Rend. Lincei Mat. Appl. 22 (2011), 453–486
DOI 10.4171/RLM/609

Algebraic Geometry — *Arrangements of rational sections over curves and the varieties they define*, by Giancarlo Urzúa, communicated on 15 April 2010.

Abstract. — We introduce arrangements of rational sections over curves. They generalize line arrangements on $\mathbb{P}^2$. Each arrangement of d sections defines a single curve in $\mathbb{P}^{d-2}$ through the Kapranov's construction of $\overline{M}_{0,d+1}$. We show a one-to-one correspondence between arrangements of d sections and irreducible curves in $M_{0,d+1}$, giving also correspondences for two distinguished subclasses: transversal and simple crossing. Then, we associate to each arrangement $\mathcal{A}$ (and so to each irreducible curve in $M_{0,d+1}$) several families of nonsingular projective surfaces X of general type with Chern numbers asymptotically proportional to various log Chern numbers defined by $\mathcal{A}$. For example, for the main families and over $\mathbb{C}$, any such X is of positive index and $\pi_1(X) \simeq \pi_1(\bar{A})$, where $\bar{A}$ is the normalization of A. In this way, any rational curve in $M_{0,d+1}$ produces simply connected surfaces with Chern numbers ratio bigger than 2. Inequalities like these come from log Chern inequalities, which are in general connected to geometric height inequalities (see Appendix). Along the way, we show examples of étale simply connected surfaces of general type in any characteristic violating any sort of Miyaoka–Yau inequality.

Key words: Arrangement of curves, moduli space, surface of general type, Miyaoka–Yau inequality, geometric height inequalities.

Mathematics Subject Classification (2000): 14J29, 14J10, 52C30.

1. Introduction

Arrangements of rational sections over curves set up a new class of arrangements of curves on algebraic surfaces. Given a nonsingular projective curve C and an invertible sheaf $\mathcal{L}$ on C, they are defined as finite collections of sections of the corresponding $\mathbb{A}^1$-bundle. The simplest example is line arrangements on $\mathbb{P}^2$, where $C = \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$. In Section 2, we systematically study arrangements of rational sections over curves. Although in somehow they can be managed as line arrangements, the big difference relies on possible tangencies among their curves, introducing more geometric liberties. We partially organize this via transversal and simple crossing arrangements (Definition 2.4). Throughout Sections 3, 4 and 5, we show one-to-one correspondences between arrangements of d sections and irreducible curves in $M_{0,d+1}$, the moduli space of curves of genus zero with $d+1$ marked ordered points. This is done for each fixed pair $(C, \mathcal{L})$ in the general (Theorem 5.1), transversal (Corollary 5.5), and simple crossing (Corollary 5.6) cases. Because of Kapranov's description of $\overline{M}_{0,d+1}$ [11, 12], this produces bijections between arrangements and curves in $\mathbb{P}^{d-2}$ (Corollary 5.2). For instance, arrangements of d lines in $\mathbb{P}^2$ correspond to lines in $\mathbb{P}^{d-2}$ (as in [21]), arrangements of d conics in $x^2+y^2+z^2=0$ correspond to conics in $\mathbb{P}^{d-2}$, etc. To exemplify, we show in Section 6 a way to produce explicit arrangements of sections from irreducible curves in $\mathbb{P}^2$. This is based on [6, Section 7], where the authors show how to cover $M_{0,d+1}$ with blow-ups of $\mathbb{P}^2$ at $d+1$ points. We use their rigid conic as concrete example (see Examples 6.1 and 7.1).

Given an arrangement of sections $\mathcal{A}$, we define two types of arrangements: the extended $\mathcal{A}_\Delta$ and some partially extended $\mathcal{A}_{p\Delta}$. Their definitions and log properties are exposed in Section 7. Over $\mathbb{C}$, they satisfy certain log Miyaoka–Yau inequalities which are no longer combinatorial as in the case of line arrangements (Remarks 7.3 and 7.4). For line arrangements we know an optimal log inequality (Hirzebruch-Sakai's in Remark 7.3), but for any other class we only have the coarse bound 3. Arrangements attaining upper bounds should be special, and would produce interesting algebraic surfaces by means of Theorem 8.1. We remark that questions about sharp upper bounds of log Chern ratios are related to questions on effective height inequalities [15, pp. 149–153] (see Appendix, where we slightly extend and give another proof of Liu's inequality [16, Theorem 0.1], which naturally implies strict Tan's height inequality [19, Theorem A]).

In Section 8, we associate various families of nonsingular projective surfaces to any given arrangement of sections $\mathcal{A}$. These surfaces share the random nature of the surfaces X constructed in [22], having Chern invariants asymptotically proportional to the log Chern invariants of $\mathcal{A}_\Delta$ and $\mathcal{A}_{p\Delta}$'s. In this way, we are able to show for a more general class of arrangements (and so singularities) that the behavior of Dedekind sums and continued fractions used in [22] can also be applied. In this paper, any such X is of general type and satisfies $c_1^2(X), c_2(X) > 0$. Putting it all together, and over $\mathbb{C}$, we have the following.

THEOREM. Let A be an irreducible curve in $\mathbb{P}^n$ not contained in $\prod_i x_i \prod_{i<j}(x_j - x_i) = 0$. Let $\bar{A}$ be the normalization of A. Then, there exist nonsingular projective surfaces X of general type such that $2 < \frac{c_1^2(X)}{c_2(X)} < 3$, having $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to $\frac{\bar{c}_1^2(A)}{\bar{c}_2(A)}$, a well-defined positive rational number depending on A and its position in $\mathbb{P}^n$. In addition, there is an induced connected fibration $\pi' : X \to \bar{A}$ which gives an isomorphism $\pi_1(X) \simeq \pi_1(\bar{A})$. In this way, $\mathrm{Alb}(X) \simeq \mathrm{Jac}(\bar{A})$ and π' is the Albanese fibration of X.

With this in hand, we aim to answer the still open question: are there simply connected nonsingular projective surfaces of general type with $\frac{c_1^2}{c_2}$ arbitrarily close to the Miyaoka–Yau bound 3? Hence, at least when A is a rational curve, it is important for us to know about sharp upper bounds for $\frac{\bar{c}_1^2(A)}{\bar{c}_2(A)}$ (also for $\mathcal{A}_{p\Delta}$, see Remark 8.3). So far, we only know that this bound is strictly smaller than 3 (Corollaries 7.2 and 7.4). On the other hand, in positive characteristic, we use our method to produce étale simply connected nonsingular projective surfaces of general type which violate any sort of Miyaoka–Yau inequality for any given characteristic (Example 2.4, Remark 7.5, Example 8.2).

ACKNOWLEDGMENTS. I am grateful to I. Dolgachev, S. Keel, F. Knudsen, K. Liu, N. Mok, A. Momot, J. Tevelev, T. Weston and S. Wong for valuable conversations and e-mail correspondence. I would like to thank the referee for such a complete and helpful review which improved much many parts of this article.

