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Algebraic Geometry — *Irreducibility of the space of dihedral covers of the projective line of a given numerical type*, by Fabrizio Catanese, Michael Lönne and Fabio Perroni[1], communicated on 11 February 2011.

Dedicated to the memory of Professor Giovanni Prodi.

Abstract. — We show in this paper that the set of irreducible components of the family of Galois coverings of $\mathbb{P}^1_{\mathbb{C}}$ with Galois group isomorphic to D_n is in bijection with the set of possible numerical types.

In this special case the numerical type is the equivalence class (for automorphisms of D_n) of the function which to each conjugacy class $\mathscr{C}$ in D_n associates the number of branch points whose local monodromy lies in the class $\mathscr{C}$.

Key words: Moduli spaces of curves, branched coverings of Riemann surfaces, Hurwitz equivalence, braid groups, monodromy.

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Introduction

The theory of covering spaces was invented to clarify the concept of an algebraic function and its polydromy.

In the modern terminology one can describe an algebraic function f on an algebraic curve Y as a rational function f on a projective curve X admitting a non constant morphism $p : X \to Y$, and such that f generates the field extension $\mathbb{C}(Y) \subset \mathbb{C}(X)$.

The easiest example would be the one where $Y = \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$ and $f = \sqrt{P}(x)$, P being a square free polynomial.

f is in general polydromic, i.e., many valued as a function on Y, and going around a closed loop we do not return to the same value. It is a theorem of Weierstrass that f is a rational function on Y iff f is monodromic, i.e., there is no polydromy.

For strange reasons (but remember the famous explanation 'Lucus a non lucendo', the grove has a similar name to light because there is no light) what should be called polydromy is nowadays called monodromy.

[1] The present work took place in the realm of the DFG Forschergruppe 790 "Classification of algebraic surfaces and compact complex manifolds".

Given $p : X \to Y$ as above there is a finite set $\mathscr{B} \subset Y$, called the branch locus, such that, setting $Y^* := Y \backslash \mathscr{B}$, $X^* := p^{-1}(Y^*)$, then p induces a covering space $X^* \to Y^*$ which is classified by its monodromy μ, which is a homomorphism of the fundamental group (Poincaré group) of Y^*, $\pi_1(Y^*, y_0)$, into the group of permutations of the fibre $p^{-1}(y_0)$.

If X is irreducible, and δ is the degree of p, then the image of the monodromy is a transitive subgroup of $\mathfrak{S}_\delta$, and conversely Riemann's existence theorem asserts that for any homomorphism $\mu : \pi_1(Y^*, y_0) \to \mathfrak{S}_\delta$ with transitive image we obtain a morphism $p : X \to Y$ as above inducing the given monodromy μ, hence also a corresponding algebraic function on Y with branch set contained in $\mathscr{B}$.

Riemann's existence theorem is a very powerful but not constructive result: it is similar in spirit to the non constructive argument which shows that any $n \times n$ matrix A satisfies a polynomial equation of degree at most n^2; while the theorem of Hamilton Cayley constructs such a polynomial equation of degree n, namely, the characteristic polynomial P_A of A. Although P_A is not the polynomial of minimal degree which gives zero when evaluated on A, it has the advantage that it varies well with A if A varies in a family.

Similarly, one can consider families of algebraic functions, or, equivalently, families of morphisms $X_t \to Y_t$ of algebraic curves, and a natural question is whether a given parameter space T is irreducible: for this type of question Riemann's existence theorem plays a crucial role.

Usually one splits the above question by considering families where the branch locus has a fixed cardinality, obtaining in this way a stratification of the parameter space (the strata are often called Hurwitz spaces, see [Ful69]); and then asking which strata are irreducible.

The archetypal result is the theorem of Lüroth-Clebsch and Hurwitz, showing that simple coverings of the projective line form an irreducible variety (see [Cleb72], [Hur91], cf. also [BaCa97] for a simple proof). Here, simple means that the local monodromies (image under μ of small loops around the branch points) are transpositions. The theorem of Lüroth-Clebsch has been extended to projective curves Y of higher genus by several authors ([GHS02]), and there are variants ([Kluit88], [Waj96], [Kanev06], [Kanev05], [Ve06], [Ve07], [Ve08]) where for one or two distinguished branch points the local monodromy can be chosen to be a different type of permutation, or where one replaces the symmetric group by Weyl groups and the transpositions by reflections.

Observe that one can factor the monodromy $\mu : \pi_1(Y^*, y_0) \to \mathfrak{S}_\delta$ through a surjection onto a finite group G followed by a permutation representation of G, i.e., an injective homomorphism $G \to \mathfrak{S}_\delta$ with transitive image.

Geometrically this amounts to construct a morphism $Z \to Y$ (the Galois closure of p) such that G acts on Z with quotient $Z/G \cong Y$, and such that X is obtained as the quotient of Z by a non normal subgroup H of G, and we have the factorization

$$Z \to Z/H = X \to Z/G = Y.$$

In this way one separates the investigation of algebraic functions into two parts: the study of Galois covers $Z \to Y$, and the study of intermediate covers.

The study of Galois covers is however also of interest in itself, since inside the moduli space $\mathfrak{M}_g$ of curves X of genus $g \geq 2$ we have the closed proper algebraic subset of curves having a nontrivial group of automorphisms, and one would like to understand, given a finite group G, which are the irreducible components of the algebraic subset $\mathfrak{M}_{g;G}$ of curves X admitting G as a subgroup of their group of automorphisms.

The action of G on the curve X gives rise to a morphism $p : X \to X/G = Y$, and the geometry of p encodes several discrete invariants which distinguish the irreducible components of $\mathfrak{M}_{g;G}$: the genus g' of Y, the number d of branch points, and the orders $m_1, \dots, m_d$ of the local monodromies. These invariants form the primary numerical type.

Once the primary numerical type is fixed, then the determination of the irreducible components of $\mathfrak{M}_{g;G}$ with a given primary numerical type is, by Riemann's existence theorem, equivalent to the determination of the orbits of the group $Map(g', d) \times Aut(G)$ on the set of possible monodromies μ. Here $Map(g', d)$ is the mapping class group of the curve $Y \setminus \mathscr{B}$, a curve of genus g' with d points removed.

Thus the general problem is to try to determine some finer numerical invariants which determine these orbits (equivalently, the above irreducible components).

The secondary numerical type consists of the equivalence class (for automorphisms of G) of the function which to each conjugacy class $\mathscr{C}$ in G associates the number of branch points whose local monodromy lies in the class $\mathscr{C}$.

It was shown in [Cat10] that the primary and secondary numerical type suffice to determine the irreducible components $\mathfrak{M}_{g;G}$ in the case where G is cyclic.

In this paper and its sequel we shall be concerned with the case where G is a dihedral group D_n. In this case one can define the numerical type, which is nothing else than the primary and secondary numerical type unless n is even and the monodromy μ' onto the Abelianization $(\mathbb{Z}/2\mathbb{Z})^2$ of D_n determines an unramified covering of Y.

