Mathematical Physics — *Modelling social dynamics of a structured population*, by Giorgio Busoni, Antonio Fasano, Alberto Mancini and Mario Primicerio, communicated on 16 June 2011.

Dedicated to the memory of Giovanni Prodi.

Abstract. — We present a model for the dynamics of a population where the age distribution and the social structure are taken into account. This results in an integro-differential equation in which the kernel of the integral term may depend on a functional of the solution itself. We prove the well posedness of the problem with some regularity properties of the solution. Then, we consider some special cases and provide some simulations.

Key words: Social dynamics, integro-differential equations, structured populations.

Mathematics Subject Classification: 45G10, 92D25, 91D10.

1. Introduction

Mathematical models to describe social phenomena have been proposed and studied in many contexts. Quite recently, some special attention has been focussed on the evolution of criminality in the society (see e.g. the workshops [1] and the special volume [2] where a great deal of references can be found [10]).

Confining ourselves to models based on the methods of population dynamics, the society is described as composed by sub-populations (in the simplest case, the classical "triangle" model: criminals, guards, potential targets) mutually interacting and connected by fluxes of individuals (see e.g. [14], [19], [20], [17]). It is evident that more accurate models should take into account the quantities that influence e.g. the recruitment of criminals, or the effect and attractivity of crime[1]. In [16] and [15] one of such quantities was identified as the "social class". More specifically the population of law-abiding people is considered to be composed of n classes (of increasing wealth) and the rate of transition from each of these classes to the adjacent one is assumed to be linear and governed by suitable non-negative coefficients of "social promotion" α_k and of "social relegation" β_k, while the transition to a social class not adjacent is forbidden.

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[1] Here we have to point out that, when attempting to model the "crime" evolution, one has to identify the kind of illegal behavior that is considered, since the dynamics is of course very different when different crimes are taken into account.

In this perspective if we consider for simplicity the case of a closed population in which there are no criminals, and denoting by $u_k(t)$, $k = 1, \ldots, n$ the number of individuals belonging to the k-th class, one has to study the following system of linear ordinary differential equations:

(1) $\dot{u}_k(t) = \alpha_{k-1} u_k(t) - (\alpha_k + \beta_k) u_k(t) + \beta_{k+1} u_{k+1}(t), \quad k = 1, 2, \ldots, n,$

where we set conventionally $u_0(t) = u_{n+1}(t) = 0$ and $\beta_1 = \alpha_n = 0$. In this case, if α_k and β_k are constant for any k, the stationary solution has to satisfy

(2) $$\hat{u}_k = \frac{\alpha_1}{\beta_2} \frac{\alpha_2}{\beta_3} \cdots \frac{\alpha_{k-1}}{\beta_k} \hat{u}_1,$$

and hence is uniquely determined once we impose the condition

(3) $$\sum_{k=1}^{n} \hat{u}_k = N,$$

where N is known once the initial conditions for the system (1) are prescribed.

The system (1) has been studied (with or without the presence of other populations, criminals, guards, prisoners, etc) under different assumptions on the α's and the β's. These coefficients, that represent the "social mobility" of the society one is considering, are known to depend on several factors. For example in [11] it is stipulated that the mobility is increasing with the total dimension of the population—i.e. with N—whereas in [16] and [15] the coefficients of social promotion and relegation are assumed to depend on the total wealth of the population, and in turn this quantity is supposed to be a linear combination of $u_k(t)$

(4) $$W(t) = \sum_{k=1}^{n} p_k u_k(t),$$

or, more generally it is assumed that it is the solution of an ordinary differential equation of the form

(5) $$\dot{W}(t) = \sum_{k=1}^{n} p_k u_k(t) - \Psi(W(t), t),$$

where the first term on the r.h.s. represents the rate of wealth production and Ψ takes into account the expenses that the society has to face, according to a chosen budgetary policy.

2. A CONTINUOUS MODEL

Here we want to introduce a continuous model based on the following features:

(i) the influence of the age structure of the population is taken into account;

(ii) the social structure of the society is in the form of a distribution function $n(x, a, t)$ satisfying suitable integrability conditions such that, at any time t and for any $0 \leq a_1 < a_2$, $0 \leq x_1 < x_2$ $\int_{x_1}^{x_2} \int_{a_1}^{a_2} n(x, a, t) \, dx \, da$ represents the number of the individuals having age between a_1 and a_2 and "wealth"[2] between x_1 and x_2.

Concerning (i), in [8] it has been pointed out that this is a crucial factor for the dynamics of crime. Indeed, for each crime there is a sort of "age window", and the influence of age in the dynamics of "recruitment" of criminals seems to be as important as the influence of social environment.

To take age dependence into account an alternative approach would be to consider also the age dependence in terms of compartmental models. It is well known that, as far as demography is concerned, this approach (Lefkovitich matrix) is extensively used (see e.g. [7], [12]). For its applications to social dynamics see [3] and [21].

Assume that the wealth index x in the population can take values in $[0, X]$ and let

$$(6) \qquad \gamma(x, y, a, t) : [0, X]^2 \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$$

be a piecewise continuous function denoting the rate of transition from x to y. Moreover, let $\mu(x, a, t)$ be the exit rate from the population (by death or emigration) at time t for individuals of age a and wealth x.