2. ARRANGEMENTS OF RATIONAL SECTIONS OVER CURVES

Fix an algebraically closed field $\mathbb{K}$. Let C be a nonsingular projective curve defined over $\mathbb{K}$ of genus $g = h^1(C, \mathcal{O}_C)$. Hence, when $\mathbb{K} = \mathbb{C}$, the curve C is a compact Riemann surface of genus g. Let $\mathcal{L}$ be an invertible sheaf on C of degree $\deg(\mathcal{L}) = e > 0$, and let

$$\mathbb{A}_C(\mathcal{L}) := \operatorname{Spec}(S(\mathcal{L}^{-1})) \to C$$

be the line bundle associated to $\mathcal{L}$ (as in [8, II, Ex. 5.18]), where $\mathcal{L}^{-1}$ is the dual of $\mathcal{L}$. This is the so-called total space of $\mathcal{L}$. A section of $\mathbb{A}_C(\mathcal{L}) \to C$ is a morphism $C \to \mathbb{A}_C(\mathcal{L})$ such that the composition map $C \to \mathbb{A}_C(\mathcal{L}) \to C$ is the identity. The space of sections can be identified with $H^0(C, \mathcal{L})$. Since it is better to deal with a projective surface, we naturally compactify all fibers, so that we work with a $\mathbb{P}^1$-bundle. Let

$$\pi : \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^{-1}) \to C$$

be the $\mathbb{P}^1$-bundle associated to $\mathcal{O}_C \oplus \mathcal{L}^{-1}$ over C, as in [8, II, Ex. 7.10]. The nonsingular projective surface $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^{-1})$ contains $\mathbb{A}_C(\mathcal{L})$ as an open set, such that the curve $C_0 := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^{-1}) \backslash \mathbb{A}_C(\mathcal{L})$ is a section of π with self-intersection $C_0^2 = -e$. It is easy to see that C_0 is the only irreducible curve with negative self-intersection in $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^{-1})$. This surface is a particular case of a geometrically ruled surface over C [8, V, Section 2], and it is in its normalized form [8, V, Proposition 2.8]. We denote by F_c the fiber over a point $c \in C$, or just F when we consider its numerical class. Any element in $\operatorname{Pic}(\mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^{-1}))$ can be written as $aC_0 + \pi^*(\mathcal{M})$ with $a \in \mathbb{Z}$, and $\mathcal{M} \in \operatorname{Pic}(C)$. Any element in $\operatorname{Num}(\mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^{-1}))$ can be written as $aC_0 + bF$ with $a, b \in \mathbb{Z}$ [8, p. 373].

EXAMPLE 2.1. Let $C = \mathbb{P}^1$, and let $e > 0$. Consider the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(e)$ on $\mathbb{P}^1$. Then, the surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ is the Hirzebruch surface $\mathbb{F}_e$. When $e = 1$, $\mathbb{F}_1$ corresponds to the blow-up at a point of $\mathbb{P}^2$ [8, V, Exa. 2.11.4], and C_0 is the (-1)-curve.

The main objects are the following.

DEFINITION 2.1. Let $d \geq 3$ be an integer. Let C be a nonsingular projective curve, and let $\mathcal{L}$ be an invertible sheaf on C of degree $e > 0$. An *arrangement of d sections* is a labeled set of d distinct sections $\mathcal{A} = \{S_1, \ldots, S_d\}$ of $\pi : \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^{-1}) \to C$ such that

$$S_i \sim C_0 + \pi^*(\mathcal{L})$$

for all $i \in \{1, 2, \ldots, d\}$, and $\bigcap_{i=1}^{d} S_i = \emptyset$. From now on, we denote $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^{-1})$ by $\mathbb{P}_C(\mathcal{L})$.

In particular, $S_i.S_j = e$, and $S_i.C_0 = 0$ for all i, and so these arrangements are indeed formed by sections of $\mathbb{A}_C(\mathcal{L}) \to C$. The condition $\bigcap_{i=1}^{d} S_i = \emptyset$ implies that $\mathcal{L}$ is base-point free. To see this, take a point $c \in C$, and consider the corresponding fiber F_c. Since $\bigcap_{i=1}^{d} S_i = \emptyset$, there are two sections S_i, S_j such that $F_c \cap S_i \cap S_j = \emptyset$. Let $\sigma_j : C \to \mathbb{P}_C(\mathcal{L})$ be the map defining the section S_j. Then, $\mathcal{L} \simeq \sigma_j^*(\pi^*(\mathcal{L}) \otimes \mathcal{O}_{S_j}) \simeq \sigma_j^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{L})}(C_0) \otimes \pi^*(\mathcal{L}) \otimes \mathcal{O}_{S_j}) \simeq \sigma_j^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{L})}(S_i) \otimes \mathcal{O}_{S_j})$, and $\sigma_j^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{L})}(S_i) \otimes \mathcal{O}_{S_j})$ is given by an effective divisor on C not supported at c. This tell us that $\mathcal{L} \simeq \mathcal{O}_C(D)$ with D base-point free effective divisor.

If $\mathcal{A} = \{S_1, \ldots, S_d\}$ is an arrangement as in Definition 2.1, but with $\bigcap_{i=1}^{d} S_i \neq \emptyset$, then we can apply elementary transformations (see [8, V, Exa. 5.7.1]) at each of the points in $\bigcap_{i=1}^{d} S_i$ to obtain a new arrangement of d sections in $\mathbb{P}_C(\mathcal{L}')$ for some $\mathcal{L}'$. After repeating this process a finite number of times, we arrive to an arrangement $\mathcal{A}'$ in $\mathbb{P}_C(\mathcal{L}')$ with $\bigcap_{i=1}^{d} S_i' = \emptyset$. If $\deg(\mathcal{L}') = 0$, then $\mathcal{L}' = \mathcal{O}_C$ since, as we showed above, $\mathcal{L}' = \mathcal{O}_C(D)$ for some effective divisor D. In this case, $\mathbb{P}_C(\mathcal{O}_C) = C \times \mathbb{P}^1$, and the arrangement is trivially formed by a collection of d "horizontal" fibers (it just corresponds to an arrangement of d points in $\mathbb{P}^1$). If $\mathcal{A} = \{S_1, \ldots, S_d\}$ is a collection of arbitrary d sections in $\mathbb{P}_C(\mathcal{L})$, we perform elementary transformations on the points in $C_0 \cap S_i$ for all i, and we repeat this process until all sections are disjoint from the new curve C_0' in $\mathbb{P}_C(\mathcal{L}')$ (proper transform of C_0). In this way, *arbitrary arrangements of sections* can always be considered, after some elementary transformations, as the ones in Definition 2.1.

We now define the morphisms between our objects.

Definition 2.2. Fix an integer $d \geq 3$. Let C, C' be nonsingular projective curves, and let $\mathcal{L}$, $\mathcal{L}'$ be invertible sheaves of positive degrees on C, C' respectively. Let $\mathcal{A}$, $\mathcal{A}'$ be arrangements of d sections in $\mathbb{P}_C(\mathcal{L})$, $\mathbb{P}_{C'}(\mathcal{L}')$ respectively. A *morphism of arrangements* is the existence of a finite map $g : C \to C'$, and a commutative diagram

$$\begin{array}{ccc} \mathbb{P}_C(\mathcal{L}) & \xrightarrow{G} & \mathbb{P}_{C'}(\mathcal{L}') \\ \pi \downarrow & & \downarrow \pi' \\ C & \xrightarrow{g} & C' \end{array}$$

so that $\mathbb{P}_C(\mathcal{L})$ is isomorphic to the base change by g, and $S_i = G^*(S_i')$ for all i. If g is an isomorphism, then the arrangements are said to be isomorphic.

In particular, a curve C with an automorphism g produces isomorphic arrangements via the pull back of g.

Lemma 2.1. *A morphism of arrangements satisfies* $C_0 = G^*(C_0')$ *and* $g^*(\mathcal{L}') \simeq \mathcal{L}$. *We have* $C_0^2 = \deg(g) C_0'^2$ *and* $S_i^2 = \deg(g) S_i'^2$ *for all* i.

PROOF. Since $0 = G^*(C_0').G^*(S_i') = G^*(C_0').S_i$, we have that $G^*(C_0') = C_0$. We know that $\pi_*(C_0) = \mathcal{O}_C \oplus \mathcal{L}^{-1}$ and $\pi'_*(C_0') = \mathcal{O}_{C'} \oplus \mathcal{L}'^{-1}$ (see [8, II, Proposition 7.11]). By flat base change [8, III, Proposition 9.3], we have

$$g^*\pi'_*(C_0') \simeq \pi_* G^*(C_0'),$$

and so $g^*(\mathcal{L}') \simeq \mathcal{L}$. Therefore, $\deg(\mathcal{L}) = \deg(g)\deg(\mathcal{L}')$, and so $C_0^2 = \deg(g)C_0'^2$, and $S_i^2 = \deg(g)S_i'^2$ for all i. □

One wants to consider arrangements of sections which do not come from others via base change, and so the following definition.

DEFINITION 2.3. Let us fix the data $(C, \mathcal{L}, d)$ as above. An arrangement of d sections $\mathcal{A}$ is said to be *primitive* if whenever we have an arrangement $\mathcal{A}'$ for some data $(C', \mathcal{L}', d)$, and a morphism g as in Definition 2.2, then g is an isomorphism. The set of *isomorphism classes* of primitive arrangements is denoted by $\mathscr{A}(C, \mathcal{L}, d)$. This is clearly independent of the isomorphism class of C and $\mathcal{L}$.

