_n(x,y,t)\sin(n\pi z), \tag{5.6}$$

with (by virtue of periodicity in the x and y directions)

$$\Delta_1 f_n = -a^2 f_n, \quad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad a^2 = a_x^2 + a_y^2. \tag{5.7}$$

In particular setting $\omega = \sum_{n=1}^{\infty}\omega_n$, $\theta = \sum_{n=i}^{\infty}\theta_n$, $\Phi_\alpha = \sum_{n=1}^{\infty}\Phi_{\alpha n}$, it follows that

$$\begin{cases} \omega_n = \tilde{\omega}_n(x,y,t)\sin(n\pi z), & \Delta\omega_n = -(a^2+n^2\pi^2)\omega_n, \\ \theta_n = \tilde{\theta}_n(x,y,t)\sin(n\pi z), & \Delta\theta_n = -(a^2+n^2\pi^2)\theta_n, \\ \Phi_{\alpha n} = \tilde{\Phi}_{\alpha n}(x,y,t)\sin(n\pi z), & \Delta\Phi_{\alpha n} = -(a^2+n^2\pi^2)\Phi_{\alpha n}. \end{cases} \tag{5.8}$$

The double curl of (5.1) multiplied by $\mathbf{k}$ gives easily

$$\omega_n = \eta_n(R\theta_n - R_1\Phi_{1n} - R_2\Phi_{2n}), \quad \xi_n = a^2 + n^2\pi^2, \quad \eta_n = \frac{a^2}{\xi_n}, \tag{5.9}$$

and hence

$$\begin{cases} \theta_t = \sum_{n=1}^{\infty}(a_{11}^{(n)}\theta_n + a_{12}^{(n)}\phi_{1n} + a_{13}^{(n)}\phi_{2n}) - \mathbf{u}\cdot\nabla\theta, \\ \phi_{1t} = \sum_{n=1}^{\infty}(a_{21}^{(n)}\theta_n + a_{22}^{(n)}\phi_{1n} + a_{23}^{(n)}\phi_{2n}) - \mathbf{u}\cdot\nabla\phi_1, \\ \phi_{2t} = \sum_{n=1}^{\infty}(a_{31}^{(n)}\theta_n + a_{32}^{(n)}\phi_{1n} + a_{33}^{(n)}\phi_{2n}) - \mathbf{u}\cdot\nabla\phi_2, \end{cases} \tag{5.10}$$

with

$$\begin{cases} a_{11}^{(n)} = HR^2\eta_n - \xi_n, \quad a_{12}^{(n)} = -HRR_1\eta_n, \quad a_{13}^{(n)} = -HRR_2\eta_n, \\ a_{21}^{(n)} = \dfrac{H_1RR_1\eta_n}{P_1}, \quad a_{22}^{(n)} = -\dfrac{(H_1R_1^2\eta_n + \xi_n)}{P_1}, \quad a_{23}^{(n)} = -\dfrac{H_1R_1R_2\eta_n}{P_1}, \\ a_{31}^{(n)} = \dfrac{H_2RR_2\eta_n}{P_2}, \quad a_{32}^{(n)} = -\dfrac{H_2R_1R_2\eta_n}{P_2}, \quad a_{33}^{(n)} = -\dfrac{(H_2R_2^2\eta_n + \xi_n)}{P_2}. \end{cases} \tag{5.11}$$

In view of

$$\begin{cases} a_{12}^{(n)}a_{21}^{(n)} = -\dfrac{HH_1R^2R_1^2\eta_n^2}{P_1}, \quad a_{13}^{(n)}a_{31}^{(n)} = -\dfrac{HH_2R^2R_2^2\eta_n^2}{P_1P_2}, \\ a_{23}^{(n)}a_{32}^{(n)} = \dfrac{H_1H_2R_1^2R_2^2\eta_n^2}{P_1P_2}, \\ a_{12}^{(n)}a_{23}^{(n)}a_{31}^{(n)} = \dfrac{HH_1H_2}{P_1P_2}R^2R_1^2R_2^2\eta_n^3 = a_{13}^{(n)}a_{21}^{(n)}a_{32}^{(n)}, \end{cases} \tag{5.12}$$

it follows that in the case $\{H_1H_2 > 0, HH_1 < 0, HH_2 < 0\}$ the assumptions of Lemma 4.1 are verified. We consider now the case $\{H > 0, H_i < 0, (i = 1, 2)\}$ which is the most destabilizing case, since the fluid in this case is heated from below and salted from above by the two salts.

