Solid Mechanics — *Illustrations of Zanaboni's formulation of Saint-Venant's principle*, by Robin J. Knops and Piero Villaggio, communicated on 16 June 2011.

Dedicated to Giovanni Prodi
in appreciation of his mathematical talent and civic commitment.

Abstract. — Zanaboni in 1937 published three notes in the *Rendiconti* in which he proposed a modified version and new proof of Saint-Venant's principle applicable to bodies not necessarily cylindrical in shape. Examples are presented to illustrate that Zanaboni's version is not universally applicable.

1. Introduction

Saint-Venant's principle, first proposed in 1855, has been accepted by most engineers as an heuristic argument for disposing of edge effects in the elementary theory of beams and similar mechanical structures. Nevertheless, its validity was challenged from the outset, and its history (briefly recounted in [11], [7]) has been characterised by attempts to provide a precise mathematical statement accompanied by rigorous proof. Early contributors include Boussinesq, Dougall, Southwell and Goodier, the last two using the potential energy to measure the spatial decay of edge effects to justify the principle. Counter-examples are provided by Toupin [17] (twisted thin walled I-beam), and, of direct relevance to the present study, by Hoff [6] for a body of narrowing central cross-sections loaded by equilibrated surface forces distributed over opposite almost contiguous surfaces. Zanaboni [18, 19, 20], although aware of these contributions, separately developed an alternative approach that concerns a bounded elastic body $\Omega_{(1)}$ in equilibrium subject to zero body force and a given system of self-equilibrated loads P_i, $i = 1, 2, 3$, distributed over a part Γ of the otherwise free smooth surface $\partial \Omega_{(1)}$. His version of Saint-Venant's principle states that the strain energy contained in the sub-region remote from the load surface diminishes to zero with increasing distance of the sub-region from the load surface. (See Fig. 1.)
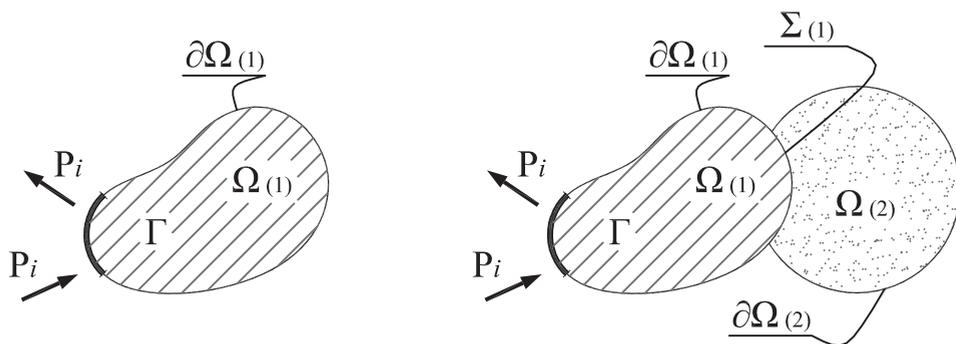
Figure 1. (a) Isolated body $\Omega_{(1)}$ (b) Bonded bodies $\Omega_{(1)}$ and $\Omega_{(2)}$.

Note that a self-equilibrated system of forces P_i on the surface Γ satisfies the conditions

$$(1.1) \qquad \int_\Gamma P_i \, dS = \int_\Gamma (x_i P_j - x_j P_i) \, dS = 0,$$

where x_i, $i = 1, 2, 3$ are orthogonal Cartesian coordinates. Let $u_i^{(1)}(x)$, to within a rigid body displacement, be the Cartesian components of the small displacement produced in $\Omega_{(1)}$ by the loads P_i, and define components of the corresponding linear strain to be

$$(1.2) \qquad e_{ij}^{(1)} = \frac{1}{2}(u_{i,j}^{(1)} + u_{j,i}^{(1)}),$$

where here, and throughout, the standard comma notation for partial differentiation is employed, along with the usual convention of summation over repeated subscripts.

Later sections consider examples in which the distributed load system is specialised to various point forces. Meanwhile, we extend the formulation of Zanaboni's analysis to include linear anisotropic nonhomogeneous elasticity for which the cartesian components of stress $\sigma_{ij}^{(1)}$ satisfy the constitutive relations

$$(1.3) \qquad \sigma_{ij}^{(1)} = c_{ijkl} e_{ij}^{(1)}.$$

The nonhomogeneous elastic moduli $c_{ijkl}(x)$ possess the symmetries

$$(1.4) \qquad c_{ijkl} = c_{klij} = c_{jikl},$$

and are positive-definite in the sense that

$$(1.5) \qquad c_0 \psi_{ij} \psi_{ij} \le c_{ijkl} \psi_{ij} \psi_{kl} \le c_1 \psi_{ij} \psi_{ij}, \quad \forall \psi_{ij} = \psi_{ji},$$

where c_0, c_1 are positive constants.

Under these assumptions, the boundary value problem to be treated is specified by

$$(1.6) \qquad \sigma^{(1)}_{ij,j} = 0, \quad x \in \Omega_{(1)},$$

$$(1.7) \qquad \sigma^{(1)}_{ij} n_j = P_i, \quad x \in \Gamma,$$

$$(1.8) \qquad \sigma^{(1)}_{ij} n_j = 0, \quad x \in \partial\Omega_{(1)} \backslash \Gamma,$$

where n_i are the Cartesian components of the unit outward normal on $\partial\Omega_{(1)}$.