We conjecture that each numerical type determines only one irreducible component, and we present the proof here for the case $g' = 0$; we have also a proof in the unramified case with $g' > 0$.

Of course one can ask similar questions for more general groups, abelian groups should be relatively easy, whereas more general solvable groups could lead to remarkable difficulties.[2]

On the opposite side, there is the case where G is a simple group: for this case we would like to call attention to the stability result of [Du-Th06].

The stability result of [Du-Th06] states that, in the unramified case (where primary and secondary invariants boil down to only one invariant, namely the genus g'), for every finite group G the number of irreducible components becomes a constant independent of g' for g' sufficiently large.

[2] Added in proof: we discovered that some partial results have been obtained in [EdmI] and [EdmII].

A very interesting question is whether a similar stability result holds fixing the secondary numerical type but letting the genus g' become sufficiently large.

1. Preliminaries

Dihedral groups. The *dihedral group* D_n of order $2n$ is the group of symmetries of a regular polygon with n edges. We assume $n \geq 3$, else we get the group $(\mathbb{Z}/2\mathbb{Z})^2$.

A simple representation of D_n is as the normal subgroup of the affine group $A(1, \mathbb{Z}/n\mathbb{Z})$ consisting of transformations of the form

$$m \mapsto \pm m + j, \quad j \in \mathbb{Z}/n\mathbb{Z}.$$

It has generators x such that $x(m) = m + 1$, and y such that $y(m) = -m$; x^j corresponds to a rotation of $2\pi j/n$ around its barycenter and each element $x^j y$ (such that $x^j y(m) = -m + j$) corresponds to a reflection with respect to a line of symmetry.

D_n can be defined by generators and relations as follows:

$$\mathrm{D}_n = \langle x, y \,|\, x^n = y^2 = (xy)^2 = 1 \rangle.$$

The above presentation shows right away that the Abelianization of D_n has the presentation

$$\mathrm{D}_n^{Ab} = \langle x, y \,|\, xy = yx, y^2 = (x)^{GCD(2,n)} = 1 \rangle$$

hence we get $\mathbb{Z}/2\mathbb{Z}$ for n odd, $(\mathbb{Z}/2\mathbb{Z})^2$ for n even.

The *n reflections* $y, xy, \dots, x^{n-1}y$ will also be denoted either by $s_0, s_1, \dots, s_{n-1}$ or by their indices $0, 1, \dots, n-1$.

For any rotation x^i, its conjugacy class consists exactly of the elements x^i and x^{-i} (if $n = 2i$ we obtain in this way the only central element).

If n is odd all the reflections belong to the same conjugacy class, while if n is even two reflections $x^i y$ and $x^j y$ are conjugate if and only if $i \equiv j \pmod 2$.

These two cases are distinguished by the property of the corresponding affine transformation to have fixed points, since $x^i y(m) = m \Leftrightarrow i \equiv 2m \ (mod\, n)$, and this equation has no solution if n is even and i is odd.

The *automorphism group* $\mathrm{Aut}(\mathrm{D}_n)$ is identified with $A(1, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^*$ as follows: the map $\mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(\mathrm{D}_n)$, which assigns $i \in \mathbb{Z}/n\mathbb{Z}$ to the automorphism defined by $y \mapsto x^i y$ and $x \mapsto x$, identifies $\mathbb{Z}/n\mathbb{Z}$ with the normal subgroup of $\mathrm{Aut}(\mathrm{D}_n)$, consisting of those automorphisms which act trivially on the subgroup of rotations.

The quotient $\mathrm{Aut}(\mathrm{D}_n)/(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to the subgroup of $\mathrm{Aut}(\mathrm{D}_n)$ consisting of automorphisms of the form $y \mapsto y$, $x \mapsto x^i$ for $i \in (\mathbb{Z}/n\mathbb{Z})^*$.

Observe that, D_n being a normal subgroup of $A(1, \mathbb{Z}/n\mathbb{Z})$, we get by conjugation a homomorphism $A(1, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Aut}(\mathrm{D}_n)$ which is an isomorphism exactly for n odd.

Dihedral coverings of curves. Let Y be a compact connected Riemann surface of genus g'. A *dihedral covering* of Y is a Galois covering $\pi : X \to Y$ with Galois group $G = \mathrm{D}_n$ and with X connected. We will also say that π is a D_n-covering.

Riemann's existence theorem allows us to use combinatorial methods to study D_n-coverings, or more generally any G-covering (a Galois covering with an arbitrary finite Galois group G).

Let $\mathscr{B} = \{y_1, \ldots, y_d\} \subset Y$ be the branch locus of π.

Fix a base point $y_0 \in Y \backslash \mathscr{B}$ and a point $x_0 \in \pi^{-1}(y_0)$.

Monodromy gives a surjective group-homomorphism

$$\mu : \pi_1(Y \backslash \mathscr{B}, y_0) \to G. \tag{1}$$

We recall now the definition of a *geometric basis of* $\pi_1(Y \backslash \mathscr{B}, y_0)$.

Let $\alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}$ be simple nonintersecting (except in y_0) closed arcs in $Y \backslash \mathscr{B}$ which are based on y_0 and whose classes in $H_1(Y; \mathbb{Z})$ form a symplectic basis.

Let $\tilde{\gamma}_i$ be an arc connecting y_0 with y_i, contained in $(Y \backslash \{\alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}\}) \cup \{y_0\}$ and such that $\tilde{\gamma}_i$ intersects $\tilde{\gamma}_j$ only in y_0 for $i \neq j$. Require moreover that $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_d$ stem out of y_0 with distinct tangents and following each other in counterclockwise order.

Let $\gamma_1, \ldots, \gamma_d \subset Y \backslash (\mathscr{B} \cup \{\alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}\}) \cup \{y_0\}$ be arcs defined as follows: γ_i begins at y_0, travels along $\tilde{\gamma}_i$ to a point near y_i, makes a small simple counterclockwise loop around y_i and then returns to y_0 along $\tilde{\gamma}_i$.

Then we have chosen a geometric basis, and we have a presentation:

$$\pi_1(Y \backslash \mathscr{B}, y_0) = \langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}, \gamma_1, \ldots, \gamma_d \mid \Pi_{i=1}^{g'} [\alpha_i, \beta_i] \cdot \gamma_1 \cdot \ldots \cdot \gamma_d = 1 \rangle.$$

Let $\mathbf{T}(g', d)$ be the group defined abstractly by generators and relations as follows:

$$\mathbf{T}(g', d) := \langle A_1, B_1, \ldots, A_{g'}, B_{g'}, \Gamma_1, \ldots, \Gamma_d \mid \Pi_{i=1}^{g'} [A_i, B_i] \cdot \Gamma_1 \cdot \ldots \cdot \Gamma_d = 1 \rangle.$$

The choice of a geometric basis yields an obvious isomorphism $\pi_1(Y \backslash \mathscr{B}, y_0) \cong \mathbf{T}(g', d)$ and under this identification the homomorphism (1) corresponds to an epimorphism:

$$\boldsymbol{\mu} : \mathbf{T}(g', d) \to G. \tag{2}$$

Conversely, given a surjective homomorphism $\boldsymbol{\mu}$ as in (2) such that $\boldsymbol{\mu}(\Gamma_i) \neq 1 \ \forall i$, by Riemann's existence theorem the choice of a geometric basis as above ensures the existence of a G-covering $\pi : X \to Y$ branched on $\mathscr{B}$ and whose monodromy is $\boldsymbol{\mu}$.