Then the dynamics of the society in terms of the state variables is found to be expressed by

$$(7) \qquad \frac{\partial}{\partial t} n(x, a, t) + \frac{\partial}{\partial a} n(x, a, t) = -n(x, a, t) \int_0^X \gamma(x, y, a, t) \, dy$$

$$+ \int_0^X n(y, a, t) \gamma(y, x, a, t) \, dy$$

$$- \mu(x, a, t) n(x, a, t).$$

Equation (7) has to be complemented with initial condition

$$(8) \qquad n(x, a, 0) = n_0(x, a), \quad x \in [0, X], \, a \geq 0,$$

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[2] Whatever this might mean (yearly income, property, taxes paid, etc.), according to the specific phenomenon that is relevant to our model.

and by a condition on the birth rate that could be simply

$$(9) \qquad n(x, 0, t) = n_1(x, t), \quad x \in [0, X], \, t \geq 0,$$

or, more naturally, given in terms of the fertility of the population $\phi(x, a, t)$

$$(10) \qquad n(x, 0, t) = \int_0^\infty \phi(x, a, t) n(x, a, t) \, da, \quad x \in [0, X], \, t \geq 0,$$

where (no sex distinction is made) it is assumed that the newborns have the same wealth index as the parents.

It is clear that the dependence of the social mobility on the total dimension of the population and/or the total wealth can be expressed postulating that γ depends in a given way on $N(t)$ and/or $W(t)$ where

$$(11) \qquad N(t) = \int_0^\infty \int_0^X n(x, a, t) \, dx \, da, \quad t \geq 0,$$

and where, in analogy with (4), (5) either

$$(12) \qquad W(t) = \int_0^\infty \int_0^X \Pi(x, a, t) n(x, a, t) \, dx \, da, \quad t \geq 0,$$

where $\Pi(x, a, t)$ is a suitable weight function, or it is the solution of the O.D.E.

$$(13) \qquad \dot{W}(t) = \int_0^\infty \int_0^X \Pi(x, a, t) n(x, a, t) \, dx \, da - \Psi(W(t), t), \quad t \geq 0,$$

with $\Psi \geq 0$ representing the global expense rate.

From now on, we set $X = 1$ with no loss of generality.

REMARK 1. *Before proceeding further, for the sake of simplicity, we neglect the dependence an age and mortality. Then, compartmental model (1) clearly corresponds to the integro-differential equation (7) when the social mobility γ is defined according to the following scheme*

and where the k-th "compartment" on model (1) *is formed by individuals of wealth between c_{k-1} and c_k.*

3. Well-posedness of the problem

As we pointed out in the previous sections, the aim of this paper is to present the model and to analyze its basic mathematical features rather than to study in detail the most general cases. Consequently, in this section we will confine ourselves to considering the case in which the dependence on age is neglected and the following simplified problem is studied.

Problem (P)

Let $L^1(0,1)$ the Banach space of the classes of equivalence of the Lebesgue integrable functions on the interval $(0,1)$ with the usual norm $\|\cdot\|_1$. Find a continuously differentiable function of $t \in [0,\infty)$ with values in $L^1(0,1)$ satisfying the differential equation

$$\text{(14)} \qquad \frac{dn}{dt} = F(t;n),$$

and the initial condition $n(x, t = 0) = n_0(x)$, where $F(t,n)$ which is the time dependent nonlinear operator with values in $L^1(0,1)$ acting on n in the following way

$$\text{(15)} \quad F(t;n)(x) = -n(x,t) \int_0^1 \gamma(x,y,t;W)\,dy + \int_0^1 \gamma(y,x,t;W)n(y,t)\,dy.$$

Here W is the linear operator depending on $t \in [0,\infty]$ defined as

$$\text{(16)} \qquad W(t) = \int_0^1 \theta(x,t)n(x,t)\,dx.$$

The assumptions on γ and θ will be listed later.

REMARK 2. *Age dependence introduces in general the additional complication of the boundary condition for $n(x,a,t)$ for $a = 0$. If the latter is simply given as a known function $n_1(x,t)$, the essential difference w.r.t. Problem* (P) *is the substitution on the l.h.s. of* (14) *of the operator $\frac{\partial}{\partial t}$ by the directional derivative in the plane (a,t): $\frac{\partial}{\partial t} + \frac{\partial}{\partial a}$. If the boundary equation is given in terms of the fertility of the population, the model becomes much more complicated and its analysis is beyond the scope of this paper.*

We list the assumptions we will use

(H1)

$$\text{(17)} \qquad\qquad\qquad n_0 \in L^1(0,1);$$

(H2) $\gamma : \Omega = (0, 1) \times (0, 1) \times [0, \infty] \times \mathbb{R} \to \mathbb{R}$ is such that

(18)
$$\operatorname*{ess\,sup}_{\Omega} |\gamma(x, y, t; W)| = \hat{\gamma} < \infty;$$

(19) $\dfrac{\partial \gamma}{\partial W}$ exists for $W \in \mathbb{R}$ a.e. $(x, y, t) \in (0, 1) \times (0, 1) \times [0, \infty)$,

and it is such that $\operatorname*{ess\,sup}_{\Omega} \left| \dfrac{\partial \gamma}{\partial W}(x, y, t; W) \right| = \gamma_1 < \infty;$

(20) $\left| \dfrac{\partial \gamma}{\partial W}(x, y, t; W) - \dfrac{\partial \gamma}{\partial W}(x, y, t; W_1) \right| \leq l_1 |W - W_1|, \quad l_1 < \infty,$

$\forall W, W_1 \in \mathbb{R}$, a.e. $(x, y, t) \in (0, 1) \times (0, 1) \times [0, \infty);$

