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Probability Theory — *Stochastic viability for regular closed sets in Hilbert spaces*, by Piermarco Cannarsa and Giuseppe da Prato, communicated on 13 May 2011.

Dedicated to Giovanni Prodi.

Abstract. — We present necessary and sufficient conditions to guarantee that at least one solution of an infinite dimensional stochastic differential equation, which starts from a regular closed subset K of an Hilbert space, remains in K for all times.

Key words: Stochastic viability, stochastic differential equations in infinite dimensions, distance function.

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1. Introduction and setting of the problem

Let H be a separable Hilbert space with scalar product $\langle \cdot , \cdot \rangle$ and norm $|\cdot|$. We are concerned with a stochastic differential equation of the form

$$\begin{cases} dX = b(X)\,dt + \sigma(X)\,dW(t), \\ X(0) = x \in H. \end{cases} \tag{1}$$

We shall assume the following:

Hypothesis 1. (i) *$b : H \to H$ is continuous.*
(ii) *$\sigma : H \to L_2(H)$ is continuous, where $L_2(H)$ is the space of all Hilbert–Schmidt operators on H.*
(iii) *$W(t)$, $t \geq 0$, is a H-valued cylindrical Wiener process defined in a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.*

We recall that $W(t)$ is formally defined by

$$W(t) := \sum_{k=1}^{\infty} e_k W_k(t), \tag{2}$$

where (e_k) is an orthonormal basis of K and (W_k) a sequence of one-dimensional Brownian motions in $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, mutually independent and adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Notice that, although the series in (2) is not convergent in $L^2(\Omega, \mathscr{F}, \mathbb{P})$, $SW(t) := \sum_{k=1}^{\infty} Se_k W_k(t)$ (where $S \in L_2(H)$) is.

We say that an adapted continuous stochastic process $X(t)$, $t \in [0, T]$, is a *solution* of (1) if

$$X(t) = x + \int_0^t b(X(s))\,ds + \int_0^t \sigma(X(s))\,dW(s), \quad t \geq 0,\ \mathbb{P}\text{-a.s.}. \tag{3}$$

Existence and uniqueness results for equation (1) are easily found in the literature, see, e.g., [5].

In this paper, we are interested in viability for equation (1) with respect to a regular closed set K. We recall that K is *viable* if, for any $x \in K$, there exists a solution $X(\cdot, x)$ of (1) which remains in K for all times.

When H is finite dimensional, stochastic viability for closed sets has been extensively studied. In connection with this paper, let us quote [3], [4], where a characterization of viability of K is given in terms of the distance function

$$d_K(x) := \inf\{|x - y| : y \in K\},$$

and several references on related works can be found.

Recently, strong interest in viability for stochastic partial differential equations was motivated by mathematical finance problems, see e.g. [1] and [6]. In that case, it is important to see if there exist viable finite dimensional subspaces for the stochastic flow. A new approach to viability for SPDEs was developed in [7], [11] and [8]. Such an approach is based on an infinite dimensional generalization of the *support theorem*, proved in the finite dimensional case in [10]. The avantage of this method is that it reduces the problem to the well known *Nagumo condition* for deterministic systems. The price to pay is that one has to assume the coefficient σ to be at least C^1.

When trying to extend finite dimensional results to the Hilbert space setting, one has to confront the major difficulty that, in euclidean space, the distance function $d_K^2(x)$ is twice differentiable almost everywhere (with respect to Lebesgue measure), which allows to apply Itô's formula to some power of d_K. In infinite dimensional Hilbert spaces, on the contrary, one only has that $d_K^2(x)$ is twice differentiable on a dense set in general (except for very special K), see [9]. Therefore, suitable regularity assumptions have to be imposed on ∂K to be able to use Itô's formula (that holds true for functions of class C^2 in Hilbert spaces, see, e.g., [5]). We do so as follows:

HYPOTHESIS 2. *$d_K^4 \in C^2(U)$ for some open neighborhood U of K with bounded first and second derivatives.*

Observe that Hypothesis 2 allows for closed sets K with empty interior.