Varying a covering in a flat family with connected base, there are some numerical invariants which remain unchanged, the first ones being the respective genera g, g' of the curves X, Y, which are related by the Hurwitz formula:

$$2(g-1) = |G| \left[2(g'-1) + \sum_i \left(1 - \frac{1}{m_i}\right) \right], \quad m_i := ord(\mu(\gamma_i)).$$

Observe moreover that a different choice of the geometric basis changes the generators γ_i, but does not change their conjugacy classes (up to permutation), hence another numerical invariant is provided by the number of elements $\mu(\gamma_i)$ which belong to a fixed conjugacy class in the group G.

We formalize these invariants through the following definitions.

Definition 1. *A G-**Hurwitz vector** is an ordered sequence*

$$\mathbf{v} = (a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots, c_d) \in G^{2g'+d} \tag{3}$$

such that the following conditions are satisfied:

(i) *$c_i \neq 1$ for all i;*
(ii) *G is generated by the components of $\mathbf{v}$, $G = \langle \mathbf{v} \rangle$;*
(iii) *$\Pi_{i=1}^{g'}[a_i, b_i] \cdot c_1 \cdot \ldots \cdot c_d = 1$.*

To any D_n-Hurwitz vector $\mathbf{v}$ we associate a tuple of positive integers $\nu(\mathbf{v})$ defined as follows.

If $n = 2n' + 1$ is odd, $\nu(\mathbf{v}) = (k, k_1, \ldots, k_{n'})$, where k (resp. k_i) is the number of the c_i's in the conjugacy class of y (resp. x^i).

If $n = 2n'$ is even, $\nu(\mathbf{v}) = (k_y, k_{xy}, k_1, \ldots, k_{n'})$, where k_y (resp. k_{xy}, k_i) is the number of the c_i's in the conjugacy class of y (resp. xy, x^i).

The group $\mathrm{Aut}(\mathrm{D}_n)$ acts diagonally on the set of Hurwitz vectors. This action induces an action of $\mathrm{Aut}(\mathrm{D}_n)$ on the set $\mathcal{N} := \{\nu(\mathbf{v}) \mid \mathbf{v} \text{ is a Hurwitz vector}\}$ such that the map ν is $\mathrm{Aut}(\mathrm{D}_n)$-equivariant.

The equivalence class of $\nu(\mathbf{v})$ in $\mathcal{N}/\mathrm{Aut}(\mathrm{D}_n)$ will be denoted by $[\nu(\mathbf{v})]$.

Definition 2. *The **numerical type** of the Hurwitz vector $\mathbf{v}$ is defined as follows.*

If $n = 2n' + 1$ is odd, it is the pair $(g', [\nu(\mathbf{v})])$, where g' is the genus of Y and $[\nu(\mathbf{v})] \in \mathcal{N}/\mathrm{Aut}(\mathrm{D}_n)$ is as above.

If $n = 2n'$ is even, then let H be the normal subgroup normally generated by $c_1, \ldots, c_d$ and set $G' = \mathrm{D}_n/H$. Then either $G' \cong \mathbb{Z}/2\mathbb{Z}$ or $G' \cong \mathrm{D}_m$, where $m \geq 2$, $m|n$. In the case where $G' \cong \mathrm{D}_m$ and $m \geq 2$ is even, the associated surjection $\mu' : \pi_1(Y) \to \mathrm{D}_m$ determines, since Y is the classifying space for $\pi_1(Y)$, a continuous map $M' : Y \to K(\mathrm{D}_m, 1)$, hence a map in homology $H_2(M') : \mathbb{Z}[Y] \cong H_2(Y, \mathbb{Z}) \to H_2(\mathrm{D}_m, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

We define $\iota \in \{0, 1\}$ to be the image element $H_2(M')([Y])$.

Then the numerical type is defined as the triple $(g', [\nu(\mathbf{v})], \iota)$.[3]

Topological type. We recall a result contained in [Cat00], see also [Cat08].

Define the orbifold fundamental group $\pi_1^{orb}(Y \backslash \mathcal{B}, y_0; m_1, \ldots m_d)$ of the covering as

$$\langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}, \gamma_1, \ldots, \gamma_d \mid \Pi_{i=1}^{g'}[\alpha_i, \beta_i] \cdot \gamma_1 \cdot \ldots \cdot \gamma_d = 1, \gamma_j^{m_j} = 1 \ \forall j = 1, \ldots d \rangle.$$

[3] for $g' = 0$ the group G' is trivial and hence the numerical type is just $[\nu(\mathbf{v})]$.

We have then an exact sequence

$$1 \to \pi_1(X, x_0) \to \pi_1^{orb}(Y \backslash \mathscr{B}, y_0; m_1, \dots m_d) \to G \to 1$$

which is completely determined by the monodromy, and which in turn determines, via conjugation, a homomorphism

$$\rho : G \to Out(\pi_1(X, x_0)) = Map(X) := Diff^+(X)/Diff^0(X)$$

which is fully equivalent to the topological action of G on X.

We have that, by Proposition 4.13 of [Cat00], all the curves X of a fixed genus g which admit a given topological action ρ of the group G, specified up to an automorphism of G, are parametrized by a connected complex manifold; arguing as in Theorem 2.4 of [Cat10] we get

THEOREM 1. *The triples (X, G, ρ) where X is a complex projective curve of genus $g \geq 2$, and G is a finite group acting on X with a topological action of type ρ are parametrized by a connected complex manifold $\mathscr{T}_{g;G,\rho}$ of dimension $3(g'-1)+d$, where g' is the genus of $Y := X/G$, and d is the cardinality of the branch locus $\mathscr{B}$.*

The image $\mathfrak{M}_{g;G,\rho}$ of $\mathscr{T}_{g;G,\rho}$ inside the moduli space $\mathfrak{M}_g$ is an irreducible closed subset of the same dimension $3(g'-1)+d$.

The next question which the above result motivates is: when do two Galois monodromies $\mu_1, \mu_2 : \pi_1^{orb}(Y \backslash \mathscr{B}, y_0; m_1, \dots m_d) \to G$ have the same topological type?