Introducing the scalings—independent of n—

$$\mu_1 = 1, \quad \mu_2 = \left|\frac{H_1}{HP_1}\right|^{1/2}, \quad \mu_3 = \left|\frac{H_2}{HP_2}\right|^{1/2}, \tag{5.13}$$

and setting

$$\begin{cases} \theta_n = X_n, \quad \Phi_{1n} = \mu_2 Y_n, \quad \Phi_{2n} = \mu_3 Z_n, \\ X = \sum_{n=1}^{\infty} X_n, \quad Y = \frac{1}{\mu_2}\Phi_1, \quad Z = \frac{1}{\mu_3}\Phi_2, \end{cases} \tag{5.14}$$

it follows that

$$\begin{cases} X_t = \sum_{n=1}^{\infty}(a_{1n}X_n + a_{2n}\mu_2 Y_n + a_{3n}\mu_3 Z_n) - \mathbf{u}\cdot\nabla X, \\ Y_t = \sum_{n=1}^{\infty}\left(\frac{b_{1n}}{\mu_2}X_n + b_{2n}\mu_2 Y_n + b_{3n}\frac{\mu_3}{\mu_2}Z_n\right) - \mathbf{u}\cdot\nabla Y, \\ Z_t = \sum_{n=1}^{\infty}\left(\frac{c_{1n}}{\mu_3}X_n + \frac{c_{2n}\mu_2}{\mu_3}Y_n + c_{3n}Z_n\right) - \mathbf{u}\cdot\nabla Z, \end{cases} \tag{5.15}$$

with

$$\begin{cases} a_{1n} = HR^2\eta_n - \xi_n, \quad a_{2n}\mu_2 = -\left|\frac{HH_1}{P_1}\right|^{1/2} RR_1\eta_n, \\ a_{3n}\mu_3 = -\left|\frac{HH_2}{P_2}\right|^{1/2} RR_2\eta_n, \\ \frac{b_{1n}}{\mu_2} = -\left|\frac{HH_1}{P_1}\right|^{1/2} RR_1\eta_n, \quad b_{2n} = -\frac{(H_1R_1^2\eta_n + \xi_n)}{P_1}, \\ b_{3n} = \left|\frac{H_1H_2}{P_1P_2}\right|^{1/2} R_1R_2\eta_n, \\ \frac{c_{1n}}{\mu_3} = -\left|\frac{HH_2}{P_2}\right|^{1/2} RR_2\eta_n, \quad c_{2n} = \left|\frac{H_1H_2}{P_1P_2}\right|^{1/2} R_1R_2\eta_n, \\ c_{3n} = -\frac{(H_2R_2^2\eta_n + \xi_n)}{P_2}. \end{cases} \tag{5.16}$$

Setting

$$E = \frac{1}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \tag{5.17}$$

and taking into account that

$$\langle \sin(n\pi z), \sin(m\pi z) \rangle = 0, \quad n \neq m, \quad \langle \mathbf{u} \cdot \nabla F^2 \rangle = 0, \quad F \in (X, Y, Z);$$

along (5.15) it follows that

$$\dot{E} = \sum_{n=1}^{\infty} \int_{\Omega} Q_n \, d\Omega, \tag{5.18}$$

with

$$\begin{aligned} Q_n = {} & c_{3n} Z_n^2 + b_{2n} Y_n^2 + a_{1n} X_n^2 - 2\left|\frac{HH_1}{P_1}\right|^{1/2} RR_1 X_n Y_n \\ & - 2\left|\frac{HH_2}{P_2}\right|^{1/2} RR_2 X_n Z_n + 2\left|\frac{H_1 H_2}{P_1 P_2}\right|^{1/2} \eta_n Y_n Z_n. \end{aligned} \tag{5.19}$$

Since Q_n is symmetric, Q_n is definite negative if and only if

$$\begin{cases} D_{1n} = c_{3n} < 0, \quad D_{2n} = c_{3n} b_{2n} - H_1 H_2 \dfrac{R_1^2 R_2^2}{P_1 P_2} \eta_n > 0, \\ D_{3n} = \begin{vmatrix} a_{1n} & -\left|\dfrac{HH_1}{P_1}\right|^{1/2} RR_1 \eta_n & -\left|\dfrac{HH_2}{P_2}\right|^{1/2} RR_2 \eta_n \\ -\left|\dfrac{HH_1}{P_1}\right|^{1/2} RR_1 \eta_n & b_{2n} & \left|\dfrac{H_1 H_2}{P_1 P_2}\right|^{1/2} R_1 R_2 \eta_n \\ -\left|\dfrac{HH_2}{P_2}\right|^{1/2} RR_2 \eta_n & \left|\dfrac{H_1 H_2}{P_1 P_2}\right|^{1/2} R_1 R_2 \eta_n & c_{3n} \end{vmatrix} < 0. \end{cases} \tag{5.20}$$

Conditions (5.20) coincide with the stability conditions (108) of [13] where the following theorem has been shown.