As in [3] and [4], our analysis relies on the use of the Kolmogorov operator restricted to K, that is,

$$L_K\varphi(x) := \frac{1}{2}\,\mathrm{tr}[a(\Pi_K(x))D^2\varphi(x)] + \langle b(\Pi_K(x), D\varphi(x)\rangle, \tag{4}$$

where

$$a(x) := \sigma(x)\sigma^*(x) \quad \forall x \in H, \tag{5}$$

and $\Pi_K(x)$ denotes the projection of x onto K, defined for each $x \in U$ as the unique element $\bar{x} \in K$ such that $|x - \bar{x}| = d_K(x)$. We observe that the fact $\Pi_K(x)$ is nonempty and reduces to a singleton is a (nontrivial) consequence of the C^1-smoothness of d_K in $U \backslash K$. Indeed, in view of Hypothesis 2, d_K is of class C^1 on $U \backslash K$. Then, the Density Theorem (see e.g., [2]) yields the existence and uniqueness of the projection onto K for a dense subset of $U \backslash K$, hence for all points of $U \backslash K$ by an approximation argument.

Before stating our main results, let us introduce further notation. Recall that, under Hypothesis 2, $d_K^2 \in C^1(U)$, $d_K \in C^1(U \backslash K)$, and so

$$Dd_K^2(x) = 2(x - \Pi_K(x)), \quad \forall x \in U, \tag{6}$$

and

$$n(x) := Dd_K(x) = \frac{x - \Pi_K(x)}{d_K(x)} \quad \forall x \in U \backslash K. \tag{7}$$

Now, define

$$V(x) := \frac{1}{4} d_K^4(x) \quad \forall x \in U. \tag{8}$$

Then

$$DV(x) = (x - \Pi_K(x)) d_K^2(x) = d_K^3(x) n(x), \quad \forall x \in U \tag{9}$$

and

$$D^2 V(x) = 3 d_K^2(x) n(x) \otimes n(x) + d_K^3(x) Dn(x), \quad \forall x \in U \tag{10}$$

where $D^2 V(x)$ is a bounded linear operator on H in view of Hypothesis 2. So,

$$D^2 V(x) = \begin{cases} 3 d_K^2(x) n(x) \otimes n(x) + d_K^3(x) Dn(x) & \forall x \in U \backslash K \\ 0 \quad \forall x \in K. \end{cases} \tag{11}$$

Finally, let us point out, for future use, the useful inequality

$$d_K(x - \Pi_K(x) + y) \le d_K(x), \quad \forall x \in U, \forall y \in K \tag{12}$$

which is derived as follows:

$$d_K(x - \Pi_K(x) + y) = \inf_{z \in K} |x - \Pi_K(x) + y - z| \le |x - \Pi_K(x)| = d_K(x).$$

It is convenient to introduce the following modified system,

$$\begin{cases} dX(t) = b(\Pi_K(X(t)))\,dt + \sigma(\Pi_K(X(t)))\,dW(t), \\ X(0) = x \in K. \end{cases} \tag{13}$$

It is obvious that K is viable for system (1) if and only if it is viable for system (13). So, we shall restrict our considerations from now on to system (13), a generic solution of which will be denoted by $X_K(t,x)$. Recall that the corresponding Kolmogorov operator is L_K defined by (4).

We can now state the main results of the paper. Hypothesis 1 will be assumed hereafter, without further notice.

Theorem 3. *Let Hypothesis 2 be fulfilled. Then K is viable if and only if*

$$L_K V(x) \le 0, \quad \forall x \in U. \tag{14}$$

Under a stronger regularity assumption on K we can characterize viability imposing a simpler set of conditions just on ∂K. We shall express such conditions in terms of the signed distance from ∂K, that is,

$$\bar{d}_K(x) := d_K(x) - d_{H \setminus \mathring{K}}(x) \quad \forall x \in H$$

where $\mathring{K}$ denotes the interior of K. Notice that the smoothness of $\bar{d}_K$ requires K to be the closure of its interior. Also, if $\bar{d}_K$ is differentiable, then the exterior normal at every point $x \in \partial K$ is given by $D\bar{d}_K(x)$.