The answer is theoretically easy, since if the two covering spaces have the same topological type then they are homeomorphic, hence this means that the two monodromies differ by:

- An automorphism of G.
- And a different choice of a geometric basis. This is performed by the mapping class group (the first equality follows since the points of $\mathscr{B}$ are the ends of $Y \backslash \mathscr{B}$):

$$Map(Y, \mathscr{B}) \cong Map(Y \backslash \mathscr{B}) := Diff^+(Y \backslash \mathscr{B})/Diff^0(Y \backslash \mathscr{B}).$$

Moduli spaces. Fixing a genus g and a finite group G we have a finite number of closed irreducible subsets $\mathfrak{M}_{g;G,\rho} \subset \mathfrak{M}_g$ corresponding to the choice of a topological type ρ for the action of G.

A first invariant for the topological type ρ is provided by the pair (g', d) where g' is as above the genus of $Y := X/G$, and d is the cardinality of the branch locus $\mathscr{B} \subset Y$.

A further numerical invariant is the $Aut(G)$ equivalence class of the class function v which, for each conjugacy class $\mathscr{C}$ in G, counts the number of local monodromies $c_i := \mu(\gamma_i)$ which belong to the conjugacy class $\mathscr{C}$.

In particular, a weaker numerical invariant is given by the sequence of multiplicities m_i of the branch points (m_i is the order of $\mu(\gamma_i)$; one can assume w.l.o.g. $m_1 \leq m_2 \leq \cdots \leq m_d$).

One can consider then the set of equivalence classes of pairs (X, a), where X is a projective curve of genus g and a is an effective action of G on X with primary numerical invariants $(g; m_1, \dots m_d)$.

Two such pairs (X, a) and (X', a') are considered equivalent iff there exists a biholomorphic map $F : X \to X'$ and an automorphism $\varphi \in \mathrm{Aut}(G)$ such that $F(gx) = \varphi(x)F(x)$, for any $x \in X$ and $g \in G$.

The set of such irreducible components $\mathfrak{M}_{g;G,\rho}$ with the given primary numerical invariants $(g; m_1, \dots m_d)$ is then computed as the number of orbits of $Map(g', d) \times Aut(G)$ on the set of surjective homomorphisms

$$\mu : \mathbf{T}(g', d; m_1, \dots m_d) \to G$$

where

$$\begin{aligned}&\mathbf{T}(g', d; m_1, \dots m_d)\\ &\quad := \langle A_1, B_1, \dots, A_{g'}, B_{g'}, \Gamma_1, \dots, \Gamma_d \mid \Pi_{i=1}^{g'}[A_i, B_i] \cdot \Gamma_1 \cdot \dots \cdot \Gamma_d = 1, \Gamma_i^{m_i} = 1 \ \forall i\rangle.\end{aligned}$$

The geometrical insight is that the union of such components $\mathfrak{M}_{g;G,\rho}$ has a finite map $Q : \mathfrak{M}_{g;G,\rho} \to \mathfrak{M}_{g',d}$ onto the (coarse) moduli space $\mathfrak{M}_{g',d}$ of smooth curves of genus g' with d unordered marked points. This is a topological covering and the fundamental group of the base is a quotient of the mapping class group $Map(g', d)$.

Hence the components $\mathfrak{M}_{g;G,\rho}$ are detected by the orbits of the monodromy of this covering space.

The case of the dihedral group. Let n be a positive integer $n \geq 3$ and let $(g, [\nu(\mathbf{v})])$ (resp. $(g, [\nu(\mathbf{v})], \imath)$ be a numerical type.

Let $\mathscr{H}_{\mathrm{D}_n}(g, [\nu(\mathbf{v})])$ (resp. $\mathscr{H}_{\mathrm{D}_n}(g, [\nu(\mathbf{v})], \imath)$) be the set of equivalence classes of pairs (X, a), where X is a Riemann surface of genus g and a is an effective action of D_n on X such that the D_n-covering $X \to X/\mathrm{D}_n$ is of numerical type $(g, [\nu(\mathbf{v})])$ (resp. $(g, [\nu(\mathbf{v})], \imath)$.

The main question we address is whether the spaces $\mathscr{H}_{\mathrm{D}_n}(g, [\nu(\mathbf{v})])$, respectively $\mathscr{H}_{\mathrm{D}_n}(g, [\nu(\mathbf{v})], \imath)$ are irreducible, i.e., are spaces $\mathfrak{M}_{g;\mathrm{D}_n,\rho}$ for a unique topological type ρ. This can be proved by showing the transitivity of $Map(g', d) \times Aut(\mathrm{D}_n)$ on the set of monodromies of given (full) numerical type.

This is the same thing as bringing each monodromy with a given numerical type to a normal form.

2. The case $g' = 0$

In this Section we assume $g' = 0$.

The moduli space $\mathfrak{M}_{0,d}$ is a quotient of $(\mathfrak{S}^d\mathbb{P}^1)\backslash\Delta$ by the action of the projective linear group $\mathbb{P}GL(2, \mathbb{C})$, where $\mathfrak{S}^d\mathbb{P}^1$ is the d-th symmetric product of $\mathbb{P}^1$, and Δ is the subset of $\mathfrak{S}^d\mathbb{P}^1$ consisting of points with two or more equal coordinates.

We have $\mathfrak{S}^d\mathbb{P}^1 \cong \mathbb{P}^d$, therefore we consider the action of the *braid group of the sphere* $\mathscr{S}\mathscr{B}_d := \pi_1(\mathbb{P}^d \backslash \Delta, \underline{y})$ on the fibre over $\underline{y}$ of the above map Q.

The group $\mathscr{S}\mathscr{B}_d$ is a quotient of Artin's braid group $\mathscr{B}_d$ which is generated by the so-called *elementary braids* $\sigma_1, \dots, \sigma_{d-1}$ acting on the Hurwitz vector $\mathbf{v} = (c_1, \dots, c_d)$ as follows:

$$(c_1, \dots, c_d)\sigma_i = (c_1, \dots, c_i c_{i+1} c_i^{-1}, c_i, \dots, c_d),$$
$$(c_1, \dots, c_d)\sigma_i^{-1} = (c_1, \dots, c_{i+1}, c_{i+1}^{-1} c_i c_{i+1}, \dots, c_d).$$

Recall moreover the diagonal action of the group $\mathrm{Aut}(\mathrm{D}_n)$ on the set of Hurwitz vectors.

Since the two actions commute, we have an action of the group $\mathscr{B}_d \times \mathrm{Aut}(\mathrm{D}_n)$.