Zanaboni considers a second bounded elastic body $\Omega_{(2)}$ of the same composition that is bonded to $\Omega_{(1)}$ along a part Σ_1 of the surface $\partial\Omega_{(1)} \backslash \Gamma$ so that there is continuity of displacement and traction across $\Sigma_1 = \partial\Omega_{(1)} \cap \partial\Omega_{(2)}$. The enlarged body $\Omega^{(2)} = \Omega_{(1)} \cup \Omega_{(2)}$ is in equilibrium subject to the same self-equilibrated system distributed over Γ now regarded as belonging to $\partial\Omega^{(2)}$. Accordingly, the stress $\sigma^{(2)}_{ij}$ in $\Omega^{(2)}$ is related to the strain $e^{(2)}_{ij}$ by

$$(1.9) \qquad \sigma^{(2)}_{ij} = c_{ijkl} e^{(2)}_{ij}, \quad x \in \Omega^{(2)},$$

while the appropriate boundary value problem becomes

$$(1.10) \qquad \sigma^{(2)}_{ij,j} = 0, \quad x \in \Omega^{(2)} = \Omega_{(2)} \cup \Omega_{(1)},$$

$$(1.11) \qquad \sigma^{(2)}_{ij} n_j = P_i, \quad x \in \Gamma,$$

$$(1.12) \qquad \sigma^{(2)}_{ij} n_j = 0, \quad x \in \partial\Omega^{(2)} \backslash \Gamma.$$

The solutions, unique to within a rigid body displacement, to the respective problems (1.6)–(1.8), and (1.10)–(1.12) enable various strain energies to be determined. Thus, the strain energy stored in $\Omega_{(1)}$, considered as isolated, is given by

$$(1.13) \qquad V_{\Omega_{(1)}}(u^{(1)}) = \frac{1}{2} \int_{\Omega_{(1)}} \sigma^{(1)}_{ij} e^{(1)}_{ij} \, dx;$$

while the strain energy stored in the enlarged body $\Omega^{(2)}$ is:

$$(1.14) \qquad V_{\Omega^{(2)}}(u^{(2)}) = \frac{1}{2} \int_{\Omega^{(2)}} \sigma^{(2)}_{ij} e^{(2)}_{ij} \, dx;$$

and the strain energy stored in the added volume $\Omega_{(2)}$ is:

$$(1.15) \qquad V_{\Omega_{(2)}}(u^{(2)}) = \frac{1}{2} \int_{\Omega_{(2)}} \sigma^{(2)}_{ij} e^{(2)}_{ij} \, dx.$$

Mainly heuristic arguments are employed by Zanaboni to relate these quantities by the inequality

$$(1.16) \qquad V_{\Omega_{(2)}}(u^{(2)}) \le V_{\Omega_{(1)}}(u^{(1)}) - V_{\Omega^{(2)}}(u^{(2)}).$$

Inequality (1.16) can be extended by considering successive accretions that create a sequence of enlarged bodies, $\Omega^{(1)}(=\Omega_{(1)})$, $\Omega^{(n)}$, $n = 2, 3, \ldots p$, each loaded by the same self-equilibrated system of forces P_i distributed over the common boundary Γ. In an obvious notation, repeated application of (1.16) yields the general bound

$$(1.17) \qquad V_{\Omega_{(n+1)}}(u^{(n+1)}) \le V_{\Omega^{(n)}}(u^{(n)}) - V_{\Omega^{(n+1)}}(u^{(n+1)}), \quad n = 1, \ldots,$$

where $\Omega_{(n+1)} = \Omega^{(n+1)} \backslash \Omega^{(n)}$. A new simplified mathematical proof of (1.17), based solely on the positive-definiteness assumption (1.5), has been constructed by the authors [9].

Observe that inequality (1.17) implies that $V_{\Omega^{(n)}}(u^{(n)})$, $n = 1, 2, \ldots$, form a decreasing sequence which by (1.5) is bounded below and therefore convergent by Cauchy's theorem. Consequently, $V_{\Omega_{(p)}}(u^{(p)})$ decreases to zero with increasing p and Saint-Venant's principle is confirmed.

It must be emphasised that the full inequality (1.17) is crucial for Saint-Venant's principle. The property that the respective energies form a decreasing bounded sequence, equivalent to the well-known result that increasing the size of a loaded elastic body decreasess the strain energy. (cp., [15, p. 103]), is a necessary but not sufficient condition for the proof since no information is provided *per se* regarding the distribution of strain energy between component sub-regions.

Inequality (1.17) may be extended to bounded regions whose surfaces have a single point in common that is subjected to the same force singularity, and, in particular, to a force dipole of interest here. A further generalisation includes unbounded regions provided that the displacement, rotation, and stress vanish in the neigbourhood of infinity.

A version of Saint-Venant's principle dual to Zanaboni's formulation was proposed by Aymerich [1]. The system of loads P_i is replaced by a given distribution of displacements and similar conclusions are established. Books that include accounts of Zanaboni's approach are [2], [4], and [16], while his work is referenced in the surveys [11], [5], and [7]. The classic paper [17] also discusses Zanaboni's contributions, but subsequent to these often brief descriptions, there is scant mention in the literature of Zanaboni's method which appears to have become overshadowed by techniques based upon differential inequalities pioneered by Toupin, Payne, and others. The advantage of such methods is the ability to estimate decay rates of the energy not immediately accessible by Zanaboni's arguments even for cylindrical bodies.

This paper seeks to partially remedy this deficiency by examining solutions to selected plane elastic problems. The strain energy is compared between bodies of increasing size but loaded by the same traction on the same part of the common boundary. Some problems involve a series of bodies touching at a common point which is subjected to a force dipole. The main conclusion indicates that the decay rate of the energy in regions remote from the load region strongly depends upon the particular geometries of the regions under study. We construct, however, two examples that contravene Saint-Venant's principle.