Theorem 4. *Let $\bar{d}_K$ be of class C^2 with bounded first and second derivatives on some neighborhood of ∂K. Then, K is viable if and only if*

$$\frac{1}{2}\,\mathrm{tr}[a(y)D^2\bar{d}_K(y)] + \langle b(y), D\bar{d}_K(y)\rangle \le 0 \quad \forall y \in \partial K \tag{15}$$

and

$$\langle a(y)D\bar{d}_K(y), D\bar{d}_K(y)\rangle = 0 \quad \forall y \in \partial K. \tag{16}$$

2. Proofs and examples

2.1. *Proof of Theorem* 3

Necessity. Suppose K is viable. Let $x \in U$ and let $X_K(t, \Pi_K(x))$ be a solution of (13) which remains in K for all $t > 0$. Then, owing to (12),

$$V(x - \Pi_K(x) + X_K(t, \Pi_K(x))) \le V(x), \quad \forall t \ge 0. \tag{17}$$

Now, by Itô's formula we have, for any $t \geq 0$,

$$
\begin{aligned}
& dV(x - \Pi_K(x) + X_K(t, \Pi_K(x))) \\
&\quad = \langle DV(x - \Pi_K(x) + X_K(t, \Pi_K(x))), dX_K(t, \Pi_K(x)) \rangle \\
&\qquad + \frac{1}{2} \operatorname{tr}[a(X_K(t, \Pi_K(x))D^2 V(x - \Pi_K(x) + X_K(t, \Pi_K(x))] \, dt \\
&\quad = L_K V(x - \Pi_K(x) + X_K(t, \Pi_K(x))) \, dt \\
&\qquad + \langle DV(x - \Pi_K(x) + X_K(t, \Pi_K(x))), \sigma(X_K(t, \Pi_K(x))) \, dW(t) \rangle.
\end{aligned}
$$

Hence, integrating between 0 and t and taking expectation,

$$
\begin{aligned}
& \mathbb{E}[V(x - \Pi_K(x) + X_K(t, \Pi_K(x))) - V(x)] \\
&\quad = \mathbb{E}\left[\int_0^t L_K V(x - \Pi_K(x) + X_K(s, \Pi_K(x))) \, ds \right] \leq 0.
\end{aligned}
$$

Consequently,

$$
\frac{1}{t} \mathbb{E}\left[\int_0^t L_K V(x - \Pi_K(x) + X_K(s, \Pi_K(x))) \, ds \right] \leq 0,
$$

which, letting $t \to 0$, yields (14).

SUFFICIENCY. Assume that (14) holds and let $x \in K$. Consider the exit time from U

$$
\tau_U(x) := \inf\{t \geq 0 : X_K(t, x) \in \partial U\}.
$$

We claim that $\tau_U(x) = \infty$ almost surely. Indeed, applying Itô's formula with a stopping time, for every $t \geq 0$ we have

$$
\begin{aligned}
V(X_K(t \wedge \tau_U(x), x)) &= \int_0^{t \wedge \tau_U(x)} L_K V(X_K(s, x)) \, ds \\
&\quad + \int_0^{t \wedge \tau_U(x)} \langle DV(X_K(s, x)), \sigma(X_K(s, x)) \, dW(s) \rangle,
\end{aligned}
$$

or

$$
\begin{aligned}
V(X_K(t \wedge \tau_U(x), x)) &= \int_0^t \mathbb{1}_{\{\tau_U(x) \geq s\}} L_K V(X_K(s, x)) \, ds \\
&\quad + \int_0^t \mathbb{1}_{\{\tau_U(x) \geq s\}} \langle DV(X_K(s, x)), \sigma(X_K(s, x)) \, dW(s) \rangle.
\end{aligned}
$$

Hence, taking expectation,

$$\mathbb{E}[V(X_K(t \wedge \tau_U(x), x))] = \mathbb{E}\left[\int_0^t \mathbb{1}_{\{\tau_U(x) \geq s\}} L_K V(X_K(s, \Pi_K(x)))\, ds\right] \leq 0$$

for every $t \geq 0$ on account of (14). This implies $X_K(t \wedge \tau_U(x), x) \in K$ a.s. for every $t \geq 0$, so that

$$\mathbb{P}(\tau_U(x) < \infty) = \lim_{i \to \infty} \mathbb{P}(\tau_U(x) \leq i) = 0.$$