Definition 3. *Two Hurwitz vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *are said to be Hurwitz equivalent, or Braid-equivalent* (*resp. automorphism-equivalent, braid-automorphism-equivalent*) *if there exist* $\sigma \in \mathscr{B}_d$ (*resp.* $\varphi \in \mathrm{Aut}(\mathrm{D}_n)$, $(\sigma, \varphi) \in \mathscr{B}_d \times \mathrm{Aut}(\mathrm{D}_n)$) *such that* $\mathbf{w} = \mathbf{v}\sigma$ (*resp.* $\mathbf{w} = \mathbf{v}\varphi$, $\mathbf{w} = \mathbf{v}(\sigma, \varphi)$). *In this case we write* $\mathbf{v} \overset{B}{\sim} \mathbf{w}$ (*resp.* $\mathbf{v} \overset{A}{\sim} \mathbf{w}$, $\mathbf{v} \overset{BA}{\sim} \mathbf{w}$).

Notation 1. Identify a reflection $s_i(m) = -m + i$ with its index $i \in \mathbb{Z}/n\mathbb{Z}$.

The main result of this section is the following

Theorem 2. *The group* $\mathscr{B}_d \times \mathrm{Aut}(\mathrm{D}_n)$ *acts transitively on the set of Hurwitz vectors of a fixed numerical type, hence dihedral covers of* $\mathbb{P}^1$ *of a fixed numerical type form an irreducible closed subvariety of the moduli space.*

More precisely, given $\mathbf{v}$ *with* $\nu(\mathbf{v}) = (k, k_1, \dots, k_{n'})$ (*resp.* $\nu(\mathbf{v}) = (k_y, k_{xy}, k_1, \dots, k_{n'})$), *set* $R := \sum_i k_i$, *and assume* (*w.l.o.g.*) $\{h, k\} = \{k_y, k_{xy}\}$, $h \le k$ (*observe that* k, *resp.* $k + h$ *is even*).

We have then, assuming throughout $0 < r_i \le r_{i+1} \le n'$, $\underline{r} = (r_1, \dots, r_R)$ *and setting* $|\underline{r}| :\equiv \sum_i r_i$ (*mod* n):

(i) $\mathbf{v} \overset{BA}{\sim} (\underbrace{0, \dots, 0, 1, 1 + |\underline{r}|}_{k}, x^{r_1}, \dots, x^{r_R})$, *if* $n = 2n' + 1$.

(ii) $\mathbf{v} \overset{BA}{\sim} (\underbrace{0, \dots, 0}_{h}, \underbrace{1, \dots, 1, \lambda}_{k}, x^{r_1}, \dots, x^{r_R})$, *if* $n = 2n'$ *and* $h \neq 0$.

Here $\lambda = |\underline{r}| + \varepsilon$, *where* $\varepsilon \in \{0, 1\}$, $\varepsilon + k \equiv 1$ (*mod* 2).

(iii) $\mathbf{v} \overset{BA}{\sim} (\underbrace{1, \dots, 1, 3, \lambda}_{k}, x^{r_1}, \dots, x^{r_R})$, *if* $n = 2n'$ *and* $h = 0$.

Here $\lambda = |\underline{r}| + 3$.

We collect in the next section some preliminary results that shall be used in the proof.

REMARK 1. *It was brought to our attention after the paper was completed that a rather complicated but more general classification of Hurwitz orbits on* D_n^d *was done in* [Sia09]. *It is however not clear to us whether one can deduce our theorem above from these results.*

2.1. Auxiliary results

REMARK 2. Identifying a reflection $s_i(m) = -m + i$ with its index $i \in \mathbb{Z}/n\mathbb{Z}$, then conjugation corresponds to the action of another reflection on i:

$$s_i \mapsto s_j s_i s_j \quad \text{corresponds to } i \mapsto 2j - i = j - (i - j).$$

REMARK 3. The action of σ_1 on a pair of reflections (i, j) leaves their product invariant, hence leaves the difference $i - j$ invariant: for instance $(i, j)\sigma_1 = (2i - j, i)$.

LEMMA 2.1 (Normalization of reflection triples). *Given a sequence of reflections* (i, j, k) *in* D_n *which generate a dihedral subgroup* D_m*, its Hurwitz orbit contains a sequence of type* (i', j', j') *and a sequence of type* (i'', i'', j'').

In particular the subgroup D_m *is generated by the first two entries of a suitable sequence in the Hurwitz orbit.*

PROOF. First we consider the action of the four elements σ_1, σ_2, σ_1^{-1}, σ_2^{-1} on the triple.

$$(i, j, k)\sigma_1 = (2i - j, i, k), \quad (i, j, k)\sigma_2 = (i, 2j - k, j),$$
$$(i, j, k)\sigma_1^{-1} = (j, 2j - i, k), \quad (i, j, k)\sigma_2^{-1} = (i, k, 2k - j).$$

The corresponding transformations on the differences of consecutive elements are

$$\begin{aligned}
(j - i, k - j)\sigma_1 &= (j - i, (k - j) + (j - i)), \\
(j - i, k - j)\sigma_2 &= ((j - i) - (k - j), k - j), \\
(j - i, k - j)\sigma_1^{-1} &= (j - i, (k - j) - (j - i)), \\
(j - i, k - j)\sigma_2^{-1} &= ((j - i) + (k - j), k - j).
\end{aligned}$$

As long as both differences are non-zero (modulo n), we can reduce the maximal difference by one of these transformations.

This process must terminate, hence we reach a situation where one of the differences is zero.

We can arrange for the other difference to become zero by at most two additional transformations. Then we end up with a triple such that the last two entries are equal, and also with a triple such that the first two entries are equal.

Note that we can compute the necessary transformations using the Euclidean algorithm for the two differences. □

LEMMA 2.2. *Let* $(s_i, s_j, x^m) \in \mathrm{D}_n^3$. *Then, for all* $\ell \in \mathbb{Z}$ *we have:*

$$(s_i, s_j, x^m) \overset{B}{\sim} (s_{i+2\ell m}, s_{j+2\ell m}, x^m).$$

PROOF. For any $\ell \in \mathbb{Z}$ the following formula can be verified:

$$(s_i, s_j, x^m)(\sigma_2\sigma_1\sigma_1\sigma_2)^\ell = (s_{i+2\ell m}, s_{j+2\ell m}, x^m).$$

This proves the claim. □

LEMMA 2.3 (Double exchange). *The following equivalence holds:*

$$(j, i, i) \overset{B}{\sim} (i, i, j).$$

PROOF. This follows from the following equality:

$$(j, i, i)\sigma_1^{-1}\sigma_2^{-1} = (i, 2i - j, i)\sigma_2^{-1} = (i, i, j).$$ □

LEMMA 2.4 (Normalization of pair sequences). *The following equivalences hold.*

(i) $(0, 0, i, i) \overset{B}{\sim} (0, 0, -i, -i)$,
(ii) $(i, i, j, j) \overset{B}{\sim} (j, j, i, i)$,
(iii) $(i, i, j, j) \overset{B}{\sim} (i + \ell(j - i), i + \ell(j - i), j + \ell(j - i), j + \ell(j - i))$, $\forall \ell \in \mathbb{Z}$.
(iv) $(0, 0, i, i, j, j) \overset{B}{\sim} (0, 0, i, i, j - 2\ell i, j - 2\ell i)$, *for any* $\ell \in \mathbb{N}$

PROOF. We achieve equivalence (i) by

$$\begin{aligned}(0, 0, i, i)\sigma_2\sigma_3^2\sigma_2 &= (0, -i, -i, 0)\sigma_3\sigma_2\\ &= (0, -i, -2i, -i)\sigma_2\\ &= (0, 0, -i, -i).\end{aligned}$$

We achieve equivalence (ii) by applying twice Lemma 2.3.