2. Plane isotropic elasticity

Known exact explicit solutions to boundary value problems in two- and three-dimensional linear elasticity are not numerous, and furthermore, few are suitable to illustrate the various formulations of Saint-Venant's principle mentioned in the literature. Other difficulties are created when the load is concentrated in the form of point forces or couples for which the exact solution possesses a displacement, rotation, and stress that become singular. This feature must be taken into account when deriving the external work and strain energy whose spatial behaviour determines decay rates appropriate to Zanaboni's formulation of Saint-Venant's principle.

We analyse several examples from plane isotropic elasticity and for definiteness restrict consideration to problems in plane stress. Several methods of solution are available, but we select the complex variable approach for its conciseness.

Let x, y be rectangular Cartesian coordinates, and introduce the complex variable $z = x + iy$ and its conjugate $\bar{z} = x - iy$. In terms of the notation adopted in [12, §1.22], the scalar functions Θ and Φ represent the fundamental stress combinations expressed as

$$(2.1) \qquad \Theta = \sigma_{xx} + \sigma_{yy}, \quad \Phi = \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy},$$

where σ_{xx}, σ_{yy}, σ_{xy} are the symmetric plane stress components.

When the elastic material is homogoneous and isotropic, the functions Θ, Φ can be expressed in terms of two sectionally holomorphic functions $W(z)$, $w(z)$ such that ([12, §20]:

$$(2.2) \qquad \Theta = W(z) + \overline{W}(\bar{z}), \quad \Phi = \bar{z}W'(z) + w(z),$$

where a superposed prime indicates differentiation with respect to the indicated argument, and a superposed bar denotes the complex conjugate. It may be shown that the Cartesian components u, v of the displacement are determined from the relation

$$(2.3) \qquad 4\mu(u + iv) = \kappa W^*(z) - z\overline{W}(\bar{z}) - \overline{w}^*(\bar{z}),$$

where a superposed asterisk denotes indefinite integration with respect to the argument of the function, μ is the shear modulus, $\kappa = (3 - 4\nu)$ is the Kosolov constant, and ν is Poisson's ratio. Indefinite integration introduces an arbitrary rigid body displacement neglected for the purposes of our analysis.

3. Explicit solutions

3.1. First example

The first example concerns an elastic circular disc of radius a loaded by a force dipole at a point on its circumference. (See Figure 2.) The dipole, alternatively
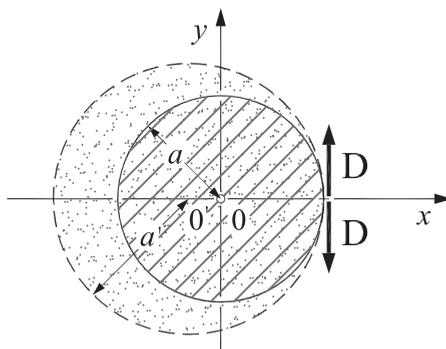
Figure 2. Disc loaded by a dipole.

termed a "nucleus of strain" (cf., [10, §151], is defined as the limit when $\varepsilon \to 0$ of two equal and opposite forces, say $\pm iY$, applied at the circumferential points $z = ae^{i\varepsilon}$ and $z = a$. In the limit, Y tends to infinity in such a way that the product $Ya\varepsilon$ remains finite at the constant value D. For this problem, the function $W(z)$ is known to be (see, [12, §5.27])

$$(3.1) \qquad\qquad W(z) = \frac{D}{\pi(z-a)^2} + A,$$

where A is a complex constant. The second stress function $w(z)$, derived from the expression (see [12, §5.21])

$$(3.2) \qquad\qquad w(z) = \frac{a^2}{z^2}\left(W(z) + \overline{W}\left(\frac{a^2}{z^2}\right) - zW'(z)\right),$$

consequently becomes

$$(3.3) \qquad w(z) = \frac{a^2}{z^2}\left[\frac{D}{\pi(z-a)^2} + A + \frac{Dz^2}{\pi a^2(z-a)^2} + \bar{A} + \frac{2Dz}{\pi(z-a)^3}\right].$$

Because this function is supposed holomorphic inside the disc, we may eliminate singular terms by setting

$$(3.4) \qquad\qquad A = -\frac{D}{2\pi a^2}.$$

After substitution of (3.4) in (3.3), we obtain the simple expression

$$(3.5) \qquad\qquad w(z) = \frac{2Da}{\pi(z-a)^3}.$$

Furthermore, on insertion of the expressions (3.5) and (3.1) into (2.3), we conclude after appropriate integration that the displacement is given by

$$(3.6) \qquad 4\mu(u + iv) = -\kappa \left(\frac{D}{\pi(z - a)} + \frac{Dz}{z\pi a^2} \right) - z \left(\frac{D}{\pi(\bar{z} - a)^2} - \frac{D}{2\pi a^2} \right)$$

$$+ \frac{Da}{\pi(\bar{z} - a)^2} + 4\mu(\alpha + i\beta + i\gamma z),$$

where the last terms on the right represent an arbitrary rigid body displacement and are subsequently ignored. As z approaches a, the non-singular lower order terms in (3.6) likewise may be discarded to yield

$$(3.7) \qquad 4\mu(u + iv) \cong -\kappa \frac{D}{\pi(z - a)} - z \frac{D}{\pi(\bar{z} - a)^2} + \frac{Da}{\pi(\bar{z} - a)^2}.$$