For instance, if $\mathcal{L}$ has a base-point, then $\mathscr{A}(C, \mathcal{L}, d) = \emptyset$.

EXAMPLE 2.2. Let $d \geq 3$ be an integer. An *arrangement of d lines* in the plane is a set of d labeled lines $\mathcal{A} = \{L_1, \dots, L_d\}$ in $\mathbb{P}^2$ such that $\bigcap_{i=1}^d L_i = \emptyset$. As in [21], we introduced ordered pairs $(\mathcal{A}, P)$, where $\mathcal{A}$ is an arrangement of d lines, and P is a point in $\mathbb{P}^2 \backslash \bigcup_{i=1}^d L_i$. If $(\mathcal{A}, P)$ and $(\mathcal{A}', P')$ are two such pairs, we say that they are isomorphic if there exists an automorphism T of $\mathbb{P}^2$ such that $T(L_i) = L_i'$ for every i, and $T(P) = P'$. Given $(\mathcal{A}, P)$, we blow up the point P to obtain an arrangement of d sections for the data $(\mathbb{P}^1, \mathcal{O}(1), d)$, and given such an arrangement of sections, we blow down C_0 to get a pair $(\mathcal{A}, P)$, where P is the image of C_0 and $\mathcal{A}$ is formed by the images of the sections. One sees that the set of pairs up to isomorphism of pairs is precisely $\mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$. By Lemma 2.1, any arrangement of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(1)) \to \mathbb{P}^1$ is primitive (degree considerations). This is the simplest case for arrangements of rational sections over curves. Notice that

$$\mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), 3) = (\{x_1 = 0, x_2 = 0, x_3 = 0\}, [1 : 1 : 1]),$$

where $[x_1 : x_2 : x_3]$ are coordinates for $\mathbb{P}^2$.

In the next sections, we will classify all primitive arrangements, and some distinguished subclasses which are defined through intersection properties of their members. We now look at these intersections. In what follows, until the end of this section, we fix the data $(C, \mathcal{L}, d)$.

DEFINITION 2.4. Let $\mathcal{A} = \{S_1, \dots S_d\}$ be an arrangement of sections in $\mathbb{P}_C(\mathcal{L})$. Let P be a point in $\mathbb{P}_C(\mathcal{L})$, and let f, g be local equations defining S_i, S_j at P. As in [8, V, Section 1], we define the *intersection multiplicity* $(S_i.S_j)_P$ of S_i and S_j at P to be the length of $\mathcal{O}_{P,\mathbb{P}_C(\mathcal{L})}/(f, g)$. If P is not in S_i or S_j, then $(S_i.S_j)_P = 0$. Notice that, since $S_i.S_j = e$, we have $0 \leq (S_i.S_j)_P \leq e$. We distinguish two classes of arrangements:

(t) We say that $\mathcal{A}$ is *transversal* if for any $i \neq j$ and any point P in $S_i \cap S_j$, there is $k \neq i, j$ such that $(S_i.S_k)_P = (S_i.S_j)_P - 1$. The set of isomorphism classes of primitive transversal arrangements is denoted by $\mathscr{A}_t(C, \mathcal{L}, d)$.
(s) We say that $\mathcal{A}$ is *simple crossing*[1] if for any $i \neq j$ and any point P in $S_i \cap S_j$, we have $(S_i.S_j) = 1$. This is, the members of the arrangement are pairwise transversal. The set of isomorphism classes of primitive simple crossing arrangements is denoted by $\mathscr{A}_s(C, \mathcal{L}, d)$.

REMARK 2.1. In (t) above, we have the requirement $(S_i.S_k)_P = (S_i.S_j)_P - 1$. This implies $(S_i.S_k)_P = (S_j.S_k)_P$, and so the definition is symmetric on i, j. To see this, let $\sigma : Bl_P(\mathbb{P}_C(\mathcal{L})) \to \mathbb{P}_C(\mathcal{L})$ be the blow-up at P. Let $\tilde{S}_a$ be the strict transforms of S_a, so that $\tilde{S}_a \sim \sigma^*(S_a) - E$, for $a = i, j, k$. Here E is the exceptional curve of σ. In this way, we have

$$\tilde{S}_a.\tilde{S}_b = S_a.S_b - 1$$

since S_a is nonsingular at P. Since σ is an isomorphism outside of E, we have that $(\tilde{S}_a.\tilde{S}_b)_{\tilde{P}} = (S_a.S_b)_P - 1$, where $\tilde{P} = \tilde{S}_a \cap E$. If $(S_i.S_j)_P = 2$, then $(S_i.S_k)_P = 1$, and so $(S_j.S_k)_P = 1$. One proves the general assertion by induction on $(S_i.S_j)_P$.

This gives the stratification

$$\mathscr{A}_s(C, \mathcal{L}, d) \subseteq \mathscr{A}_t(C, \mathcal{L}, d) \subseteq \mathscr{A}(C, \mathcal{L}, d).$$

Notice that for line arrangements $\mathscr{A}_s(\mathbb{P}^1, \mathcal{O}(1), d) = \mathscr{A}_t(\mathbb{P}^1, \mathcal{O}(1), d) = \mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$, but already for $(\mathbb{P}^1, \mathcal{O}(2), d)$ we have different sets, as the next example shows.

EXAMPLE 2.3. Consider collections of curves in $\mathbb{P}^2$ given by $A_i = \{C_1, C_2, C_3, C_4\}$, as shown in Figure 1. Here, C_1 is a conic, and C_2, C_3, C_4 are lines. For distinct i's, we have different intersections among C_j's. Each A_i

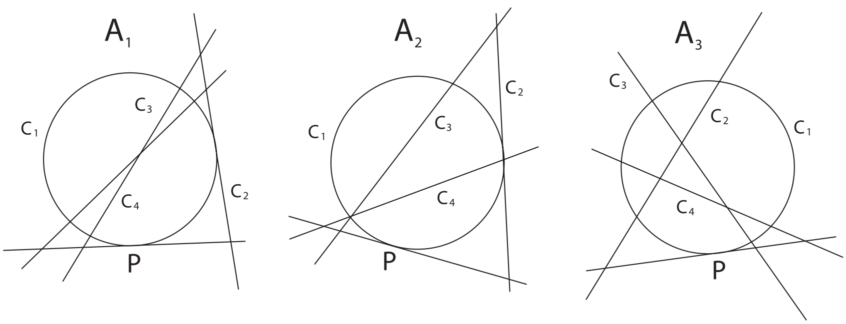


Figure 1. Configurations of curves in $\mathbb{P}^2$ which produce arrangements in $\mathbb{F}_2$.

[1] These are the type of singularities for arrangements in [22].

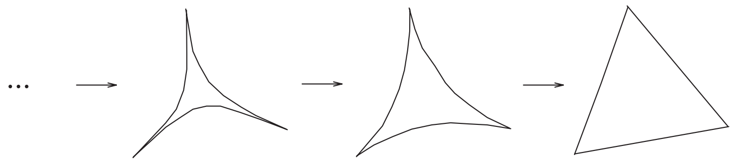


Figure 2. Evolution of a triangle under Frobenius maps in Example 2.4.

has a marked point P in C_1. Out of these configurations, we produce three arrangements of sections in $\mathbb{F}_2$. We blow up P, and then we perform an elementary transformation at $\tilde{P}$, which is the intersection of the strict transform of C_1 with the exceptional divisor E. Then, we have an arrangement of sections $\mathcal{A}_i = \{S_1, S_2, S_3, S_4\}$ in $\mathbb{F}_2$, where S_j corresponds to the strict transform of C_j.