Equivalence (iii) follows from the formula

$$\begin{aligned}&(i, i, j, j)(\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2)^\ell\\ &\quad = (i + \ell(j - i), i + \ell(j - i), j + \ell(j - i), j + \ell(j - i)), \quad \forall \ell \in \mathbb{Z},\end{aligned}$$

which can be proved e.g. by induction.

For equivalence (iv) we have:

$$\begin{aligned}(0, 0, i, i, j, j)\sigma_4\sigma_5\sigma_5\sigma_4 &= (0, 0, i, 2i - j, 2i - j, i)\sigma_5\sigma_4\\ &= (0, 0, i, i, 2i - j, 2i - j)\\ &\overset{B}{\sim} (0, 0, i, i, j - 2i, j - 2i) \quad \text{by (i) and (iii)}.\end{aligned}$$

Iterating this procedure we get the claim for all $\ell \in \mathbb{N}$. □

Lemma 2.5. *Let n be an integer $n \geq 3$ and let $\mathbf{v} = (i_1, \dots, i_{2N}) \in \mathrm{D}_n^{2N}$ be a Hurwitz vector whose components are all reflections. Then there exists $j \in \mathbb{Z}/n\mathbb{Z}$ such that:*

$$\mathbf{v} \overset{BA}{\sim} \begin{cases} (0, \dots, 0, j, j) & \textit{if } n \textit{ is odd}, \\ (0, \dots, 0, j, \dots, j) & \textit{if } n \textit{ is even}. \end{cases}$$

Moreover, the automorphisms involved in the previous equivalences are all of the form $y \mapsto x^{\ell} y$, $x \mapsto x$.

Proof. Using Lemma 2.1 inductively we get

$$\mathbf{v} \overset{B}{\sim} (i_1, i_1, i_2, i_2, \dots, i_{N-1}, i_{N-1}, i_N, j_N).$$

Then also $i_N = j_N$, since the product is the identity and we have in fact obtained

$$\mathbf{v} \overset{B}{\sim} (i_1, i_1, i_2, i_2, \dots, i_{N-1}, i_{N-1}, i_N, i_N).$$

By the automorphism $y \mapsto x^{-i_1} y$, $x \mapsto x$ which is of the form given in the claim of the lemma we get the following form

$$\overset{BA}{\sim} (0, 0, i_2, i_2, \dots, i_N, i_N)$$

and we may assume $i_\nu \geq 0$ by choosing suitable representatives.

The assertion of the Lemma follows now from the following:

CLAIM: Unless the sequence is already in the asserted form there is another sequence of pairs of non-negative integers representing a Hurwitz vector in the same braid equivalence class which is strictly smaller with respect to the lexicographical ordering.

Since we may reorder the sequence of pairs according to Lemma 2.4 (ii), we may assume $0 \leq i_2 \leq \cdots \leq i_N$.

Assume now we have three different kinds of entries $0 < i < j$. Then by using once more Lemma 2.4 (ii) we can bring these entries next to each other and have then a subsequence of the form $(0, 0, i, i, j, j)$.

By Lemma 2.4 (iv) and (i), (ii) we have:

$$(0, 0, i, i, j, j) \overset{B}{\sim} (0, 0, i, i, j - 2i, j - 2i) \tag{4}$$

$$\overset{B}{\sim} (0, 0, i, i, -j + 2i, -j + 2i) \tag{5}$$

Now, either $j - 2i \geq 0$ or $j - 2i < 0$. In the first case $j > j - 2i \geq 0$ and the r.h.s. of (4) is smaller than the l.h.s. In the second case $0 < i < j$ implies $2i - j < j$ and the r.h.s. of (5) is smaller than the l.h.s of (4).

Therefore we can reduce to a sequence of pairs where all entries are either 0 or a positive integer j. This concludes the claim in case where n is even.

In the case where n is odd we may have reached a situation with at least four entries equal to j. But, according to Lemma 2.4 (iv) with $\ell = -(n-1)/2$, we have

$$\begin{aligned}(0,0,j,j,j,j) &\overset{B}{\sim} (0,0,j,j,j-2\ell j, j-2\ell j)\\ &= (0,0,j,j,nj,nj)\\ &= (0,0,j,j,0,0)\\ &\overset{B}{\sim} (0,0,0,0,j,j)\end{aligned}$$

Hence also in this case our claim holds true. □

Lemma 2.6 (Normalization of reflection pair). *Given a sequence of reflections (i_0, j_0) in D_n which generate a dihedral subgroup D_m, its Hurwitz orbit consists of the pairs (i,j) with $i \equiv i_0 \left(\mathrm{mod}\ \frac{n}{m}\right)$ and $j - i = j_0 - i_0$.*

2.2. Proof of Theorem 2

1. We first bring all the rotations to the right by elementary braids, obtaining:

$$\mathbf{v} \overset{B}{\sim} (s_{i_1}, \dots, s_{i_{2N}}, x^{r_1}, \dots, x^{r_R}), \tag{6}$$

where $2N = k$ if n is odd, $2N = h + k$ if n is even and $R = \sum_i k_i$.

Observe moreover that we can arbitrarily permute the rotations among themselves, a fact that at a later moment will allow us to assume $r_i \le r_{i+1}$, $\forall i$.

2. If $r_i > n'$, we bring r_i next to the reflection $s_j = s_{i_{2N}}$ and then apply a full twist with this reflection s_j, obtaining:

$$(s_j, x^{r_i})\sigma_1^2 = (s_{j-2r_i}, x^{-r_i}).$$

Hence we can assume $0 < r_i \le n'$ for all i.

3. If n is even, without loss of generality, we may further assume that

$$k = |\{s_i \mid s_i \text{ is conjugate to } s_{i_{2N}}\}|.$$

4. By Lemma 2.3 we can assume that $i_{2f} = i_{2f-1}$ for any $f \in \{1, \dots, N-1\}$.

Then we set $j_f = i_{2f}$ for $f \in \{1, \dots, N-1\}$ and $j_N = i_{2N-1}$, thus we have:

$$\mathbf{v} \overset{B}{\sim} (j_1, j_1, j_2, j_2, \dots, j_{N-1}, j_{N-1}, j_N, j_N + |\underline{r}|, x^{r_1}, \dots, x^{r_R}).$$

Notice that the condition that $k = |\{s_i \mid s_i \text{ is conjugate to } s_{j_N + |\underline{r}|}\}|$ still holds.