We conclude that the displacement at a point on the circumference close to the dipole, say $z = ae^{i\varepsilon} \cong a(1 + i\varepsilon)$, is given by

$$(3.8) \qquad 4\mu(u + iv) \cong -\frac{\kappa D}{\pi a(e^{i\varepsilon} - 1)} - \frac{e^{i\varepsilon}D}{\pi a(e^{-i\varepsilon} - 1)} + \frac{D}{\pi a(e^{-i\varepsilon} - 1)^2}$$

$$= \frac{i\kappa D}{\pi a \varepsilon} + \frac{D(1 - e^{i\varepsilon})}{\pi a(1 - e^{-i\varepsilon})^2}$$

$$= \frac{i(1 + \kappa)D}{\pi a \varepsilon},$$

to first order in ε^{-1}. The work done by the dipole is the limit

$$(3.9) \qquad L(a) = \lim_{\varepsilon \to 0} Dv$$

$$= \frac{D^2(1 + \kappa)}{4\mu\pi a} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon},$$

which demonstrates the otherwise predictable result that the work done (and hence the strain energy) becomes infinite. Nevertheless, some useful information may be extracted from the expression (3.9). Consider a second circular disc of radius $a_1 > a$, which contains the first disc and touches it at the point $z = a$ where it is subject to the same dipole D. The corresponding work done $L(a_1)$ is again given by the limit (3.9) but with a replaced by a_1. We immediately establish that the ratio

$$(3.10) \qquad \frac{L(a_1)}{L(a)} = \frac{a}{a_1} < 1,$$

is finite, and that $L(a_1) < L(a)$. The last result is predicted by the right side of Zanaboni's inequality (1.17) showing that the strain energy decreases due to additional material. As already remarked, however, this conclusion of itself is insufficient to establish Saint-Venant's principle, the chief element of which asserts that the strain energy or similar measure decays to zero in regions increasingly remote from the load surface.

To obtain the decay envisaged in Saint-Venant's principle, we successively use (1.17). Suppose a series of circular discs of radii $a_n > a_{n-1} > \cdots > a_1 > a$ enclose each other and touch at the point $z = a$ where a common dipole D is applied. Upon representing the respective strain energies of each disc by $L(a_p)$, $p = 1, 2, \ldots n$, Zanaboni's inequality (1.17), in an obvious notation, may be written as

$$(3.11) \qquad V_{\Omega_{(n+1)}}(u^{(n+1)}) \le L(a_n) - L(a_{n+1})$$

$$= L(a_n)\left(1 - \frac{L(a_{n+1})}{L(a_n)}\right)$$

$$= L(a_n)\left(1 - \frac{a_n}{a_{n+1}}\right)$$

$$(3.12) \qquad \le \frac{a}{a_n}L(a)\left(1 - \frac{a_n}{a_{n+1}}\right)$$

$$(3.13) \qquad = \frac{D^2(1+\kappa)}{4\mu\pi\varepsilon}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right),$$

from which $V_{\Omega_{(n+1)}}(u^{(n+1)}) \to 0$ as $a_n \to \infty$, in confirmation of Saint-Venant's principle. To be explicit, we set, for example,

$$(3.14) \qquad a_{n+1} - a_n = n\varepsilon = k,$$

for positive constant k, and when $a_n = O(n)$ obtain from (3.13) the estimate

$$(3.15) \qquad V_{\Omega_{(n+1)}}(u^{(n+1)}) \le \frac{D^2(1+\kappa)}{4\mu\pi}\frac{1}{(a_n + k)},$$

demonstrating a decay rate that is at most linear.

An alternative bound for the decay rate, obtained from (3.12), is given by

$$(3.16) \qquad V_{\Omega_{(n+1)}}(u^{(n+1)}) \le \frac{aL(a)}{a_n},$$

which also is at most linear. Note that in the limit $a_n \to \infty$, the circular disc tends to the half-plane subjected to a dipole at a point on the otherwise traction free straight boundary. We conclude that the strain energy vanishes in the neighbourhood of infinity to the order at most $o(a_n)$.

Inequality (1.17) provides further information. Suppose $\Omega^{(n+1)}$ denotes a sequence of plane regions with smooth boundary such that $\Omega^{(n+1)}$ is contained

between the discs $C^{(n)}$, $C^{(n+1)}$ respectively of radii a_n and a_{n+1} and touches these discs at the common contact point $z = a$. As before, we suppose a dipole D is separately applied at $z = a$ to $\Omega_{(n+1)}$ and the discs. On recalling $\Omega_{(n+2)} = \Omega^{(n+2)} \backslash \Omega^{(n+1)}$ and other previous notation, we obtain from (1.17) the bound

$$(3.17) \qquad V_{\Omega_{(n+2)}}(u^{(n+2)}) \leq V_{\Omega^{(n+1)}}(u^{(n+1)}) - V_{\Omega^{(n+2)}}(u^{(n+2)})$$
$$\leq L(a_n) - L(a_{n+2}),$$

which is of the same form as (3.11) and similar conclusions may be deduced.

3.2. Second example

As a second example, we examine a similar problem but for a nested sequence of regions exterior to a family of parabolas with common vertex $z = x_0$ to which is applied a dipole of magnitude D. (See Figure 3). With respect to the origin O of coordinates, let us introduce the conformal mapping (cp., [14, §4.2.5])

$$(3.18) \qquad z = m(\zeta) = \zeta^2 = (\xi + i\eta)^2,$$

that transforms the point (ξ, η) of the ζ-plane into the point (x, y) of the z-plane according to the transformation

$$(3.19) \qquad x = \xi^2 - \eta^2, \quad y = 2\xi\eta.$$

Lines $\xi = \xi_0$, a constant, in the ζ-plane are transformed into the parabola

$$(3.20) \qquad x = \xi_0^2 - \eta^2, \quad y = 2\xi_0\eta,$$

where η is a parameter. We take $\xi = \xi_0$ to be the internal parabolic boundary of the exterior region. The focus of the parabola is at the origin, and the vertex in the z-plane has coordinates $(\xi_0^2, 0)$.
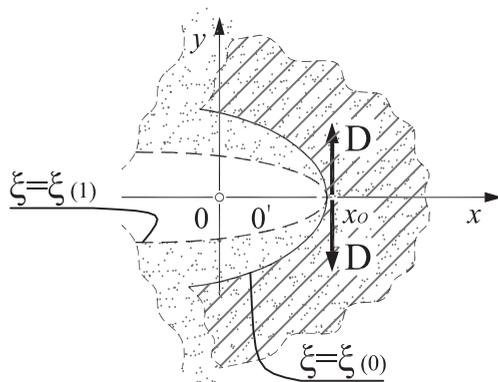

Figure 3. Exteriors of parabola with common vertex and focii at 0 and 0′.