Any possible morphism of arrangements, from $\mathcal{A}_i$ to some $\mathcal{A}_i'$, would have $(\mathbb{P}^1, \mathcal{O}(1), 4)$ as target, and the degree of g would have to be 2. For $\mathcal{A}_1$, we have 8 points in $\mathbb{F}_2$ where exactly two sections intersect, and 1 where exactly 3 intersect, so $\mathcal{A}_1$ is impossible as pull-back of 4 sections in $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(1))$. Similar arguments apply to $\mathcal{A}_2$ and $\mathcal{A}_3$, and so one easily checks that all of them a primitive. Notice that $\mathcal{A}_3$ is simple crossing, $\mathcal{A}_2$ is a transversal, and $\mathcal{A}_1$ is neither, so $\mathscr{A}_s(\mathbb{P}^1, \mathcal{O}(2), 4) \subsetneq \mathscr{A}_t(\mathbb{P}^1, \mathcal{O}(2), 4) \subsetneq \mathscr{A}(\mathbb{P}^1, \mathcal{O}(2), 4)$.

Example 2.4. Assume $\mathbb{K}$ has $\mathrm{char}(\mathbb{K}) = p > 0$. Let $\mathcal{A}' \in \mathscr{A}_s(C', \mathcal{L}', d)$, and consider the $\mathbb{K}$-linear Frobenius morphism $g : C \to C'$ [8, p. 301], so C and C' are isomorphic as abstract curves. Let $\mathcal{A} \in \mathscr{A}(C, \mathcal{L}, d)$ be the pull-back arrangement by g, as in Definition 2.2. Then, $g^*(\mathcal{L}') = \mathcal{L}$, and for any two members S_i, S_j we have $(S_i.S_j)_P = p$ when $P \in S_i \cap S_j$. The simple crossing arrangement $\mathcal{A}'$ is transformed into an arrangement $\mathcal{A}$ where any two members are tangent at e points, each of order p.

3. Some facts about $\overline{M}_{0,d+1}$

Let $d \geq 3$ be a integer. We denote by $\overline{M}_{0,d+1}$ the moduli space of $(d+1)$-pointed stable curves of genus zero [13, 11]. This is a smooth rational projective variety of dimension $d-2$. The open set $M_{0,d+1}$ parametrizes configurations of $d+1$ distinct labeled points in $\mathbb{P}^1$. The boundary $\Delta := \overline{M}_{0,d+1} \setminus M_{0,d+1}$ is formed by the following prime divisors: for each subset $I \subset \{1, 2, \ldots, d+1\}$ with $|I| \geq 2$ and $|I^c| \geq 2$, we let $\delta_I \hookrightarrow \overline{M}_{0,d+1}$ be the divisor whose generic element is a curve with two components: the points marked by I in one, and the points marked by I^c in the other. Hence $\delta_I = \delta_{I^c}$, and usually we will assume $d+1 \in I$ to avoid repetitions. These divisors are smooth, and $\Delta = \sum \delta_I$ is a simple normal crossing. The variety $\overline{M}_{0,d+1}$ represents a fine moduli space, proper and smooth over $\mathrm{Spec}(\mathbb{Z})$. For $i \in \{1, \ldots, d+2\}$, the i-th forgetful map $\pi_i : \overline{M}_{0,d+2} \to \overline{M}_{0,d+1}$, which forgets the i-th marked point and stabilizes, gives a universal family. We will mainly consider

$$\pi_{d+2} : \overline{M}_{0,d+2} \to \overline{M}_{0,d+1}.$$

It has $d+1$ distinguished sections $\delta_{1,d+2}, \ldots, \delta_{d+1,d+2}$, producing the markings on the parametrized curves.

Definition 3.1. Let X be a nonsingular projective variety, and let D be a nonsingular divisor in X. Let B be a curve in X. We say that B is *transversal* to D if locally at any $x \in D \cap B$, the curve B can be factored in $B_1, \ldots, B_n$ distinct local irreducible curves (branches of B in $\hat{\mathcal{O}}_{x,X}$) so that $B_i.D = 1$ for every i. If D is a sum of nonsingular divisors D_j, we say that B is transversal to D if it is to each D_j.

Below a well-known property for stable families, coming from the construction of $\overline{M}_{0,d+1}$.

Lemma 3.1. *Let x be a $\mathbb{K}$-point in $\overline{M}_{0,d+1}$. Let B be a local curve passing through x, i.e., B is a irreducible curve defined by functions in $\hat{\mathcal{O}}_{x,\overline{M}_{0,d+1}} \simeq \mathbb{K}[[t_1, \ldots, t_{d-2}]]$. Assume $t_1 t_2 \ldots t_k = 0$ defines $\Delta = \sum \delta_I$, so $k \leq d-2$, and that $t_j|_B$ is not identically zero for all $1 \leq j \leq k$. Consider the commutative diagram*

$$\begin{array}{ccc} R & \longrightarrow & \overline{M}_{0,d+2} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi_{d+2}} \\ \bar{B} & \xrightarrow{\ \iota\ } & \overline{M}_{0,d+1} \end{array}$$

where ι is the composition of the inclusion of B with its normalization, so $\bar{B}$ is the normalization of B, and R is defined by base change. Then, the surface R is normal, and can only have singularities of the form

$$Spec\, \mathbb{K}[u, v, t]/(uv - t^m)$$

at the nodes of the singular fiber, for some m. Moreover, the surface R is nonsingular if and only if B is transversal to Δ.

A brief outline of the proof. Let $X \to \operatorname{Spec} \mathbb{K}$ be the corresponding stable curve over $t_1 = t_2 = \cdots = t_{d-2} = 0$. Consider the deformation of X as described in [7, pp. 79–85]. At a nonsingular point of X, we have a nonsingular point for R, so we pay only attention to the nodes of X. Let y be a node of X, corresponding to the intersection of B with t_i for some $i \in \{1, \ldots, k\}$, i.e., the node y splits $\{1, \ldots, d+1\}$ in two subsets I and I^c, and $t_i = 0$ corresponds to δ_I. At the corresponding point y in $\overline{M}_{0,d+2}$ (over $\mathbb{K}$), the local rings and the universal map can be written in the projection form

$$\hat{\mathcal{O}}_{x,\overline{M}_{0,d+1}} = \mathbb{K}[[t_1, \ldots, t_{d-2}]] \to \hat{\mathcal{O}}_{y,\overline{M}_{0,d+2}} = \mathbb{K}[[u_i, v_i, t_1, t_2, \ldots, t_{d-2}]]/(u_i v_i - t_i),$$

for suitable variables u_i, v_i. Now, the composition of the inclusion of B with its normalization ι has the form

$$\iota(t) = (u_1 t^{m_1}, u_2 t^{m_2}, \ldots, u_{d-2} t^{m_{d-2}})$$

for some units u_i's and a local parameter t on $\bar{B}$. This is because B is not in $t_j = 0$ for all j. Hence, $\iota^*(\delta_I) = u_i t^{m_i}$ for $\delta_I = \{t_i = 0\}$. Since R is defined through the base change by ι, we have the isomorphism

$$\hat{\mathcal{O}}_{y,R} \simeq \mathbb{K}[[u_i, v_i, t]]/(u_i v_i - t^{m_i}),$$

and so, R is nonsingular iff $m_i = 1$ for all $i \in \{1, \ldots, k\}$, i.e., transversal to Δ.

The moduli spaces $\overline{M}_{0,d+1}$ have a beautiful construction, due to Kapranov [11, 12], as iterated blow-ups of $\mathbb{P}^{d-2}$ (see below). It follows that curves in $M_{0,d+1}$ are strict transforms of curves in $\mathbb{P}^{d-2}$, which are not contained in a certain fixed hyperplane arrangement $\mathcal{H}_d$. The following description of these spaces can be found in [11, 12].

Definition 3.2. A *Veronese curve* is a rational normal curve of degree $d-2$ in $\mathbb{P}^{d-2}$, i.e., a curve projectively equivalent to $\mathbb{P}^1$ in its Veronese embedding.

It is a classical fact that any $d+1$ points in $\mathbb{P}^{d-2}$ in general position lie on a unique Veronese curve. The points $P_1, \ldots, P_{n+2}$ are said to be in general position if no $n+1$ of them lie in a hyperplane. The main theorem in [11] says that the set of Veronese curves in $\mathbb{P}^{d-2}$ and its closure are isomorphic to $M_{0,d}$ and $\overline{M}_{0,d}$ respectively.