(21) $\left| \dfrac{\partial \gamma}{\partial W}(x, y, t; W) - \dfrac{\partial \gamma}{\partial W}(x, y, t_1; W) \right| \leq l_2 |t - t_1|, \quad l_2 < \infty,$

$\forall t, t_1 \in \mathbb{R}$, a.e. $(x, y, W) \in (0, 1) \times (0, 1) \times \mathbb{R};$

(H3) $\theta : (0, 1) \times [0, \infty] \to \mathbb{R}$ is such that

(22)
$$\operatorname*{ess\,sup}_{[0,1] \times [0,\infty)} |\theta(x, t)| = \hat{\theta} < \infty;$$

(23) $|\theta(x, t) - \theta(x, t_1)| < \varepsilon, \quad |t - t_1| < \delta(\varepsilon) \quad$ and a.e. $x \in (0, 1).$

REMARK 3. *In the present model n_0, γ, θ are non-negative elements of the respective spaces, but the proofs we give do not require positivity assumptions. We will see that n_0, γ, θ non-negative imply $n(x, t)$ non-negative too.*

We state the theorem

THEOREM 1. *Under assumptions* (17)–(23), *problem* (P) *has one and only one solution.*

PROOF. This proof follows the lines of the theory based on ε-approximated solutions (see [6]).

We consider W as a map from $L^1(0, 1)$ in a suitable set of time dependent bounded real functions. Note that

(24) $W : L^1(0, 1) \ni f \to W(f) = \displaystyle\int_0^1 \theta(x, t) f(x) \, dx, \quad \forall f \in L^1(0, 1),$

is such that $\| W(f) \|_\infty \leq \hat{\theta} \| f \|_1.$

The assumptions on W and γ assure continuity and boundedness of the map F, defined in (15) on $[0, \infty) \times L^1(0, 1)$, for any $f \in L^1(0, 1)$ such that it belongs to the closed bounded subset $\{f : \|f\|_1 \leq L\} = B(0, L)$. Therefore one can state

that it exists a function of t with values in $L^1(0,1)$, let it be ϕ, such that it is continuous and has a piecewise continuous derivative in a suitable neighborhood of $t = 0$, and that it is $(t, \phi(t)) \in [0, t_0] \times B(0, L)$ with $\|\phi'(t) - F(t, \phi(t))\|_1 \leq \varepsilon$ for any $t \in [0, t_0]$ (see Th. 3.1 in [6]).

Moreover the assumptions imply

$$(25) \quad \|F(t, f(\cdot, t)) - F(t, f_1(\cdot, t))\|_1 \leq 2(\hat{\gamma} + l\hat{\theta}\|f_1\|_1)\|f - f_1\|_1, \quad \forall t \geq 0,$$

and consequently if $f, f_1 \in B(f_0, r) = \{f \in L^1(0,1) : \|f - f_0\|_1 \leq r\} \subset B(0, L)$, the truth of the estimate

$$(26) \quad \|F(t, f(\cdot, t)) - F(t, f_1(\cdot, t))\|_1 \leq 2(\hat{\gamma} + l\hat{\theta}(\|f_1\|_1 + r))\|f - f_1\|_1, \quad \forall t \geq 0.$$

This inequality imply that two ε_1 and ε_2-approximated solutions to equation (14), corresponding to the initial values $f(\cdot, 0)$ and $f_1(\cdot, 0)$ are such that

$$(27) \quad \|f(\cdot, t) - f_1(\cdot, t)\|_1 \leq \|f(\cdot, 0) - f_1(\cdot, 0)\|_1 \exp\{kt\} + (\varepsilon_1 + \varepsilon_2)\frac{\exp\{kt\} - 1}{k}.$$

Here k is the Lipschitz constant of F in the ball $B(0, L)$.

Consequently (see Th. 1.6.1 and 1.7.1 of [6]) existence and uniqueness of the solution of problem (P) can be proved in a suitable bounded closed interval $[0, t_0]$.

The fact that the integral of the r.h.s. in (15)

$$(28) \quad -\int_0^1 n(x, t) \int_0^1 \gamma(x, y, t; W) \, dy \, dx + \int_0^1 \int_0^1 \gamma(y, x, t, W)n(y, t) \, dy \, dx = 0,$$

for any $t \geq 0$ in the time interval of existence of the solution, for any $n_0 \in L^1(0,1)$, assure that the solution can be extended to the interval $[0, \infty)$.

REMARK 4. *The solution is such that*

$$(29) \qquad \int_0^1 n(x, t) \, dx = \int_0^1 n_0(x) \, dx, \quad \forall t \geq 0, \forall n_0 \in L^1(0,1).$$

REMARK 5. *If γ, θ, n_0 are non-negative elements in the respective domains, then the approximated solutions are non-negative; hence the unique solution is non-negative.*

REMARK 6. *An alternative strategy of proving the existence theorem is based on the theory of semi-groups of operators. The proofs are essentially generalizations of the methods of successive approximations which are suitable for nonlinear evolution problems (see e.g. [4], [5]).*

Even if proofs are given there for autonomous equations, a careful reader can modify and fit them to many non autonomous problems. For instance, the case γ and θ independent of t is well described by the theory exposed in section 3.4 and/or 5.4 of [4], and in chapter 3 of [5].