We assume the rotation and stress vanish in the neighbourhood of infinity. The appropriate forms ([14, §4.25]) for the complex potentials with respect to the variables in the ζ-plane are given firstly by

$$(3.21) \qquad\qquad m'(\zeta)W(\zeta) = 2\zeta W(\zeta)$$

$$= \frac{D}{\pi} \frac{1}{(\zeta - \xi_0)}.$$

Note that the parabola is given by $\zeta + \bar{\zeta} = 2\xi_0$, and so by the principle of reflexion, the second complex potential is related to the first by

$$m'(\zeta)w(\zeta) = 2\zeta w(\zeta)$$
$$= \bar{m}'(2\xi_0 - \zeta)\big[W(\zeta) + \overline{W}(2\xi_0 - \zeta)\big] - \bar{m}(2\xi_0 - \zeta)W'(\zeta)$$
$$= \frac{D}{\pi}\left[\frac{2\xi_0}{\zeta(\zeta - \xi_0)^2} + (2\xi_0 - \zeta)^2 \frac{\{(\zeta - \xi_0)^2 + 2\zeta(\zeta - \xi_0)\}}{2\zeta^2(\zeta - \xi_0)^2}\right],$$

after we have substituted from (3.21).

The representation (2.3) of the displacement is modified to the expresssion

$$(3.22) \qquad\qquad 4\mu(u + iv) = \kappa W^*(\zeta) - m(\zeta)\overline{W}(\bar{\zeta}) - w^*(\bar{\zeta}),$$

where

$$(3.23) \quad W^*(\zeta) = \int_0^\zeta W(z)\,dz$$

$$= \int_0^\zeta W(\zeta)m'(\zeta)\,d\zeta$$

$$= -\frac{D}{\pi}\frac{1}{(\zeta - \xi_0)},$$

$$(3.24) \quad w^*(\zeta) = \int_0^\zeta w(z)\,dz$$

$$= \int_0^\zeta w(\zeta)m'(\zeta)\,d\zeta$$

$$= -\frac{D}{\pi}\left[\left(\frac{11}{2} - \frac{2\xi_0}{3}\right)\frac{1}{(\zeta - \xi_0)} + \left(\frac{9}{2}\xi_0 - 4\zeta\right)\frac{1}{(\zeta - \xi_0)^2}\right].$$

To determine the work done by the dipole, we first obtain the component v of the displacement from (3.22), (3.23), and (3.24). On setting $\zeta = \xi_0(1 + i\varepsilon)$, where $\varepsilon \ll 1$, and on neglecting first and higher order powers of ε, we obtain

$$(3.25) \qquad\qquad 4\mu v \cong \frac{D}{\varepsilon\pi\xi_0}(\kappa + 7/2).$$

Consequently, the work done by the dipole is

$$(3.26) \qquad L(\xi_0) = \lim_{\varepsilon \to 0} \frac{D}{4\mu\pi\varepsilon\xi_0}(\kappa + 7/2).$$

Now consider the parabolas generated by $\xi = \xi_n$, $n = 1, 2 \ldots$ and successively displace the origin to the focii of the respective parabolas whose vertices are translated to coincide with that for the parabola $\xi = \xi_0$. The parabola ξ_n bounds an exterior region containing regions exterior to the parabolas ξ_m, $m = 0, \ldots n - 1$. The same dipole D is applied at the common vertex to all exterior regions, and we assume that the strain energy is finite in all regions thus constructed. Accordingly, the work done by the dipole on the region bounded by the curve $\xi = \xi_n (\leq \xi_0)$ is the expression (3.26) but with ξ_0 replaced by ξ_n, where ξ_n^2 is the distance between the common vertex and the displaced origin O_n. Note that as $\xi_n \to 0$ for $n \to \infty$, the boundary becomes a straight line and the region is the whole plane with the negative x-axis deleted. Upon taking the ratios of the work done by the dipole on the respective regions, we have

$$(3.27) \qquad \frac{L(\xi_0)}{L(\xi_n)} = \frac{\xi_n}{\xi_0} < 1.$$

But *decreasing ξ_n increases* the area of the exterior region and we conclude from (3.27) that additional material *increases* the strain energy in the ratio $\xi_n/\xi_0 = \sqrt{x_n/x_0}$, where x_0, x_n are the distances of the foci from the common vertex. The conclusion is perhaps counter-intuitive until it is realised that the operation of increasing ξ_n enlarges the part of the exterior region in the vicinity of the dipole singularity. As a consequence, the magnitude of the stress and strain components and therefore the strain energy density in such parts is large, which contributes to the increasing total strain energy in the region as $\xi_n \to 0$. The example contradicts inequality (1.17) and therefore contravenes Zanaboni's version of Saint-Venant's principle.

4. Internal boundary

Zanaboni also considered enlargements that possess a commom internal closed load boundary Γ, such that each body in the sequence contains its predeccessors; for example, a sequence of rings with increasing diameters that are loaded on the common internal boundary. The problem is discussed by Aymerich [1]. Here, we present another, less elementary, example to help further elucidate Zanaboni's version of Saint-Venant's principle.