The proof is thus complete. □

2.2. *Proof of Theorem* 4

To begin with, we observe that our assumption on $\bar{d}_K$ implies that d_K^4 is of class C^2 with bounded first and second derivatives on some neighborhood of K, say U. So, Theorem 3 can be applied. Denoting by n the gradient $D\bar{d}_K$, in light of (9) and (11) we have

$$L_K V(x) = \frac{1}{2} d_K^3(x) \operatorname{tr}[a(\Pi_K(x)) Dn(x)] + d_K^3(x) \langle b(\Pi_K(x), n(x) \rangle + \frac{3}{2} d_K^2(x) \langle a(\Pi_K(x)) n(x), n(x) \rangle, \quad \forall x \in U. \tag{18}$$

Suppose K is viable. Then, by Theorem 3,

$$\frac{1}{2} d_K^3(x) \operatorname{tr}[a(\Pi_K(x)) Dn(x)] + d_K^3(x) \langle b(\Pi_K(x)), n(x) \rangle + \frac{3}{2} d_K^2(x) \langle a(\Pi_K(x)) n(x), n(x) \rangle \leq 0, \quad \forall x \in U \backslash K. \tag{19}$$

Hence, dividing both sides of (19) by $d_K^2(x)$ to obtain (16) as $x \to \partial K$, i.e.,

$$\langle a(y) n(y), n(y) \rangle = 0, \quad \forall y \in \partial K. \tag{20}$$

Moreover, since $n(\Pi_K(x)) = n(x)$, the above equality yields

$$\langle a(\Pi_K(x)) n(x), n(x) \rangle = 0, \quad \forall x \in U \backslash K. \tag{21}$$

Therefore, (19) reduces to

$$\frac{1}{2} d^3(x) \operatorname{tr}[a(\Pi_K(x)) Dn(x)] + d^3(x) \langle b(\Pi_K(x)), n(x) \rangle \leq 0, \quad \forall x \in U \backslash K.$$

Consequently, as $x \to \partial K$, we obtain (15).

Next, assume (15) and (16). Then, (21) also holds true. So, by (15),

$$\begin{aligned} L_K V(x) &= \frac{1}{2} d_K^3(x) \operatorname{tr}[a(\Pi_K(x)) Dn(x)] + d_K^3(x) \langle b(\Pi_K(x), n(x) \rangle \\ &= d_K^3(x) \left\{ \frac{1}{2} \operatorname{tr}[a(\Pi_K(x)) Dn(\Pi_K(x))] + \langle b(\Pi_K(x), n(\Pi_K(x)) \rangle \right\} \leq 0 \end{aligned}$$

for all $x \in U$. Hence, Theorem 3 ensures that K is viable. □

2.3. *Examples*

In this section, we apply the above theory to three examples characterizing viability for a ball, a half-space, and a subspace of H.

Example 5 (The ball B_1). Let $K = B_1 := \{x \in H : |x| \leq 1\}$. Then,

$$n(x) = \frac{x}{|x|}, \quad \forall x \in B_1^c.$$

Therefore,

$$Dn(x) = \frac{1}{|x|} - \frac{x \otimes x}{|x|^3}, \quad \forall x \in B_1^c.$$