4. SOME REMARKS ON THE REGULARITY OF THE SOLUTION

We begin by considering the simplest evolution model

$$(30) \qquad n_t(x, t) = -n(x, t) \int_0^1 \gamma(x, y) \, dy + \int_0^1 \gamma(y, x) n(y, t) \, dy.$$

THEOREM 2. *If the functions $n_0(x)$, $\gamma(x, y)$ are k-times continuously differentiable in $[0, 1]$ and $[0, 1]^2$, respectively, then the unique solution $n(x, t)$ of (30) with the initial condition $n(x, 0) = n_0(x)$ has k continuous derivatives with respect to x, and all of them are continuously differentiable w.r.t. t.*

PROOF. By formal differentiation of (30) w.r.t. x we obtain the Cauchy problem

$$(31) \quad \xi_t = -\xi(x, t) \int_0^1 \gamma(x, y) \, dy - n(x, t) \int_0^1 \gamma_x(x, y) \, dy + \int_0^1 \gamma_y(y, x) n(y, t) \, dy,$$

$$(32) \qquad\qquad\qquad\qquad \xi(x, 0) = n_0'(x),$$

where $n(x, t)$ is the known solution of (30). Such a problem has a unique solution, which is continuous, with a continuous t-derivative and that coincides necessarily with n_x. Its explicit expression is

$$(33) \qquad n_x(x, t) = n_0'(x) e^{-g(x)t} + \int_0^t S(x, \tau) e^{-g(x)(t-\tau)} \, d\tau,$$

where

$$(34) \qquad\qquad\qquad g(x) = \int_0^1 \gamma(x, y) \, dy,$$

and $S(x, y)$ denotes the free term in (31).

Under the differentiability assumptions on $n_0(x)$, $\gamma(x, y)$ the procedure can be iterated to calculate $\partial_x^k n$.

THEOREM 3. *Under the assumptions of Theorem 1 the function $n(x, t)$ is C^∞ w.r.t. t, for almost all $x \in (0, 1)$.*

PROOF. From Theorem 1 we already know that n_t exists and is continuous in t for x a.e. in $(0, 1)$. Moreover we can write

$$(35) \qquad n_t(x, 0) = n_1(x) = -n_0(x) g(x) + \int_0^1 \gamma(y, x) n_0(y) \, dy.$$

The function $\eta(x, t)$ satisfying the Cauchy problem

$$(36) \qquad\qquad \eta_t(x, t) = -\eta(x, t) g(x) + \int_0^1 \gamma(y, x) \eta(y, t) \, dy,$$

$$(37) \qquad\qquad\qquad\qquad \eta(x, 0) = n_1(x),$$

which coincides with the original problem for n with the only change of the initial data, necessarily coincides with n_t. Not only it exists, but has the derivative $\partial_t^2 n$ continuous in t. Next we can calculate the initial value of

$$(38) \qquad \partial_t^2 n(x, 0) = n_2(x) = -n_1(x)g(x) + \int_0^1 \gamma(y, x)n_1(y)\, dy,$$

and iterate the procedure infinitely many times.

In the particular case in which the initial distribution $n_0(x)$ is bounded we can say more.

THEOREM 4. *If* ess $\sup_\Omega n_0(x) \leq A_0$ *the function* $n(x, t)$ *is analytic with respect to* t, *a.e. in* x.

PROOF. Besides the function $g(x)$ we define $h(x) = \int_0^1 \gamma(y, x)\, dy$, and we denote by $\hat{g}$ and $\hat{h}$ the L^∞ norm of g, h. From (34) we deduce that $|n_1| \leq A_0(\hat{g} + \hat{h})$, and from (38) that $|n_2| \leq A_0(\hat{g} + \hat{h})^2$, and so on. Thus we have the estimates

$$(39) \qquad |\partial_t^k|_{t=0}| \leq A_0(\hat{g} + \hat{h})^k,$$

showing that $n(x, t)$ has a uniformly convergent Taylor expansion for $t \geq 0$, a.e. in x. Moreover, we have the estimates

$$(40) \qquad |\partial_t^m| \leq A_0(\hat{g} + \hat{h})^m e^{(\hat{g}+\hat{h})t}, \quad m = 0, 1, 2, \ldots.$$

Theorems 2, 3 and 4 were proved in the case (30) i.e. when social mobility does not depend on the total wealth $W(t)$.

For the more complicated model

$$(41) \quad n_t(x, t) = -n(x, t) \int_0^1 \gamma(x, y, W(t))\, dy + \int_0^1 \gamma(y, x, W(t))n(y, t)\, dy$$

$$(42) \qquad W(t) = \int_0^1 \theta(x)n(x, t)\, dx$$

the argument for the differentiability of n w.r.t. x goes exactly as in the prof of Theorem 2.

Proving higher-order differentiability w.r.t. t is still possible but estimates of the kind (39) cannot be obtained.

5. SOME PARTICULAR CASES

This section will be devoted to the analysis of some qualitative properties of the solution of problem (P). To be specific we will consider first the case

W-independent that is not so far from the concrete situation[3]. Then we will discuss a case of uniform promotion/relegation depending on W.

The scheme of the session is the following.