We consider the whole x, y-plane pierced by a circular hole of radius a, centered at the origin, and loaded at the point $z = a$ by a dipole of magnitude D parallel to the y-axis. (See Figure 4.) The displacement, rotation, and stress are assumed to vanish in the neighbourhood of infinity.
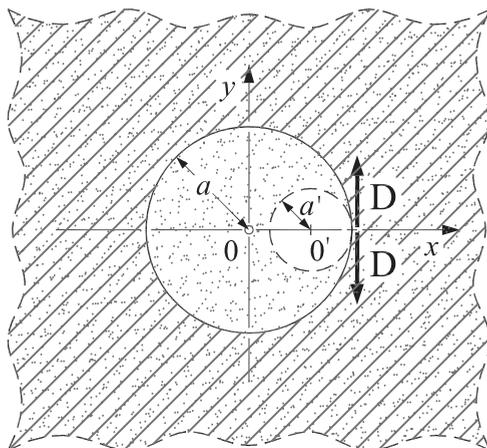
Figure 4. Exteriors to holes of radii a and a_1.

Similar arguments leading to (3.1) yield the representations

$$(4.1) \qquad W(z) = \frac{D}{\pi} \frac{1}{(a-z)^2},$$

$$(4.2) \qquad w(z) = \frac{a^2 D}{\pi z^2} \left[\frac{(a^2 + z^2)}{a^2 (a-z)^2} - \frac{2z}{(a-z)^3} \right],$$

which on integration give

$$(4.3) \qquad W^*(z) = \frac{D}{\pi} \frac{1}{(a-z)},$$

$$(4.4) \qquad w^*(z) = \frac{D}{\pi} \left[-\frac{a^2}{z(a-z)^2} + \frac{1}{(a-z)} \right].$$

We ignore arbitrary rigid body displacements, and apply (2.3) to obtain

$$(4.5) \qquad 4\mu(u + iv) = \frac{D}{\pi} \left[\frac{\kappa}{(a-z)} - \frac{1}{(a-\bar{z})} + \frac{(a^2 - z\bar{z})}{\bar{z}(a-\bar{z})^2} \right].$$

On setting $z = a \exp i\varepsilon$, or $z \cong a(1 + i\varepsilon)$ when $\varepsilon \ll 1$, the last expression reduces to

$$(4.6) \qquad 4\mu(u + iv) \cong \frac{iD}{\pi} \frac{(\kappa + 1)}{a\varepsilon},$$

and the work done by the dipole becomes

$$(4.7) \qquad L(a) = \lim_{\varepsilon \to 0} Dv = \frac{D^2(\kappa + 1)}{4\mu\pi a} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}.$$

Now consider a hole of smaller radius $a_1 < a$ which touches the first at the point $z = a$, and which is loaded by the same dipole D at $z = a$. To determine the work done, $L(a_1)$, we replace a by a_1 in the expression (4.7) to give the ratio

$$(4.8) \qquad \frac{L(a)}{L(a_1)} = \frac{a_1}{a} < 1.$$

Recall that the region exterior to the hole of radius a_1 is *larger* than that exterior to the hole of radius a, so that $a_n \ll 1$ enlarges the parts of the exterior region in the neighbourhood of the dipole. Thus, we have circumstances similar to those encountered in Section 3.2 and accordingly, (4.8) provides a second example for which additional material *increases* the work done. Condition (1.17) is contradicted and Saint-Venant's principle according to Zananboni again is contravened.

5. MULTIPLY-CONNECTED LOAD BOUNDARY

The formulation of Saint-Venant's principle due to Zanaboni admits several interesting refinements, prominent among which is the important case of a multi-connected load surface or boundary. The loads on the component surfaces may not be self-equilibrated, although, of course, the total load taken over all these components is in overall equilibrium. This broad category of problems, to which belongs the Hoff counter-example [6], includes the cylinder loaded at each end. These problems have been studied by Fichera [3], who extended the differential inequality technique introduced by Toupin [17]. In [8], the authors discussed how Zanaboni's treatment may be adapted to general problems in this category.

We describe two examples of this type for the half-plane, both of which conform to the principle.

Consider the half-plane $y \geq 0$ loaded by dipoles of magnitude D concentrated at points $z = \pm a$ (Figure 5), and with rotation and stress vanishing in the neighbourhood of infinity.
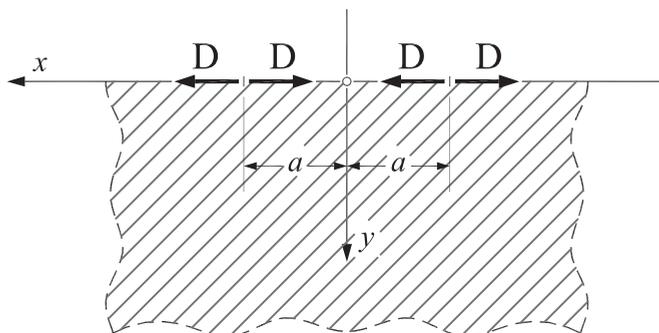

Figure 5. Two equal symmetrically placed dipoles.

The load boundary Γ consists of separate points $z = \pm a$, to each of which is applied a self-equilibrated force dipole. The corresponding stress functions are ([13, p. 386]

$$(5.1) \qquad W(z) = -\frac{D}{\pi}\left[\frac{1}{(z-a)^2} + \frac{1}{(z+a)^2}\right],$$

$$(5.2) \qquad w(z) = -\frac{2aD}{\pi}\left[\frac{1}{(z-a)^3} - \frac{1}{(z+a)^3}\right],$$

and on substitution in (2.3) we obtain for the displacement

$$(5.3) \qquad 4\mu(u+iv) = \frac{D}{\pi}\left[\frac{2\kappa z}{(z^2-a^2)} + \frac{2(\bar{z}(z\bar{z}-a^2)+a^2(z-\bar{z}))}{(\bar{z}^2-a^2)^2}\right].$$