Thus, (15) and (16) become, respectively,

$$\frac{1}{2} \operatorname{tr}[a(y)] + \langle b(y), y \rangle - \frac{1}{2} \langle a(y) y, y \rangle \leq 0 \quad \forall y \in \partial B_1,$$

and

$$\langle a(y) y, y \rangle = 0 \quad \forall y \in \partial B_1.$$

So,

$$\frac{1}{2} \operatorname{tr}[a(y)] + \langle b(y), y \rangle \leq 0 \quad \text{and} \quad \langle a(y) y, y \rangle = 0 \quad \forall y \in \partial B_1$$

are necessary and sufficient conditions for B_1 to be viable. □

Example 6 (Half-space). Let $\{e_k\}$ be an orthonormal basis of H, let $x_k = \langle x, e_k \rangle$, $k \in \mathbb{N}$, and define

$$K = \{x \in H : x_1 \geq 0\}.$$

Then, $\partial K = \{x \in H : x_1 = 0\}$ and $d_K(x) = x_1^- = -\min\{x_1, 0\}$. So,

$$n(x) = -e_1 \mathbb{1}_{x_1 \leq 0}, \quad x \in (\overset{\circ}{K})^c,$$

and

$$Dn(x) = 0, \quad x \in (\overset{\circ}{K})^c.$$

Therefore, the two conditions

$$b_1(y) \le 0, \quad \text{and} \quad a_{1,1}(y) = 0, \quad \forall y \in \partial K,$$

where $b_1(y) = \langle b(y), e_1 \rangle$ and $a_{1,1}(y) = \langle a(y)e_1, e_1 \rangle$, are necessary and sufficient for the viability of K. □

Example 7 (Subspace). Let Z be a closed subspace of H and let P be the orthogonal projector onto Z. Then, $\Pi_Z(x) = Px$,

$$n(x) = \frac{x - Px}{|x - Px|},$$

and

$$Dn(x) = \frac{I - P}{|x - Px|} - \frac{(x - Px) \otimes (x - Px)}{|x - Px|^3}$$

for all $x \in V$. Thus, Z is viable if and only if

$$\frac{1}{2} d_Z^2(x)\, \mathrm{tr}[a(Px)(I - P)] + d_Z^2(x) \langle b(Px), x - Px \rangle$$
$$+ \langle a(Px)(x - Px), x - Px \rangle \le 0 \quad \forall x \in H.$$

3. Equations in mild form

Let us consider the stochastic differential equation

$$\begin{cases} dX = (AX + b(X))\, dt + \sigma(X)\, dW(t), \\ X(0) = x \in H, \end{cases} \tag{22}$$

where $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA}, and b, σ and W are assumed to satisfy Hypothesis 1. Let K be a closed convex set in H such that d_K^4 is of class C^2.

A mild solution of equation (22) is a stochastic process X which solves the integral equation

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} b(X(s))\, ds + \int_0^t e^{(t-s)A} \sigma(X(s))\, dW(s). \tag{23}$$

It is useful to consider the approximate equation

$$\begin{cases} dX_n = (A_n X_n + b(X_n))\,dt + \sigma(X_n)\,dW(t), \\ X_n(0) = x \in H, \end{cases} \tag{24}$$

where for each $n \in \mathbb{N}$, $A_n = nA(nI - A)^{-1}$ is the Yosida approximation of A.

The following result follows from Theorem 3.

PROPOSITION 8. *Let Hypothesis* 2 *be fulfilled. Then K is viable for problem* (24) *if and only if*

$$\frac{1}{2}\operatorname{tr}[a(\Pi_K x)D^2 V(x)] + \langle A_n \Pi_K x + b(\Pi_K x), DV(x)\rangle \le 0, \quad \forall x \in U. \tag{25}$$

Moreover, if (25) *holds for any $n \in \mathbb{N}$, then K is viable for problem* (22).

Applying Theorem 4 we obtain the following.

PROPOSITION 9. *Let $\bar{d}_K$ be of class C^2 with bounded first and second derivatives on some neighborhood of ∂K. Then K is viable for problem* (24) *if and only if, for every $y \in \partial K$,*

$$\frac{1}{2}\operatorname{tr}[a(y)Dn(y)] + \langle A_n y + b(y), n(y)\rangle \le 0 \tag{26}$$

and

$$\langle a(y)n(y), n(y)\rangle = 0. \tag{27}$$

Moreover, if (26) *holds for any $n \in \mathbb{N}$, then K is viable for problem* (22).

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P. Cannarsa
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica 1
00133 Roma (Italy)
cannarsa@mat.uniroma2.it

G. Da Prato
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7
I-56125 Pisa (Italy)
daprato@sns.it