Theorem 3.2 (Kapranov [11]). *Take d points $P_1, \ldots, P_d$ of projective space $\mathbb{P}^{d-2}$ which are in general position. Let $V_0(P_1, \ldots, P_d)$ be the space of all Veronese curves in $\mathbb{P}^{d-2}$ through these d points P_i. Consider it as a subvariety in the Hilbert scheme $\mathcal{H}$ parametrizing all subschemes on $\mathbb{P}^{d-2}$. Then,*

1. *We have $V_0(P_1, \ldots, P_d) \cong M_{0,d}$.*
2. *If $V(P_1, \ldots, P_d)$ is the closure of $V_0(P_1, \ldots, P_d)$ in $\mathcal{H}$, then $V(P_1, \ldots, P_d) \cong \overline{M}_{0,d}$. The subschemes representing limit positions of curves from $V_0(P_1, \ldots, P_d)$ are, considered together with P_i, stable d-pointed curves of genus* 0, *which represent the corresponding points of $\overline{M}_{0,d}$.*
3. *The analogs of statements* (a) *and* (b) *hold also for Chow variety instead of Hilbert scheme.*

Theorem 3.3 (Kapranov [12]). *Choose d general points $P_1, \ldots, P_d$ in $\mathbb{P}^{d-2}$. The variety $\overline{M}_{0,d+1}$ can be obtained from $\mathbb{P}^{d-2}$ by a series of blow-ups of all the projective spaces spanned by P_i. The order of these blow ups can be taken as follows:*

1. *Points $P_1, \ldots, P_{d-1}$ and all the projective subspaces spanned by them in order of the increasing dimension;*
2. *The point P_d, all the lines $\langle P_1, P_d \rangle, \ldots, \langle P_{d-2}, P_d \rangle$ and subspaces spanned by them in order of the increasing dimension;*
3. *The line $\langle P_{d-1}, P_d \rangle$, the planes $\langle P_i, P_{d-1}, P_d \rangle$, $i \neq d-2$ and all subspaces spanned by them in order of the increasing dimension, etc, etc.*

Let us denote the Kapranov's map in Theorem 3.3 by $\psi_{d+1} : \overline{M}_{0,d+1} \to \mathbb{P}^{d-2}$.

Some conventions and notations for the rest of the paper. Let us fix d points in general position in $\mathbb{P}^{d-2}$. We take $P_1 = [1 : 0 : \ldots : 0]$, $P_2 = [0 : 1 : 0 : \ldots : 0], \ldots, P_{d-1} = [0 : \ldots : 0 : 1]$ and $P_d = [1 : 1 : \ldots : 1]$. The symbol $\langle Q_1, \ldots, Q_r \rangle$ denotes the projective linear space spanned by the points Q_i. Let

$$\Lambda_{i_1,\ldots,i_r} = \langle P_j : j \notin \{i_1, \ldots, i_r\} \rangle$$

where $1 \le r \le d-1$ and $i_1, \ldots, i_r$ are distinct numbers, and let $\mathscr{H}_d$ be the union of all the hyperplanes $\Lambda_{i,j}$. Hence, $\Lambda_{i,j} = \{[x_1 : \ldots : x_{d-1}] \in \mathbb{P}^{d-2} : x_i = x_j\}$ for $i, j \neq d$, $\Lambda_{i,d} = \{[x_1 : \ldots : x_{d-1}] \in \mathbb{P}^{d-2} : x_i = 0\}$ and

$$\mathscr{H}_d = \left\{ [x_1 : \ldots : x_{d-1}] \in \mathbb{P}^{d-2} : x_1 x_2 \ldots x_{d-1} \prod_{i<j} (x_j - x_i) = 0 \right\}.$$

Example 3.1. For $d = 4$, Theorem 3.3 says that the map $\psi_5 : \overline{M}_{0,5} \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at the points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, $P_3 = [0 : 0 : 1]$, and $P_4 = [1 : 1 : 1]$. The hyperplane arrangement $\mathscr{H}_4$ is given by the complete quadrilateral

$$x_1 x_2 x_3 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = 0.$$

The universal map $\pi_5 : \overline{M}_{0,5} \to \overline{M}_{0,4} = \mathbb{P}^1$ is induced by the pencil of conics (Veronese curves in $\mathbb{P}^2$) containing P_1, P_2, P_3, and P_4.

4. Arrangements of d sections and curves in $M_{0,d+1}$

Let B be an irreducible projective curve in $\overline{M}_{0,d+1}$ with $B \cap M_{0,d+1} \neq \emptyset$. By using Kapranov's map $\psi_{d+1} : \overline{M}_{0,d+1} \to \mathbb{P}^{d-2}$, this is the same as giving an irreducible projective curve A in $\mathbb{P}^{d-2}$ not contained in $\mathscr{H}_d$. The proper transform of A under ψ_{d+1} is B. Consider the base change diagram

$$\begin{array}{ccc} R & \longrightarrow & \overline{M}_{0,d+2} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi_{d+2}} \\ \bar{B} & \xrightarrow{\ \imath\ } & \overline{M}_{0,d+1} \end{array}$$

where $\imath$ is the composition of the inclusion of B with its normalization. Let us denote $\bar{B} = C$. Notice that the distinguished sections $\delta_{1,d+1}, \delta_{2,d+1}, \ldots, \delta_{d+1,d+1}$ of π_{d+2} induce $d+1$ sections $\tilde{S}_{1,d+2}, \ldots, \tilde{S}_{d+1,d+2}$ for ρ. Also, by Lemma 3.1, the surface R is a normal projective surface with only canonical singularities of type $uv = t^m$ for various integers m, and only at nodes of singular fibers. We now resolve these singularities minimally to obtain a fibration $\tilde{\rho} : \tilde{R} \to C$, so that $\tilde{R}$ is nonsingular. Notice that $\tilde{\rho}$ has only reduced trees of $\mathbb{P}^1$'s as fibers, and it has $d+1$ distinguished sections.

Let F be a singular fiber of $\tilde{\rho}$. Consider the curves E in F with $E.(F-E)=1$, and which do not intersect the $(d+1)$-th section (the proper transform of $\tilde{S}_{d+1,d+2}$). Then, the E's are disjoint with self-intersection -1. We now blow down all of these E's to obtain a new fibration over C with $d+1$ distinguished sections, and reduced trees of $\mathbb{P}^1$'s as fibers. If there is a singular fiber F, we repeat the previous procedure. After finitely many steps, this stops in a fibration $\rho_0 : R_0 \to C$ with nonsingular fibers, and $d+1$ labeled sections $\{S_1, S_2, \ldots, S_{d+1}\}$, where $\tilde{S}_{i,d+2}$ is the proper transform of S_i.

Proposition 4.1. *The fibration $\rho_0 : R_0 \to C$ is isomorphic over C to $\pi : \mathbb{P}_C(\mathcal{L}) \to C$ with $\mathcal{L} \simeq \imath^*(\psi_{d+1}^*(\mathcal{O}_{\mathbb{P}^{d-2}}(1)))$, and so* $\deg(\mathcal{L}) = e = \deg(\psi_{d+1}(B))$. *The labeled set $\{S_1, \ldots, S_d\}$ is a primitive arrangement of d sections.*

Proof. By [8, V, Proposition 2.8], the ruled surface $\rho_0 : R_0 \to C$ is isomorphic over C to $\mathbb{P}_C(\mathcal{E}) \to C$, where $\mathcal{E}$ is a rank two locally free sheaf on C with the property that $H^0(\mathcal{E}) \neq 0$ but for all invertible sheaves $\mathcal{M}$ on C with $\deg \mathcal{M} < 0$, we have $H^0(\mathcal{E} \otimes \mathcal{M}) = 0$. So, we assume $R_0 = \mathbb{P}_C(\mathcal{E})$. Since, by construction, $S_i.S_{d+1} = 0$ for all $i \neq d+1$, we have that $\mathcal{E}$ is decomposable, say $\mathcal{E} \simeq \mathcal{O}_C \oplus \mathcal{L}^{-1}$, where $\mathcal{L}$ is unique up to isomorphism. Moreover, $S_i \sim S_{d+1} + \pi^*(\mathcal{L})$ for all $i \neq d+1$. In particular, $0 < S_i.S_j = e = \deg \mathcal{L}$ for all $i, j \neq d+1$ and $S_{d+1}^2 = -e$. Notice that $\bigcap_{i=1}^d S_i = \emptyset$. Hence, $\mathcal{A} = \{S_1, \ldots, S_d\}$ is an arrangement of d sections of $\pi : \mathbb{P}_C(\mathcal{L}) \to C$.