5. Consider the vector

$$\mathbf{w} := (j_1, j_1, j_2, j_2, \dots, j_N, j_N) \in \mathrm{D}_n^{2N}.$$

The subgroup $\langle \mathbf{w} \rangle \leq \mathrm{D}_n$ generated by $\mathbf{w}$ is isomorphic either to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or to D_m with $m \geq 3$.

We show now, in each of these three cases, that $\mathbf{v}$ is equivalent to one of the vectors in the statement of Theorem 2.

I. $\langle \mathbf{w} \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Then $j_1 = j_2 = \cdots = j_N =: j$ and

$$\mathbf{v} \overset{B}{\sim} (j, \ldots, j, j + |\underline{r}|, x^{\underline{r}}).$$

We have that $\mathrm{D}_n = \langle j, x^{\underline{r}} \rangle$ and hence $GCD(n, \underline{r}) = 1$. By Lemmas 2.2 and 2.4 (ii) it follows that

$$\mathbf{v} \overset{B}{\sim} (j + 2\ell, \ldots, j + 2\ell, j + 2m, j + 2m + |\underline{r}|, x^{\underline{r}}), \quad \forall \ell, m \in \mathbb{Z}.$$

If n is odd, 2 is a generator of $\mathbb{Z}/n\mathbb{Z}$ and hence the result follows.

If n is even, we may assume that j is odd. In the case where moreover $|\underline{r}|$ is even (i.e., $h = 0$) we choose ℓ such that $j + 2\ell = 1$ and we set $m = (3 - j)/2$. Otherwise we take m such that $j + 2m + |\underline{r}| = 0$, and ℓ such that $j + 2\ell = -1$. The result follows by a sequence of Hurwitz moves between reflections bringing the element 0 from the last position to the initial one.

II. $\langle \mathbf{w} \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Then n must be even and there exist $i, j \in \mathrm{D}_n$ with $i - j = \frac{n}{2}$ such that $j_f \in \{i, j\}$ for all $f \in \{1, \ldots, N\}$. Using Lemma 2.4 (ii) we can bring all the pairs of i's to the left and obtain:

$$\mathbf{v} \overset{B}{\sim} (i, \ldots, i, j, \ldots, j, j + |\underline{r}|, x^{\underline{r}}) \quad \text{or}$$
$$\mathbf{v} \overset{B}{\sim} (i, \ldots, i, j, \ldots, j, i, i + |\underline{r}|, x^{\underline{r}}).$$

Exchanging the roles of i and j and using again Lemma 2.4 (ii) we see that the second vector is braid-equivalent to one of the first type, hence we shall only consider the first vector.

We have that $\mathrm{D}_n = \langle i, j, x^{\underline{r}} \rangle$ and so $GCD(n, \underline{r}) \in \{1, 2\}$.

If $GCD(n, \underline{r}) = 1$, we apply Lemma 2.2 and we get:

$$\mathbf{v} \overset{B}{\sim} (i + 2\ell, \ldots, i + 2\ell, j + 2m, \ldots, j + 2m, j + 2p, j + 2p + |\underline{r}|, x^{\underline{r}}), \forall \ell, m, p \in \mathbb{Z}.$$

For an appropriate choice of $\ell, m, p \in \mathbb{Z}$ if i, j are odd we reach the required normal form (iii).

If one is even and the other is odd there are two possibilities: either the larger group of k elements (to which $j + |r|$ by our assumption belongs) contains the numbers j, or that it contains the numbers i.

In the former case with an automorphism of D_n we achieve that i is even and again for an appropriate choice of $\ell, m, p \in \mathbb{Z}$ we reach the required normal form (ii).

In the latter case since the number of occurrences of i is even we first apply repeatedly Lemma 2.3 to move all the j's to the left, then with an automorphism we achieve that j is even, and finally for an appropriate choice of ℓ, m we reach the required normal form (ii).

Assume now that $GCD(n, \underline{r}) = 2$.

Then $\frac{n}{2}$ must be odd and therefore i and j have different parities. Moreover $|\underline{r}|$ is even and so we may assume, acting with an automorphism, that h coincides with the number of i's. By Lemma 2.2 we get:

$$\mathbf{v} \overset{B}{\sim} (i+4\ell, \dots, i+4\ell, j, \dots, j, j+|\underline{r}|, x^{\underline{r}}), \quad \forall \ell \in \mathbb{Z}.$$

If $i \equiv j-1 \pmod 4$, we apply the automorphism $x^{j-1}y \mapsto y$, $x \mapsto x$ to transform the vector in the desired form (ii). Otherwise $i \equiv j+1 \pmod 4$ and we proceed as follows:

$$\begin{aligned}\mathbf{v} &\overset{B}{\sim} (j+1, \dots, j+1, j, \dots, j, j+|\underline{r}|, x^{\underline{r}})\\ &\overset{A}{\sim} (-j-1, \dots, -j-1, -j, \dots, -j, -j-|\underline{r}|, x^{-\underline{r}})\\ &\overset{A}{\sim} (0, \dots, 0, 1, \dots, 1, 1-|\underline{r}|, x^{-\underline{r}})\\ &\overset{B}{\sim} (0, \dots, 0, 1, \dots, 1, 1+|\underline{r}|, x^{\underline{r}}),\end{aligned}$$

where in the second equivalence we have used the automorphism $x \mapsto x^{-1}$, $xy \mapsto x^{-1}y$, in the third equivalence we have used the automorphism $x \mapsto x$, $xy \mapsto x^{j+2}y$, and in the fourth one we proceeded as in the reduction step 2.

III. $\langle \mathbf{w} \rangle \cong \mathrm{D}_m$, $m \geq 3$.

By Lemma 2.5 applied to $\mathbf{w}$ we reduce $\mathbf{w}$ to the form

$$\mathbf{w} \overset{BA}{\sim} (0, \dots, 0, j, \dots, j, j). \tag{7}$$

Since we want to apply the corresponding moves to $\mathbf{v}$ we avoid moves which put the last pair into a different position.

By a careful modification of the proof of Lemma 2.5 this restriction leads to

$$\begin{aligned}\mathbf{w} &\overset{BA}{\sim} (0, \dots, 0, j, \dots, j, j, j) \quad \text{or}\\ &\overset{BA}{\sim} (0, \dots, 0, j, \dots, j, 0, 0)\end{aligned} \tag{8}$$

hence we have

$$\begin{aligned}\mathbf{v} &\overset{BA}{\sim} (0, \dots, 0, j, \dots, j, j+|\underline{r}|, x^{\underline{r}}) \quad \text{or}\\ &\overset{BA}{\sim} (0, \dots, 0, j, \dots, j, 0, |\underline{r}|, x^{\underline{r}}).\end{aligned} \tag{9}$$

It is clearly enough to consider only the first case.

Observe that $\mathrm{D}_n = \langle y, x^j y, x^{\underline{r}} \rangle$.