5.1 We consider the case in which γ has constant values in the regions $A = \{(x, y) : 0 < x < y < 1\}$ and $B = \{(x, y) : 0 < y < x < 1\}$ above and below the diagonal of the square $(0, 1)^2$,

5.2 We allow dependence on just one variable in A and B,

5.3 We consider the symmetric case $\gamma(x, y) = \gamma(y, x)$,

5.4 We study the case $\gamma(x, y) = p(x)q(y)$,

5.5 We consider the case where γ is a given function of W in each of the regions A and B.

5.1. Constant social promotion and relegation

If

$$(43) \qquad \gamma(x, y) = \begin{cases} a, & (x, y) \in A, \\ b, & (x, y) \in B, \end{cases}$$

then equation (14) gives

$$(44) \qquad n_t(x, t) = -n(x, t)(a + cx) + bN - c \int_0^x n(y, t)\, dy,$$

where

$$(45) \qquad N = \int_0^1 n_0(x)\, dx, \quad c = b - a.$$

Since (44) can be rewritten

$$(46) \qquad n_t(x, t) = -c \frac{\partial}{\partial x}\left(x \int_0^x n(y, t)\, dy\right) - an + bN,$$

integrating with respect to x and setting

$$(47) \qquad M(x, t) = \int_0^x n(y, t)\, dy,$$

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[3] Indeed one can study the problem in a given time interval (say 1-year) and assume that the social mobility does not depend on time and corresponds to the total wealth of the society at the beginning of the period considered. The change of the total wealth can be thought to be relevant to the social dynamics of the following year.

we have

$$(48) \qquad M_t = -cxM - aM + bNx.$$

Therefore we find the explicit solution

$$(49) \qquad n(x, t) = \left\{ n_0(x) - \frac{bNa}{(cx + a)^2} - ct \left[M_0(x) - \frac{bN}{(cx + a)} \right] \right\} e^{-(cx+a)t}$$

$$+ \frac{bNa}{(cx + a)^2}.$$

In particular if $ab \neq 0$ the equilibrium solution is

$$(50) \qquad n_\infty = \frac{bNa}{(cx + a)^2}.$$

If a or b vanishes, the solution tends asymptotically to concentrate in $x = 0$ or $x = 1$ respectively.

We note that, with some additional work, the analogous of (50) can be found also in cases on which γ has constant values in subsets of A and B.

5.2. Dependence on just one variable in A and B

We have four cases

$$(51) \qquad \gamma(x, y) = \begin{cases} a(y), & (x, y) \in A, \\ b(y), & (x, y) \in B, \end{cases}$$

$$(52) \qquad \gamma(x, y) = \begin{cases} a(x), & (x, y) \in A, \\ b(y), & (x, y) \in B, \end{cases}$$

$$(53) \qquad \gamma(x, y) = \begin{cases} a(y), & (x, y) \in A, \\ b(x), & (x, y) \in B, \end{cases}$$

$$(54) \qquad \gamma(x, y) = \begin{cases} a(x), & (x, y) \in A, \\ b(x), & (x, y) \in B. \end{cases}$$

We can find the explicit equilibrium solution in each of the four cases. In case (51) we find

$$(55) \qquad M_\infty(x) = \frac{N \int_0^x b(y)\, dy}{\int_0^x b(y)\, dy + \int_x^1 a(y)\, dy},$$

whence $n_\infty(x)$ is found by differentiation.

In case (54) we set

$$(56) \qquad c(x) = b(x) - a(x),$$

and we find

$$(57) \qquad \log\frac{n_\infty(x)}{n_\infty(0)} = \log\frac{a(0)}{c(x)x + a(x)} - \int_0^x \frac{c(y)}{c(y)y + a(y)}\,dy,$$

where $n_\infty(0)$ has to be found by imposing

$$(58) \qquad \int_0^1 n_\infty(x) = N.$$

Finally, we find

$$(59) \qquad n_\infty(x) = N \left[\int_0^1 \frac{\exp\left[-\int_0^x \frac{c(y)}{c(y)y + a(y)}\,dy\right]}{c(x)x + a(x)}\,dx\right]^{-1}$$

$$\times \frac{\exp\left[-\int_0^x \frac{c(y)}{c(y)y + a(y)}\,dy\right]}{c(x)x + a(x)}.$$

In cases (52) and (53) the calculations are more lengthy and we give just the final result.

For case (52) we have

$$(60) \qquad n_\infty(x) = \left(\frac{\mu(x)}{\beta(x)}\right)' - Nb(x),$$

where

$$(61) \qquad \mu(x) = (1 - x)\int_0^x a'(y)M_\infty(y)\,dy,$$

$$(62) \qquad \beta(x) = \int_0^x b(y)\,dy + a(x)(1 - x).$$

For case (53) we define

$$(63) \qquad \lambda(x) = \int_x^1 a(y)\,dy + xb(x),$$

$$(64) \qquad v(x) = \lambda(x)M_\infty(x) - N\int_0^x b(y)\,dy,$$

and we find

$$(65) \qquad v(x) = N\int_0^x \frac{x}{y}\exp\int_y^x \frac{zb'(z)}{\gamma(z)}\,dz\left[\frac{1}{y}\int_0^y b(z)\,dz - b(y)\right]dy.$$

5.3. The symmetric case $\gamma(x, y) = \gamma(y, x)$

It is immediately verified that in this case we have the constant equilibrium solution

$$(66) \qquad n_\infty = N.$$

5.4. The case of factorized kernel $\gamma(x, y) = p(x)q(y)$

In the particular case $p = 1$ equation (14) can be written

$$(67) \qquad \frac{\partial}{\partial t} n(x, t) = -n(x, t)\tilde{q} + q(x)N,$$

where

$$(68) \qquad \tilde{q} = \int_0^1 q(y)\, dy.$$

Hence

$$(69) \qquad n(x, t) = n_0(x)e^{-\tilde{q}t} + \frac{q(x)N}{\tilde{q}}(1 - e^{-\tilde{q}t}),$$