Observe that $\mathcal{A}$ is primitive because it comes from the normalization of a curve in $\overline{M}_{0,d+1}$. Assume there is a morphism of arrangements from $\mathcal{A}$ to $\mathcal{A}'$, with data $(C', \mathcal{L}', d)$ and map $g : C \to C'$ (see Definition 2.2). Then, our map $\imath : C \to B$ would factor through g, induced by the natural map $\imath' : C' \to \overline{M}_{0,d+1}$. This is possible only if $\deg g = 1$, because C is the normalization of B, and so $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic arrangements.

Let c be a point in C, and consider the fiber F_c of π. Let $S_i, S_j \neq S_{d+1}$ be distinct sections which intersect at a point P in F_c. Then, through the description in Lemma 3.1, it is not hard to see that

$$(S_i.S_j)_P = C_{\text{loc}}. \sum_{\text{all } I \text{ with } i,j \in I \setminus \{d+1\}} \delta_I$$

where C_{loc} is the corresponding local branch of B at $\imath(c)$.

Let $\{P_1, \ldots, P_m\} = S_i \cap S_j$. Let $\sigma_j : C \to \mathbb{P}_C(\mathcal{L})$ be the section corresponding to S_j. Then, $\mathcal{L} \simeq \sigma_j^*(\pi^*(\mathcal{L}) \otimes \mathcal{O}_{S_j}) \simeq \sigma_j^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{L})}(S_i) \otimes \mathcal{O}_{S_j}) \simeq \mathcal{O}_C(\sum_{k=1}^m (S_i.S_j)_{P_k} \pi(P_k))$, and so, because of the previous formula, $\mathcal{L} \simeq \imath^*(\sum_{\text{all } I \text{ with } i,j \in I \setminus \{d+1\}} \delta_I)$. Now, by Kapranov's description in Theorem 3.3, we have

$$\psi_{d+1}^*(\mathcal{O}_{\mathbb{P}^{d-2}}(1)) \simeq \sum_{\text{all } I \text{ with } i,j \in I \setminus \{d+1\}} \delta_I,$$

and so $\mathcal{L} \simeq \imath^*(\psi_{d+1}^*(\mathcal{O}_{\mathbb{P}^{d-2}}(1)))$. This comes from the pull-back of the hyperplane $\Lambda_{i,j}$. By the projection formula, we have

$$\deg(\mathcal{L}) = B.\psi_{d+1}^*(\mathcal{O}_{\mathbb{P}^{d-2}}(1)) = \psi_{d+1_*}(B).\mathcal{O}_{\mathbb{P}^{d-2}}(1) = \deg \psi_{d+1}(B). \qquad \square$$

5. The one-to-one correspondences

Fix an integer $d \geq 3$, and an algebraically closed field $\mathbb{K}$.

Theorem 5.1. *We have*

$$\bigsqcup_{C,\mathcal{L}} \mathscr{A}(C,\mathcal{L},d) \equiv \{\textit{irreducible curves in } M_{0,d+1}\},$$

where the disjoint union is over all nonsingular projective curves C and line bundles $\mathcal{L}$ on C (both up to isomorphism). This equality gives a bijection between $\mathscr{A}(C,\mathcal{L},d)$ and the set of irreducible projective curves B in $\overline{M}_{0,d+1}$ with $M_{0,d+1} \cap B \neq \emptyset$, whose normalization is C and $\mathcal{L} \simeq \imath^*(\psi_{d+1}^*(\mathcal{O}_{\mathbb{P}^{d-2}}(1)))$, where $\imath : C \to B$ is the composition of the inclusion of B and its normalization.

Proof. Let B be an irreducible curve in $\overline{M}_{0,d+1}$ with $B \cap M_{0,d+1} \neq \emptyset$. By Proposition 4.1, B produces a unique element in

$$\mathscr{A}(\bar{B}, \imath^*(\psi_{d+1}^*(\mathcal{O}_{\mathbb{P}^{d-2}}(1))), d),$$

where $\imath$ is the composition of the inclusion of B and its normalization. In this way, we only need to prove that given $\mathcal{A} \in \mathscr{A}(C,\mathcal{L},d)$, there is an irreducible curve B in $\overline{M}_{0,d+1}$ intersecting $M_{0,d+1}$ so that $\mathcal{A}$ is induced by B as in Proposition 4.1.

Let $\mathcal{A} = \{S_1, \ldots, S_d\}$ be a primitive arrangement of d sections of $\pi : \mathbb{P}_C(\mathcal{L}) \to C$. The section C_0 is denoted by S_{d+1}. We repeatedly perform blow-ups at the intersections of the sections S_i and their proper transforms, until they are all disjoint. We do this in a minimal way, that is, given a (-1)-curve in a fiber, its blow-down produces an intersection of the distinguished sections. The corresponding fibration $\tilde{T} \to C$ has $(d+1)$-pointed semi-stable genus zero curves as fibers. The $d+1$ markings are produced by intersecting the proper transforms of the sections S_i's with the fibers. They may fail to be stable exactly because of the presence of fibers with $\mathbb{P}^1$'s having no markings, and intersecting the rest of the fiber at two points. These components form chains of $\mathbb{P}^1$'s, which we blow down to obtain a $(d+1)$-pointed stable family of genus zero curves $T \to C$. Therefore, we have a commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & \overline{M}_{0,d+2} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_{d+2}} \\ C & \longrightarrow & \overline{M}_{0,d+1} \end{array}$$

so that $T \simeq C \times_{\overline{M}_{0,d+1}} \overline{M}_{0,d+2}$. Notice that the image of C is a curve B because $\bigcap_{i=1}^{d} S_i = \emptyset$ (so not a point), and B intersects $M_{0,d+1}$. Let $\bar{B}$ be the normalization of B, and let $\imath : \bar{B} \to \overline{M}_{0,d+1}$ be the corresponding map. Then, the diagram above factors as

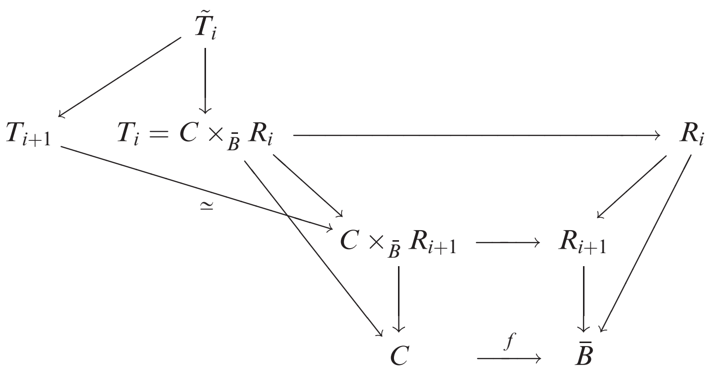
$$\begin{array}{ccccc} T & \longrightarrow & R & \longrightarrow & \overline{M}_{0,d+2} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_{d+2}} \\ C & \xrightarrow{f} & \bar{B} & \xrightarrow{\imath} & \overline{M}_{0,d+1} \end{array}$$

where R is given by pull-back, and $T \simeq C \times_{\bar{B}} R$. Let $\tilde{R}$ be the minimal resolution of the singularities of R. Let us finally consider the commutative diagram

$$\begin{array}{ccc} T_0 & \longrightarrow & \tilde{R} \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & \bar{B} \end{array}$$

where $T_0 \simeq C \times_{\bar{B}} \tilde{R}$ (it may be singular). The pull-back of the $d+1$ distinguished sections are the $d+1$ distinguished sections of $T_0 \to C$. We now inductively blow-down all (-1)-curves on the fibers of $\tilde{R} \to \bar{B}$ in the following way.