If n is odd, then by Lemma 2.5 we have:

$$\mathbf{v} \overset{BA}{\sim} (0, \dots, 0, j, j + |\underline{r}|, x^{\underline{r}})$$

with $GCD(j, n, \underline{r}) = 1$.

From Lemma 2.2 it follows that the right hand side is braid-equivalent to

$$(0, \dots, 0, j + 2\ell M, j + |\underline{r}| + 2\ell M, x^{\underline{r}}),$$

where $M := GCD(n, \underline{r})$. We have then $GCD(j, M) = 1$.

Set $\ell = n/\gamma$, with γ equal to the product of all common prime factors of n and j taken with the maximal power with which they divide n; hence $GCD(\varepsilon := j + 2\ell M, n) = 1$. In fact if $p \mid n$, then $p \neq 2$, and either $p \mid \ell$, or $p \mid j$: but if $p \mid \ell$ then $p \mid j$, contradicting that ℓ and j are relatively prime; if instead $p \mid j$, then by the same token $p \mid M$, contradicting $GCD(j, M) = 1$.

Using Lemma 2.6 we get:

$$\begin{aligned}
(0, \dots, 0, \varepsilon, \varepsilon + |\underline{r}|, x^{\underline{r}}) &\overset{B}{\sim} (0, \dots, 0, \ell\varepsilon, (\ell + 1)\varepsilon, \varepsilon + |\underline{r}|, x^{\underline{r}}) \\
(\text{for } \ell = -\varepsilon^{-1}) &= (0, \dots, 0, -1, \varepsilon - 1, \varepsilon + |\underline{r}|, x^{\underline{r}}) \\
&\overset{B}{\sim} (0, \dots, -\ell, -\ell - 1, \varepsilon - 1, \varepsilon + |\underline{r}|, x^{\underline{r}}) \\
(\text{for } \ell = -\varepsilon) &= (0, \dots, \varepsilon, \varepsilon - 1, \varepsilon - 1, \varepsilon + |\underline{r}|, x^{\underline{r}}).
\end{aligned}$$

Repeating these steps inductively we obtain a vector of the following form:

$$\begin{aligned}
(\varepsilon, \varepsilon - 1, \dots, \varepsilon - 1, \varepsilon + |\underline{r}|, x^{\underline{r}}) &\overset{B}{\sim} (\varepsilon - 1, \dots, \varepsilon - 1, \varepsilon, \varepsilon + |\underline{r}|, x^{\underline{r}}) \\
&\overset{A}{\sim} (0, \dots, 0, 1, 1 + |\underline{r}|, x^{\underline{r}}).
\end{aligned}$$

The case where n is odd is then settled.

Let now n be even.

We distinguish three cases: $h = 0$, h is equal to the number of y's in (9), or h is equal to the number of $x^j y$'s in (9).

In the first case we apply Lemma 2.2 to obtain:

$$\begin{aligned}
&(0, \dots, 0, j, \dots, j, j + |\underline{r}|, x^{\underline{r}}) \\
&\quad \overset{B}{\sim} (0, \dots, 0, j + 2\ell M, \dots, j + 2\ell M, j + 2\ell M + |\underline{r}|, x^{\underline{r}}),
\end{aligned}$$

where again $M = GCD(n, \underline{r})$. Let $\ell = n/\gamma$, with γ equal to the product of all common prime factors of n and j taken with the maximal power with which they divide n; hence $GCD(\varepsilon := j + 2\ell M, n) = 2$.

Using Lemma 2.6 we have:

$$\begin{aligned}(0,\dots,0,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}}) &\overset{B}{\sim} (0,\dots,0,\ell\varepsilon,(\ell+1)\varepsilon,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ (\text{for } \ell\varepsilon=-2) &= (0,\dots,0,-2,\varepsilon-2,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ &\overset{B}{\sim} (0,\dots,-2\ell,-2(\ell+1),\varepsilon-2,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ (\text{for } -2\ell=\varepsilon) &= (0,\dots,\varepsilon,\varepsilon-2,\varepsilon-2,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ &\overset{B}{\sim} (0,\dots,\varepsilon-2,\varepsilon-2,\varepsilon,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}}).\end{aligned}$$

Repeating these steps inductively we arrive at the following form:

$$(\varepsilon-2,\dots,\varepsilon-2,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}}) \overset{A}{\sim} (1,\dots,1,3,\dots,3,3+|\underline{r}|,x^{\underline{r}}).$$

If $N=2$ this completes the proof. Otherwise we need to transform each pair of the form (x^3y,x^3y) into (xy,xy). Notice that, since $(x^3y)^2=1$, we can move this pair everywhere inside the vector without changing the other elements. Moreover we can conjugate both elements by any of the others obtaining again a pair of the form (g,g) with $g^2=1$. It follows that we can transform (x^3y,x^3y) into $(hx^3yh^{-1},hx^3yh^{-1})$, for any $h\in\langle xy,x^3y,x^{\underline{r}}\rangle=\mathrm{D}_n$, hence the result follows (notice that this argument follows the proof of Lemma 1.9 in [Kanev06]).

We consider now the case where h is equal to the number of 0's in (9). In this situation j must be odd, therefore there exists an ℓ such that $GCD(\varepsilon:=j+2\ell M,n)=1$, where $M=GCD(n,\underline{r})$. From Lemmas 2.2 and 2.6 we have:

$$\begin{aligned}(0,\dots,0,j,\dots,j,j+|\underline{r}|,x^{\underline{r}}) &\overset{B}{\sim} (0,\dots,0,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ &\overset{B}{\sim} (0,\dots,0,\ell\varepsilon,(\ell+1)\varepsilon,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ (\text{for } \ell\varepsilon=-1) &= (0,\dots,0,-1,\varepsilon-1,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ &\overset{B}{\sim} (0,\dots,-\ell,-\ell-1,\varepsilon-1,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ (\text{for } \ell=-\varepsilon) &= (0,\dots,\varepsilon,\varepsilon-1,\varepsilon-1,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}})\\ &\overset{B}{\sim} (0,\dots,\varepsilon-1,\varepsilon-1,\varepsilon,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}}).\end{aligned}$$

Repeating this argument inductively we reach the following form:

$$(\varepsilon-1,\dots,\varepsilon-1,\varepsilon,\dots,\varepsilon,\varepsilon+|\underline{r}|,x^{\underline{r}}) \overset{A}{\sim} (0,\dots,0,1,\dots,1,1+|\underline{r}|,x^{\underline{r}}),$$

hence the claim follows.

If h is equal to the number of x^jy's, we apply the automorphism $x\mapsto x$, $x^jy\mapsto y$ and use the equivalence

$$(-j,\dots,-j,0,\dots,0,|\underline{r}|,x^{\underline{r}}) \overset{B}{\sim} (0,\dots,0,-j,\dots,-j,|\underline{r}|,x^{\underline{r}}).$$

The claim follows now from the previous case. This completes the proof of the Theorem. □

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