Along the x-axis, for which $z = \bar{z} = t$, this expresssion simplifies to

$$(5.4) \qquad 4\mu(u+iv) = \frac{D}{\pi}\frac{2(\kappa+1)t}{(t^2-a^2)},$$

so that at $t = a(1+\varepsilon)$ and $t = -a(1-\varepsilon)$ we have respectively

$$(5.5) \qquad u(a(1+\varepsilon),0) = \frac{D(\kappa+1)}{2a\mu\pi\varepsilon}\left[\frac{(1+\varepsilon)}{(2+\varepsilon)}\right],$$

$$(5.6) \qquad u(-a(1-\varepsilon),0) = \frac{D(\kappa+1)}{2a\mu\pi\varepsilon}\left[\frac{(1-\varepsilon)}{(2-\varepsilon)}\right],$$

and consequently the total work done to first order by the symmetrically disposed dipoles when $\varepsilon \ll 1$ is

$$(5.7) \qquad L(a) = \frac{(\kappa+1)D^2}{\mu\pi a\varepsilon}.$$

The half-plane is now enlarged by insertion of a semi-infinite strip of the same material between the dipoles and paralled to the y-axis. This is equivalent to increasing the distance separating the dipoles to $2a_1 > 2a$, with the strip's width being $2(a_1 - a)$. The work done is now $L(a_1)$, where the function L is given by (5.7). Accordingly, we have

$$(5.8) \qquad \frac{L(a)}{L(a_1)} = \frac{a_1}{a} > 1,$$

and we conclude that for this example, additional material decreases the strain energy consistent with (1.17) and Saint-Venant's principle. The discussion may be completed as for the example considered in Section 3.1.

The second problem again concerns the half-plane just described, but now loaded by two equal and opposite point forces F at the points $z = \pm a$ and directed along the positive x-axis. The loading is not self-equilibrated on each disjoint load boundary point, but is in aggregate. Appropriate stress functions are (cp., [12, §2.21, §4.22])

$$(5.9) \qquad W(z) = -\frac{F}{\pi}\left[\frac{1}{(z-a)} - \frac{1}{(z+a)}\right],$$

$$(5.10) \qquad w(z) = \frac{F}{\pi}\left[\frac{1}{(z-a)} - \frac{1}{(z+a)} - \frac{a}{(z-a)^2} + \frac{a}{(z+a)^2}\right],$$

and by (2.3), after appropriate integration, the displacement becomes

$$(5.11) \qquad 4\mu(u+iv) = -\frac{F}{\pi}\left[\kappa \ln\left(\frac{z-a}{z+a}\right) + \ln\left(\frac{\bar{z}-a}{\bar{z}+a}\right) - 2a\frac{(z-a)}{(\bar{z}^2-a^2)}\right].$$

On the plane boundary, we have $z = \bar{z} = t$, say, and (5.11) reduces to the expression

$$(5.12) \qquad 4\mu(u+iv)|_{z=\bar{z}=t} = -\frac{F}{\pi}\left[(1+\kappa)\ln\left(\frac{t-a}{t+a}\right) - 2a\frac{1}{(t+a)}\right],$$

which yields the following displacements at the points $z = \pm a(1+\varepsilon)$:

$$(5.13) \qquad 4\mu(u+iv)|_{z=a(1+\varepsilon)} = -(1+\kappa)\frac{F}{\pi}\ln\left(\frac{\varepsilon}{(2+\varepsilon)}\right) + \frac{2F}{\pi(2+\varepsilon)},$$

$$(5.14) \qquad 4\mu(u+iv)|_{z=-a(1+\varepsilon)} = (1+\kappa)\frac{F}{\pi}\ln\left(\frac{\varepsilon}{(2+\varepsilon)}\right) - \frac{2F}{\pi\varepsilon}.$$

The total work done by the forces $\pm F$ at $z = \pm a(1+\varepsilon)$, respectively, when $\varepsilon \ll 1$ is therefore to first order

$$(5.15) \qquad L(a) = \frac{F^2}{\mu\pi}\left[\frac{1}{\varepsilon} - \frac{(1+\kappa)}{2}\ln\varepsilon\right],$$

and by inspection is independent of the distance a. As a consequence, addition of material, or equivalently, the replacement of a by $a_1 > a$ in (5.15) does not affect the ratio $L(a)/L(a_1)$.

Nevertheless, this observation does not invalidate Saint-Venant's principle according to Zanaboni's formulation. We recall that this asserts that the strain energy in regions increasingly remote from the load region tends to zero. The fact that the strain energy is independent of the distance a provides no informa-

tion on the distribution of the strain energy in various sub-regions of the half-plane under consideration.

We can verify Zanaboni's conclusion on selecting the constituent regions to be strips parallel to the plane boundary of the half-space. Let $\Omega_{(n)}$ be the strip specified by

$$(5.16) \qquad \Omega_{(n)} = \{(x, y) \in \Omega : h_{n-1} \leq y \leq h_n\}, \quad n = 0, 1, \dots$$

We let $h_0 = 0$, $h_1 = 1$, and suppose the first strip $\Omega_{(1)}$ to be subject to the loads $\pm F$ at $z = \pm a$ on the plane boundary $y = 0$, the boundary $y = 1$, to be traction free, and let the stress asymptotically vanish as $x \to \pm\infty$. The corresponding displacement is denoted by $u^{(1)}$ in conformity with previous notation, The strain energy in $\Omega_{(1)}$ may be written in the form

$$(5.17) \qquad V_{\Omega_{(1)}}(u^{(1)}) = 1/2 \left[\int_{\Omega_{(1)}} \{(\lambda + 2\mu)(u_{1,1}^{(1)})^2 + 4\mu(u_{2,1}^{(1)})^2\} \, dx \right.$$