Let $R_i \to \bar{B}$ be the fibration produced in the i-th step, where $\tilde{R} = R_0$. Then, $T_i \simeq C \times_{\bar{B}} R_i$. We obtain the fibration $R_{i+1} \to \bar{B}$ through the commutative diagram below.



Let E be a (-1)-curve in a fiber of $R_i \to \bar{B}$, and let P be the point of intersection with the rest of the fiber. Notice that at least two distinguished sections U and V intersect E (not at P, of course). Let R_{i+1} be the blow-down of E, and $R_{i+1} \to \bar{B}$ be the corresponding fibration. Let Q be the pre-image of P and F the pre-image of E in T_i. Notice that T_i may be singular at Q, say with a singularity of type $xy = t^a$. If $a > 1$ we resolve Q to get $\tilde{T}_i$. Then we define T_{i+1} to be the blow-down of the total transform of F in $\tilde{T}_i$ (this is a chain of (-1)-curves). Let us consider P in R_{i+1}, and its pre-image in $C \times_{\bar{B}} R_{i+1}$, say Q'. Now, $C \times_{\bar{B}} R_{i+1}$ is nonsingular at Q', and there is a morphism $T_{i+1} \to C \times_{\bar{B}} R_{i+1}$, which is clearly an isomorphism.

For R_i, these procedure is what we have in Section 4. When it stops, say at the m-th step, we have $\pi' : R_m = \mathbb{P}_{\bar{B}}(\mathcal{L}') \to \bar{B}$ with $d+1$ distinguished sections, and T_m nonsingular $\mathbb{P}^1$-bundle over C. Moreover, because of the construction of T_i's, we have $T_m \simeq \mathbb{P}_C(\mathcal{L})$, and the arrangement $\mathcal{A}$ is the pull-back of the one in $\pi' : \mathbb{P}_{\bar{B}}(\mathcal{L}') \to \bar{B}$. So this is a morphism of arrangements as in Definition 2.2. But $\mathcal{A}$ primitive implies $\deg f = 1$, and so $C = \bar{B}$. □

For example, one has $\bigsqcup_{C,\mathcal{L}} \mathscr{A}(C,\mathcal{L},3) = \mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), 3)$ which corresponds to the unique curve in $M_{0,4} = \mathbb{P}^1 \backslash \{[0:1],[1:1],[1:0]\}$. An immediate corollary is the following.

COROLLARY 5.2. *Given a nonsingular projective curve C and a line bundle $\mathcal{L}$ on C, the Kapranov's map $\psi_{d+1} : \overline{M}_{0,d+1} \to \mathbb{P}^{d-2}$ gives a one-to-one correspondence between elements of $\mathscr{A}(C,\mathcal{L},d)$ and irreducible projective curves A in $\mathbb{P}^{d-2}$ not contained in $\mathscr{H}_d$ such that $\bar{A} = C$ and $\mathcal{L} \simeq \imath^*(\psi_{d+1}^*(\mathcal{O}_{\mathbb{P}^{d-2}}(1)))$. In particular,* $\deg A = \deg \mathcal{L}$.

COROLLARY 5.3. $\mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d) \equiv \{$*lines in* $\mathbb{P}^{d-2}$ *not in* $\mathscr{H}_d\}$.

PROOF. A curve of degree one in $\mathbb{P}^{d-2}$ is a line. □

COROLLARY 5.4. $\mathscr{A}(\mathbb{P}^1, \mathcal{O}(2), d) \equiv \{$*conics in* $\mathbb{P}^{d-2}$ *not in* $\mathscr{H}_d\}$.

PROOF. An irreducible curve of degree two is a conic. □

The next two corollaries identify precisely the two distinguished classes of arrangements in Definition 2.4.

COROLLARY 5.5.

$$\bigsqcup_{C,\mathcal{L}} \mathscr{A}_t(C,\mathcal{L},d) \equiv \{\textit{irreducible curves in } \overline{M}_{0,d+1} \textit{ transversal to } \Delta\},$$

where the disjoint union is over all nonsingular projective curves C and line bundles $\mathcal{L}$ on C, both up to isomorphism.

PROOF. Let $\mathcal{A} \in \mathscr{A}_t(C,\mathcal{L},d)$. Let P be a singular point of the reducible curve defined by $\mathcal{A}$ in $\mathbb{P}_C(\mathcal{L})$. Let F_c be the fiber containing P. Hence, since $\mathcal{A}$ satisfies (t) in Definition 2.4, there are two transversal sections S_i, S_j containing P, i.e., $(S_i.S_j)_P = 1$. Consider the blow-up at P, $\mathrm{Bl}_P(\mathbb{P}_C(\mathcal{L})) \to \mathbb{P}_C(\mathcal{L})$, and let E be the exceptional curve. Then, E has at least three special distinct points: the intersections with $\tilde{F}_c$, $\tilde{S}_i$, and $\tilde{S}_j$ (corresponding proper transforms). Now, it is clear that the final *stable* fibration $R \to C$ produced from $\mathcal{A}$ has the proper transform of E as a component of the fiber over c. Let $\tilde{\mathcal{A}} = \{\tilde{S}_1, \ldots, \tilde{S}_d\}$ be the collection of proper transforms of S_i's in $\mathrm{Bl}_P(\mathbb{P}_C(\mathcal{L}))$. Then, $\tilde{\mathcal{A}}$ satisfies property (t) in

Definition 2.4 (extending naturally this definition). So, we repeat the blow-ups until all sections are disjoint (in a minimal way) to obtain the stable fibration $R \to C$, where no blow-downs are needed. Since R is a nonsingular surface, the curve $\imath(C)$ is transversal to Δ by Lemma 3.1.

Now assume $\mathcal{A}$ is not in $\mathscr{A}_t(C, \mathcal{L}, d)$. Then, there are indices i, j and a point $P \in S_i \cap S_j$ such that $n = \max\{(S_i.S_k)_P : (S_i.S_k)_P \le (S_i.S_j)_P - 1\} < (S_i.S_j)_P - 1$. Let $(S_i.S_j)_P = m$, so $0 \le n \le m-2$. We blow up n times the corresponding point in $\tilde{S}_i \cap \tilde{S}_j$ for the successive proper transforms of S_i and S_j. Let X be the resulting surface, and $\tilde{P} = \tilde{S}_i \cap \tilde{S}_j$. Let E be the exceptional curve of the blow-up at $\tilde{P}$. Then, E has only two special points: the intersection with the rest of the fiber and with the section $\tilde{S}_i$. Notice that $\tilde{S}_i \cap \tilde{S}_j \neq \emptyset$ at E. So, in the process to obtain the corresponding stable fibration $R \to C$, we need to blow up again at $\tilde{S}_i \cap \tilde{S}_j$, and so at the end the proper transform of E will have to be blown down (in order to have a stable fibration). Therefore, R is singular, and by Lemma 3.1, $\imath(C)$ is not transversal to Δ. By Theorem 5.1, we have checked all irreducible curves in $M_{0,d+1}$. □

Corollary 5.6.

$$\bigsqcup_{C,\mathcal{L}} \mathscr{A}_s(C, \mathcal{L}, d) \equiv \{\textit{irreducible curves in } \mathbb{P}^{d-2} \textit{ transversal to } \mathscr{H}_d\},$$

where the disjoint union is over all nonsingular projective curves C and line bundles $\mathcal{L}$ on C, both up to isomorphism.