THEOREM 5.1. *Let $H > 0$, $H_i < 0$, $(i = 1, 2)$. Then the conduction solution is globally asymptotically stable if and only if*

$$HR^2 + |H_1| R_1^2 + |H_2| R_2^2 < 4\pi^2. \tag{5.21}$$

We confine ourselves here to remark that in view of Theorem 4.1—since the contribution of the nonlinear terms $\mathbf{u} \cdot \nabla X, \mathbf{u} \cdot \nabla Y, \mathbf{u} \cdot \nabla Z$ is zero—(5.21) guarantees that (5.20) hold $\forall n$ and hence exists a constant $d^* > 0$, independent of n, such that

$$Q_n \leq -d^*(X_n^2 + Y_n^2 + Z_n^2). \tag{5.22}$$

Finally (5.4)–(5.5) give

$$\dot{E} \leq -d^* E \Leftrightarrow E \leq E_0 e^{-d^* t}. \tag{5.23}$$

5.2. *Stability of a SIR system*

In this subsection we consider the SIR system

$$\begin{cases} \dfrac{\partial S}{\partial t} = a(1 - S) + \gamma_1 \Delta S - bIS, \\ \dfrac{\partial I}{\partial t} = -(a + c)I + \gamma_2 \Delta I + bIS, \\ \dfrac{\partial R}{\partial t} = -dR + cI + \gamma_3 \Delta R, \end{cases} \tag{5.24}$$

with R, S and I densities of removed, susceptibles and infectives respectively {cfr. [14], p. 150} and a, b, c, d, γ_i, $(i = 1, 2, 3)$, positive constants. We consider (5.24) in the domain (2.11) with $l = 1$, under the boundary conditions

$$\begin{cases} S = \overline{S}, \quad I = \overline{I}, \quad R = \overline{R}, & \text{in } x = 0, \\ \nabla S = \nabla I = \nabla R = 0, & \text{in } x = 1, \end{cases} \tag{5.25}$$

with $(\overline{S}, \overline{I}, \overline{R})$ constant solution of (5.24). As constant solutions we consider

(i) *the disease-free equilibrium state* $\{\overline{S} = 1, \overline{I} = \overline{R} = 0\}$,
(ii) *the endemic equilibrium state*

$$\left\{ \overline{S} = \frac{a + c}{b}, \overline{I} = \frac{d[b - (c + d)]}{c + d}, \overline{R} = \frac{c[b - (c + d)]}{c + d} \right\}, \quad \text{with } b > c + d.$$

Denoting by (s, i, r) the perturbation to an equilibrium state $(\overline{S}, \overline{I}, \overline{R})$, it follows that

$$\begin{cases} s_t = -(a + b\overline{I})s - b\overline{S}i + \gamma_1 \Delta s - bis, \\ i_t = b\overline{I}s - (a + c - b\overline{S})i + \gamma_2 \Delta i + bis, \\ r_t = -dr + ci + \gamma_3 \Delta r, \end{cases} \tag{5.26}$$

under the boundary conditions

$$\begin{cases} s = r = i = 0, & \text{in } x = 0, \\ \nabla s = \nabla r = \nabla i = 0, & \text{in } x = 1. \end{cases} \tag{5.27}$$

One easily verifies that (3.22) holds and Theorem 3.1 can furnish the conditions necessary and sufficient for the stability-instability of $(\overline{S}, \overline{I}, \overline{R})$. In particular, in the case of the *disease-free* equilibrium $(\overline{S} = 1, \overline{I} = \overline{R} = 0)$, denoting by

$$R_0 = \frac{b}{a + c + \bar{\alpha}\gamma_2}, \tag{5.28}$$

the *basic reproductive number* it turns out immediately that *the disease-free equilibrium is* (*locally*) *asymptotically stable if and only if* $R_0 < 1$.

6. Appendix: The basic peculiar Liapunov function for binary systems

We define *sharp* the *peculiar* Liapunov functions able to give for the nonlinear stability-instability, exactly the same conditions of the linear stability. We here recall the construction of a such Liapunov function for a binary system of O.D.Es, since is this function integrated on Ω and named V that appears in Section 3.

Let us consider the stability of the zero solution of the system

$$\begin{cases} \dfrac{dx}{dt} = ax + by + f(x, y), \\ \dfrac{dy}{dt} = cx + dy + g(x, y), \end{cases} \tag{6.1}$$

(f, g being nonlinear and such that $f(0,0) = g(0,0) = 0$) and introduce the function

$$V = \frac{1}{2} I[A(x^2 + y^2) + (ay - cx)^2 + (by - dx)^2], \tag{6.2}$$

with

$$I = a + d = \lambda_1 + \lambda_2, \quad A = ad - bc = \lambda_1 \cdot \lambda_2, \tag{6.3}$$

λ_1, λ_2 eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since

$$\begin{cases} x\dot{x} = ax^2 + bxy + xf, & y\dot{y} = cxy + dy^2 + yg, \\ y\dot{x} = axy + by^2 + yf, & x\dot{y} = cx^2 + dxy + xg, \end{cases} \tag{6.4}$$

by straightforward calculations it follows that

$$\frac{dW}{dt} = IA(x^2 + y^2) + \Psi, \tag{6.5}$$

with

$$\begin{cases} \Psi = [(\alpha_1 x - \alpha_3 y)f + (\alpha_2 y - \alpha_3 x)g], \\ \alpha_1 = A + c^2 + d^2, \quad \alpha_2 = A + a^2 + b^2, \quad \alpha_3 = ac + bd. \end{cases} \tag{6.6}$$

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