$$+ \int_{\Omega_{(1)}} \{8\mu u_{1,2}^{(1)} u_{2,1}^{(1)} + 2\lambda u_{1,1}^{(1)} u_{2,2}^{(1)}\} \, dx$$

$$\left. + \int_{\Omega_{(1)}} \{(\lambda + 2\mu)(u_{2,2}^{(1)})^2 + 4\mu(u_{1,2}^{(1)})^2\} \, dx \right]$$

$$\equiv V_{11} + V_{12} + V_{13}.$$

We also let $\Omega^{(n)} = \Omega^{(n-1)} \cup \Omega_{(n)}$, which is of width h_n, be subject to the loads $\pm F$ at $z = \pm a$ on $y = 0$, have the boundary $y = h_n$ traction-free, and the stress asymptotically vanishing as $x \to \pm\infty$. We rescale the coordinates such that $x \to x$ and $y \to h_n y$, upon which it follows that the strain energy in $\Omega^{(n)}$ is

$$(5.18) \qquad V_{\Omega^{(n)}}(u^{(n)}) = h_n V_{11} + V_{12} + h_n^{-1} V_{13}.$$

The solution for the half-plane is obtained in the limit $h_n \to \infty$. But we have just shown that the strain energy in the half-plane is bounded, and so we must have not only that both V_{12}, V_{13} are bounded, but also that $V_{11} = 0$. By virtue of (1.17), we then have in the previous notation

$$(5.19) \qquad V_{\Omega_{(n)}}(u^{(n)}) \leq V_{\Omega^{(n-1)}}(u^{(n-1)}) - V_{\Omega^{(n)}}(u^n)$$

$$(5.20) \qquad\qquad\qquad = V_{13}(h_{n-1}^{-1} - h_n^{-1})$$

$$(5.21) \qquad\qquad\qquad = V_{13} \frac{\eta}{h_{n-1}(h_{n-1} + \eta)}$$

upon supposing that $h_n = h_{n-1} + \eta$, where η is a finite positive, possibly small, constant. We conclude that the strain energy in the strip $\Omega_{(n)}$ vanishes as h_{n-1}^{-2}

in accordance with Saint-Venant's principle. A sharper decay rate of h_{n-1}^{-1} is provided by (5.20).

References

[1] G. AYMERICH, *Una proprietá dell'energia elastica*, Boll. UMI, III, (1955) 10, 332–336.

[2] C. B. BIEZENO - R. GRAMMEL, *Technische Dynamik*, Springer (1939). English Edition: *Engineering Dynamics*, Blackie Glasgow (1955).

[3] G. FICHERA, *Il principio di Saint-Venant: intuizione dell'ingegnere e rigore del matematico*, Rend. Mat. Ser. VI (1977) 10, 1–24.

[4] Y. C. FUNG, *Foundations of Solid Mechanics*, Prentice-Hall. New Jersey (1965).

[5] M. E. GURTIN, *The Linear Theory of Elasticity*, in: Handbuch der Physik (eds.: S. Flügge and C. Truesdell.) VIa/2, 1–295. Springer-Verlag. Heidelberg. Berlin (1972).

[6] N. J. HOFF, *The applicability of Saint-Venant's principle to airplane structures*, J. Aero. Sciences (1945) 12, 455–460.

[7] C. O. HORGAN - J. K. KNOWLES, *Recent developments concerning Saint-Venant's Principle*, Advances in Applied Mechanics (eds.: J. W. Hutchinson and T. Y. Wu) 23, 179–269. Academic Press. New York (1983).

[8] R. J. KNOPS - P. VILLAGGIO, *An alternative approach to Saint-Venant's principle*, STAMM 2008. Levico. (2008).

[9] R. J. KNOPS - P. VILLAGGIO, *Zanaboni's treatment of Saint-Venant's principle* (forthcoming).

[10] A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, 4th Edition. Cambridge University Press. Cambridge (1927).

[11] O. MAISSONEUVE, *Sur le principe de Saint-Venant*, Thésis presentées à l'Université de Poitiers (1971).

[12] L. M. MILNE-THOMSON, *Plane Elastic Systems*, Springer-Verlag. Berlin. Göttingen. Heildelberg. (1960).

[13] N. I. MUSKHELISHVILI, *Some Basic Problems of the Mathematical Theory of Elasticity*, (English translation by J. R. M. Radok). Noordhoff. Groningen. (1953). (Second Edition 1963.)

[14] H. NEUBER, *Kerbspannungslehre*, 3rd Edition. Springer-Verlag. Berlin. Heidelberg. New York. (1985).

[15] C. PEARSON, *Theoretical Elasticity*, Harvard University Press. Cambridge Massachusetts. (1959).

[16] A. ROBINSON, *Non-Standard Analysis*, North-Holland. Amsterdam. (1966).

[17] R. A. TOUPIN, *Saint-Venant's principle*, Arch. Rational Mech. Anal., (1965) 18, 83–96.

[18] O. ZANABONI, *Dimostrazione generale del principio del De Saint-Venant*, Rend. Accad. Lincei (1937) 25, 117–121.

[19] O. ZANABONI, *Valutazione dell'errore massimo cui dá luogo l'applicazione del principio del De Saint-Venant in un solido isotropo*, Rend. Accad. Lincei (1937) 25, 595–601.

[20] O. ZANABONI, *Sull'approssimazione dovuta al principio del De Saint-Venant nei solidi prismatici isotropi*, Rend. Acca. Lincei (1937) 26, 340–365.

R. J. Knops
School of Mathematical and Computer Sciences
and The Maxwell Institute of Mathematical Sciences
Heriot-Watt University, Edinburgh, EH14 4AS, Scotland, U.K.
r.j.knops@ma.hw.ac.uk

Piero Villaggio
Dipartimento di Ingeneria Strutturale
Universitá di Pisa, Pisa, Italy
dis@ing.unipi.it