having the asymptotic profile

$$(70) \qquad n_\infty(x) = q(x)N/\tilde{q}.$$

In the general factorized case, defining

$$(71) \qquad v(x) = p(x)n_\infty(x),$$

we find that the equilibrium solution n_∞ has to satisfy

$$(72) \qquad \tilde{q}v(x) = q(x)\tilde{v},$$

where

$$(73) \qquad \tilde{v} = \int_0^1 p(x)n_\infty(x)\, dx.$$

Hence n_∞ is given by

$$(74) \qquad n_\infty(x) = K\frac{q(x)}{p(x)},$$

where K is found by imposing that $\int_0^1 n_\infty(x)\,dx = N$

$$(75) \qquad\qquad K = N\left[\int_0^1 \frac{q(x)}{p(x)}\,dx\right]^{-1}.$$

Of course, we excluded the possibility that $p(\hat{x}) = 0$ for some $\hat{x}$. But this fact would mean that there is no possibility of leaving the state $\hat{x}$, whereas (if $q(\hat{x}) \neq 0$) there is a finite rate of "arrival" in $\hat{x}$ from other states. Of course, if $\hat{x}$ is the unique zero of $p(x)$ the final situation would be $n_\infty(x) = N\delta(x - \hat{x})$ where δ is the Dirac's delta.

We also note that this result still holds if γ is allowed to depend on W in a factorized form

$$(76) \qquad\qquad \gamma = p(x)q(y)\Phi(W).$$

Indeed, looking for a stationary solution of (14) $\Phi(W)$ cancels and we are back to the previous case.

5.5. Uniform promotion/relegation rates, depending on W

Let us go back to the case examined in Section 5.1, but letting a, b depend on W. Thus we assume

$$(77) \qquad\qquad \gamma(x, y, W) = \begin{cases} a(W), & (x, y) \in A, \\ b(W), & (x, y) \in B, \end{cases}$$

with a, b continuous and positive for $W \in [0, N\|\theta\|]$, and we look for a steady state solution[4] From Section 5.1 we know that if $n_\infty(x)$ is such a solution, then

$$(78) \qquad\qquad n_\infty(x) = N\frac{a(W_\infty)b(W_\infty)}{[a(W_\infty) + c(W_\infty)x]^2},$$

where W_∞ is the wealth index corresponding to $n_\infty(x)$, namely

$$(79) \qquad\qquad W_\infty = \int_0^1 \theta(x)n_\infty(x)\,dx.$$

In the following it will be convenient to define

$$\omega = W_\infty, \quad H(\omega) = \frac{b(\omega)}{a(\omega)},$$

so that $\dfrac{c(\omega)}{a(\omega)} = H(\omega) - 1$.

---

[4] We have seen that the fact that a or b vanish can be associated to singular solutions.

Thus equation (78) can be written as

$$(80) \qquad n_\infty(x) = N \frac{H(\omega)}{\{1 + [H(\omega) - 1]x\}^2}.$$

Imposing (79) we conclude that steady state solutions correspond to the roots of the algebraic equation

$$(81) \qquad \omega = NH(\omega) \int_0^1 \frac{\theta(x)}{\{1 + [H(\omega) - 1]x\}^2} \, dx.$$

Of course (81) has one unique solution for any $\theta(x)$ in the particular case in which a and b are proportional, while (81) is identically satisfied for $\omega = NK_0$ when $\theta = K_0$.

We will consider with some more detail the case

$$(82) \qquad \theta(x) = \lambda x,$$

which is, in some sense, the most natural way of associating the wealth index to the produced wealth W.

We define the function

$$(83) \qquad F(H) = H \int_0^1 \frac{x \, dx}{(1 + (H-1)x)^2} = \frac{H \ln H}{(H-1)^2} - \frac{1}{H-1}.$$

The function $F(H)$ is plotted in Fig. (1) and can be made C^1 by defining $F(1) = \frac{1}{2}$, $F'(1) = -\frac{1}{6}$.
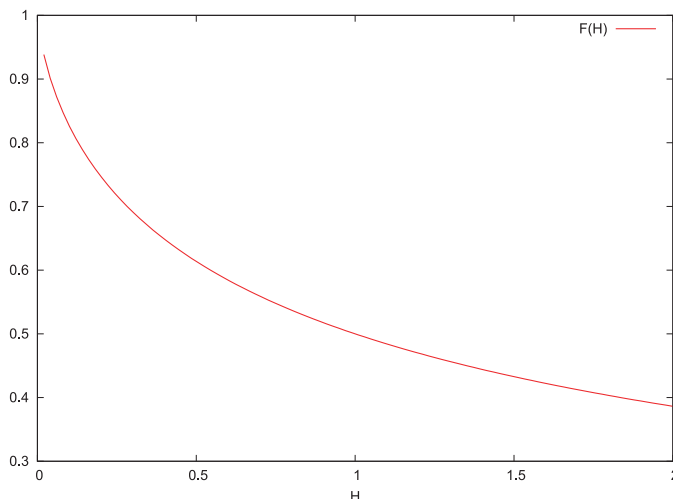

Figure 1. $F(H)$.

Thus, the solutions of (71) are the values of ω corresponding to the intersection of the curves

$$(84) \qquad\qquad y = N\lambda F(H(\omega)),$$

$$(85) \qquad\qquad y = \omega,$$

in the strip $\omega \in [0, N\lambda]$ of the (ω, y) plane.

Since the range of F in $[0, 1]$ (including also the limit case $a(W) = 0$ for some $W \in [0, N\lambda]$) we can conclude that

PROPOSITION 1. *In the assumptions (77) there exists at least one solution of the problem.*

Moreover

REMARK 7. *If* $\dfrac{dH}{dW} > 1$ *there exists one and only one stationary solution.*

To be specific we consider the following example

$$(86) \qquad\qquad a = KW(\lambda N - W), \quad b = 1,$$

so that

$$(87) \qquad\qquad H(W) = \frac{1}{KW(\lambda N - W)},$$

and we rewrite (81) as

$$(88) \qquad\qquad \frac{1}{\lambda N} W^{-1}(H) = F(H),$$

where $W^{-1}(H)$ is the inverse of the function $H(W)$ and when H is not invertible we mean one branch of its inverse graph.

In the example we are considering we have two branches

$$(89) \qquad\qquad \Omega_1(H) = \frac{1}{2}\left\{ 1 + \sqrt{1 - \frac{4}{KH\lambda^2 N^2}} \right\},$$

$$(90) \qquad\qquad \Omega_2(H) = \frac{1}{2}\left\{ 1 - \sqrt{1 - \frac{4}{KH\lambda^2 N^2}} \right\},$$

both defined for $H > H_{min} = \dfrac{4}{K\lambda^2 N^2}$ and such that

$$(91) \qquad\qquad \Omega_1(H_{min}) = \Omega_2(H_{min}) = \frac{1}{2}.$$

Note that Ω_1 takes values in $\left[\dfrac{1}{2}, 1\right]$ and Ω_2 in $\left(0, \dfrac{1}{2}\right]$.
We distinguish two cases:

- $H_{min} \leq 1$,
- $H_{min} > 1$,

In the first case the branch Ω_1 gives a finite solution, while the branch Ω_2 may have no finite intersection with the graph of $F(H)$. However it does provide the solution $\Omega = 0$ (intersection at infinity). This is related (as we have remarked in Section 5.1) with the fact that $a(0) = 0$.

In the second case we lose the intersection of the first branch, but a finite intersection will exist with the second branch, since $F(H)$ decays at infinity as $\dfrac{\ln H}{H}$, while Ω_2 decays like $\dfrac{1}{KH\lambda^2 N^2}$. The intersection at infinity still exists.

6. SOME SIMULATIONS

In this section we will display some numerical simulations that have the sole aim of showing how the model—once calibrated on data taken from some concrete situation—can give some ideas on the social dynamics.

It is commonly accepted that the policy of a society should tend to increase the *total wealth*, preserving at the same time some *equity*.

To measure the latter quantity, several indexes have been proposed. One of them is based on the so-called Lorenz curve [13]. In our model it is the curve that expresses the quantity $R(x) = \displaystyle\int_0^x \xi n(\xi)\,d\xi / W$ with respect to $M(x) = \displaystyle\int_0^x n(\xi)\,d\xi / N$.

The Gini index (see [9]) is defined as the double of the area between the diagonal of the two positive axes in the plane (M, R) and the Lorenz curve for the given society.

It is a number ranging from 0 (total "equity": a limit case in which all the individuals have the same individual wealth) to 1 (a limit case in which one has two subpopulations of wealth 0 and of wealth 1).

Another possibility of measuring the equity is the fraction of the total wealth that is in the hands—say—the richest 25% of the population: in our notation

$$(92) \qquad\qquad \rho = 1 - R(0.75).$$

Just to have an idea of the dynamics, we have considered the social mobility of our societies defined as follows

$$(93) \qquad\qquad \gamma(x, y) = \exp\left[-\frac{-(x - y)^2}{\sigma}\right]\Gamma(x, y),$$

where

$$(94) \qquad \Gamma(x, y) = \begin{cases} \exp -\dfrac{(x - x_a)^2}{\rho_a}, & x < y, \\[3ex] \exp -\dfrac{(x - x_b)^2}{\rho_b}, & x > y. \end{cases}$$

Fig. (2) displays the evolution for different time values where the initial situation is

$$(95) \qquad n(x, 0) = Mx(1 - x),$$

and the parameters are the following:

$$\sigma = 0.1 \qquad \rho_a = 0.1,$$
$$x_a = 0.55 \qquad \rho_b = 0.1,$$
$$x_b = 0.65 \qquad M = 4 * 1.0^3.$$

(a)

(b)

(c)

(d)
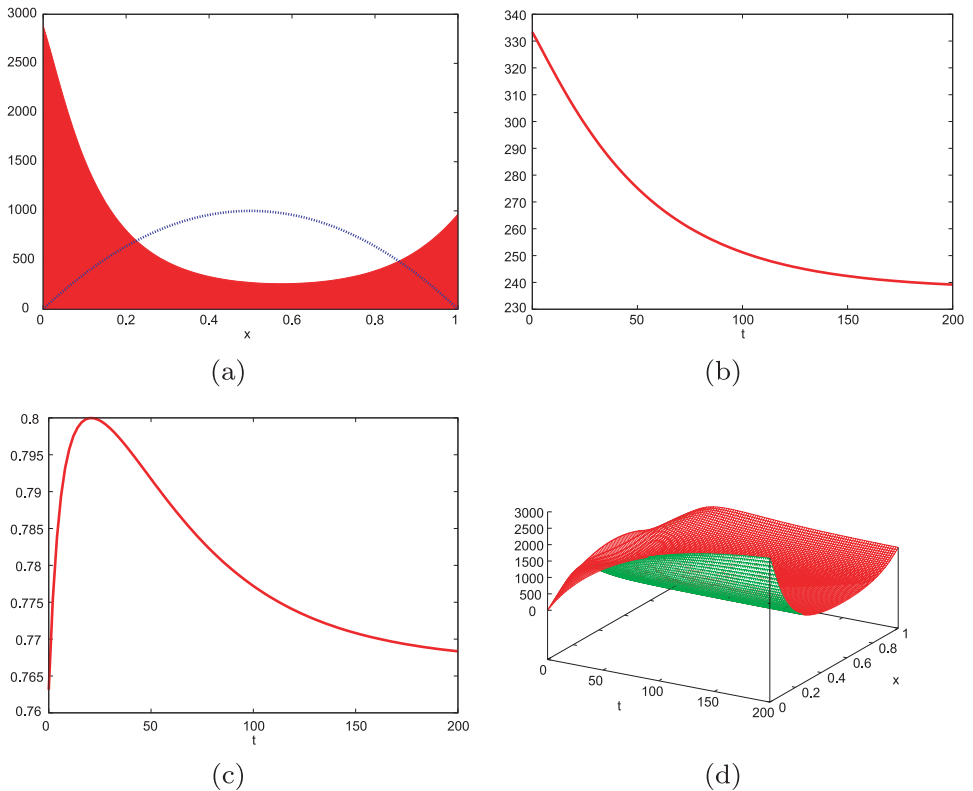
Figure 2. γ as in (93). a: $n(x, t)$ for $t = 0$ (dotted line), $t = 2.0e2$ (filled curve); b: $W(t)$; c: $\rho(t)$; d: $n(x, t)$.

It can be observed that the total wealth W decreases and that the index ρ, after a rapid increase, keeps decreasing. It means that there is more equity but the society is less productive.

In fig. (3) we assumed that the terms expressing social promotion $(x < y)$ and social relegation $(x > y)$ are affected by the total wealth and by the policy chosen as a consequence of it. More specifically we assumed

$$(96) \quad \Gamma(x, y) = \begin{cases} 10.0\,W \exp-\dfrac{(x - x_a)^2}{\rho_a} \\ \qquad \times \left(1 - \exp-\dfrac{(x - x_a)^2}{0.02}\right) \exp-\dfrac{(x - y)^2}{\sigma}, \quad x < y, \\ 10.0\,W \exp-\dfrac{(x - x_a)^2}{\rho_b} \\ \qquad \times \left(1 - \exp-\dfrac{(x - x_b)^2}{0.02}\right) \exp-\dfrac{(x - y)^2}{\sigma}, \quad x > y. \end{cases}$$
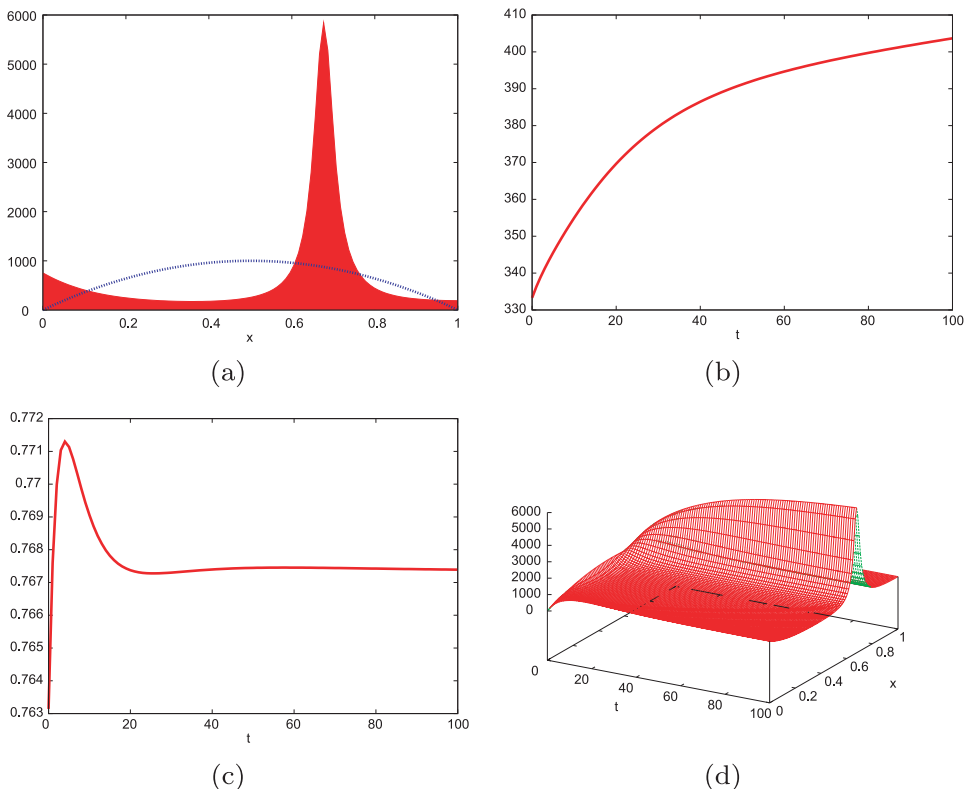

(a)

(b)

(c)

(d)

Figure 3. γ as in (96). a: $n(x, t)$ for $t = 0$ (dotted line), $t = 2.0e2$ (filled curve); b: $W(t)$; c: $\rho(t)$; d: $n(x, t)$.

Here the evolution of the society is favorable: W increases, while the index ρ actually stabilizes to a rather large value. This is the effect of creating welfare of the middle class (see the peak in panel a).

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