Proof. Let $\mathcal{A} \in \mathscr{A}_s(C, \mathcal{L}, d)$, and consider its stable fibration $\rho : R \to C$. We know that the image of C in $\overline{M}_{0,d+1}$ is transversal to Δ by the previous corollary. Let $c \in C$ be a point whose fiber is singular. Then there exists an element in $\mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$ that produces the same fiber. By Corollary 5.3, the set $\mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$ is in one-to-one correspondence with lines in $\mathbb{P}^{d-2}$ not in $\mathscr{H}_d$. Therefore, there is a line in $\mathbb{P}^{d-2}$ not in $\mathscr{H}_d$ passing through $\psi_{d+1}(\imath(c)) \in \mathbb{P}^{d-2}$. This implies that the image of $\imath(C)$ under ϕ_{d+1} is transversal to $\mathscr{H}_d$. The converse is clear using the same correspondence with lines. □

6. Producing explicit primitive arrangements

In the previous section we classified all arrangements of d sections (and two distinguished subclasses). They are in one-to-one correspondence with curves in $\mathbb{P}^{d-2}$ outside of a certain fixed hyperplane arrangement $\mathscr{H}_d$ (Theorem 5.1). In [21], we used this correspondence to explicitly find new special line arrangements in $\mathbb{P}^2$. For this, we computed the corresponding line as in Corollary 5.3. In general, it is hard to present a curve in $\mathbb{P}^{d-2}$ in the form we need to construct its corresponding arrangement. In this brief section, we show a simple way to produce arrangements via irreducible curves in $\mathbb{P}^2$. This is based on [6, Section 7], where Castravet and Tevelev describe how to cover $M_{0,d+1}$ with blow-ups of $\mathbb{P}^2$ at $d+1$ points.

PROPOSITION 6.1 [6, Proposition 7.3]. *Suppose $p_1, \ldots, p_{d+1}$ are distinct points in $\mathbb{P}^2$, and let $U \subset \mathbb{P}^2$ be the complement to the union of lines containing at least two of them. The morphism*

$$\theta : U \to M_{0,d+1}$$

obtained by projecting $p_1, \ldots, p_{d+1}$ from points of U extends to the morphism

$$\theta : Bl_{p_1,\ldots,p_{d+1}}\mathbb{P}^2 \to \overline{M}_{0,d+1}.$$

If there is no (probably reducible) conic through $p_1, \ldots, p_{d+1}$ then θ is a closed embedding. In this case the boundary divisors δ_I of $\overline{M}_{0,d+1}$ pull-back as follows: for each line $L_I := \langle p_i \rangle_{i \in I} \subset \mathbb{P}^2$, we have $\theta^*(\delta_I) = \tilde{L}_I$ (the proper transform of L_I) and (assuming $|I| \geq 3$), $\theta^*(\delta_{I \setminus \{k\}}) = E_k$, where $k \in I$ and E_k is the exceptional divisor over p_k. Other boundary divisors pull-back trivially.

In this way, we have

$$\theta^*(\psi_{d+1}^*(\mathcal{O}(1))) = (n_{d+1} - 1)H - (n_{d+1} - 2)E_{d+1} - \sum_{i=1}^{d} \varepsilon_i E_i$$

where H is the class of a general line in $\mathbb{P}^2$, n_{d+1} is the number of lines in $\mathbb{P}^2$ passing through p_{d+1} and some other p_j, and $\varepsilon_i = 0$ if there is a p_k in $\langle p_{d+1}, p_i \rangle$ $k \neq i, d+1$ or $\varepsilon_i = 1$ otherwise. Hence, the image of $Bl_{p_1,\ldots,p_{d+1}}\mathbb{P}^2$ under $\psi_{d+1} \circ \theta$ is a surface S in $\mathbb{P}^{d-2}$ of degree $2n_{d+1} - 3 - \sum_{i=1}^{d} \varepsilon_i$, and so

$$2 \leq \deg(S) \leq d - 3.$$

Therefore, S is a surface of minimal degree in some $\mathbb{P}^{\deg(S)+1} \subset \mathbb{P}^{d-2}$. Thus S is either a rational normal scroll in $\mathbb{P}^{\deg(S)+1}$ or the Veronese of $\mathbb{P}^2$ in $\mathbb{P}^5$. Moreover, S is smooth. One can check that ψ_{d+1} blows down certain d (-1)-curves in $Bl_{p_1,\ldots,p_{d+1}}\mathbb{P}^2$ (proper transforms of lines $\langle p_{d+1}, p_i \rangle$ with $\varepsilon_i = 1$, and E_i with $\varepsilon_i = 0$) having as result a Hirzebruch surface $\mathbb{F}_m$, where m depends on the configuration of points p_i such that $\varepsilon_i = 1$.

Given $p_1, \ldots, p_{d+1}$ points in $\mathbb{P}^2$, with no (probably reducible) conic through them, we consider an irreducible plane curve Γ not included in the union of lines containing $p_1, \ldots, p_{d+1}$. Then, by Proposition 6.1, we have the inclusion $\theta : B := \tilde{\Gamma} \hookrightarrow \overline{M}_{0,d+1}$ and so a primitive arrangement $\mathcal{A}$ in $\mathscr{A}(\bar{B}, \mathcal{L}, d)$ for some line bundle $\mathcal{L}$, by Theorem 5.1. The line bundle $\mathcal{L}$ depends on the specific configuration $p_1, \ldots, p_{d+1}$ and the position of the curve Γ with respect to these points. Proposition 6.1 gives a way to precisely see all possible intersections of Γ with Δ, and so this procedure indeed gives an explicit description of $\mathcal{A}$.

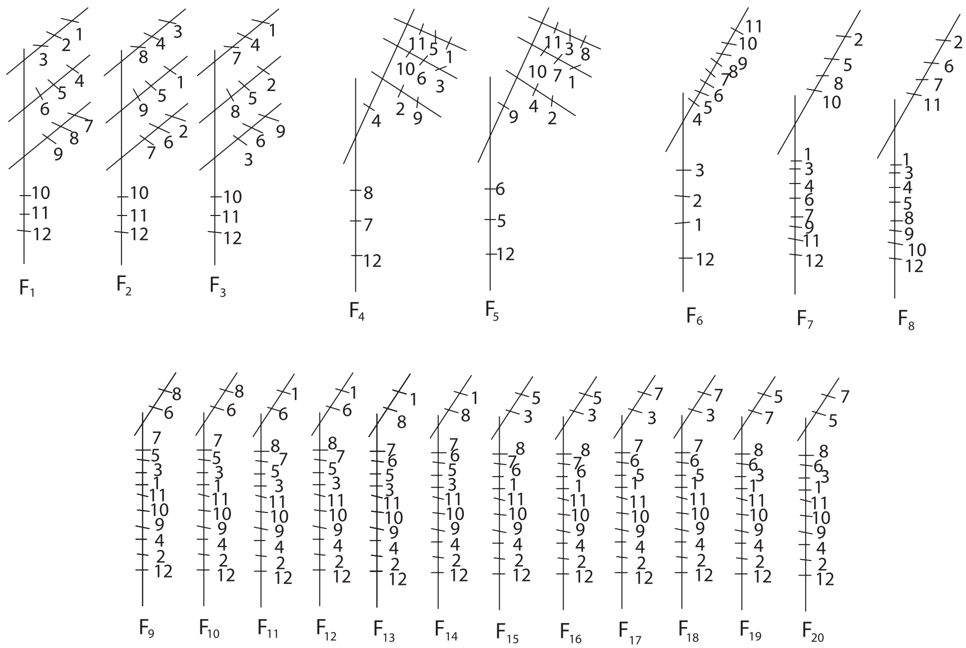


Figure 3. The singular fibers of the stable fibration induced by the conic Γ.

Example 6.1. Let $\zeta = e^{2\pi i/3}$. Consider the dual Hesse arrangement of 9 lines in $\mathbb{P}^2(\mathbb{C})$:

$$(x_1^3 - x_2^3)(x_1^3 - x_3^3)(x_2^3 - x_3^3) = 0.$$

It has 12 triple points and no other singularities. We label these points as $p_1 = [1 : \zeta : \zeta]$, $p_2 = [\zeta : \zeta^2 : 1]$, $p_3 = [1 : \zeta : 1]$, $p_4 = [\zeta : 1 : \zeta^2]$, $p_5 = [\zeta : 1 : 1]$, $p_6 = [1 : \zeta^2 : 1]$, $p_7 = [1 : 1 : \zeta]$, $p_8 = [1 : 1 : \zeta^2]$, $p_9 = [1 : 1 : 1]$, $p_{10} = [0 : 1 : 0]$, $p_{11} = [1 : 0 : 0]$, and $p_{12} = [0 : 0 : 1]$. Consider the unique conic Γ passing through p_{12}, p_{11}, p_{10}, p_9, p_4. It is given by the equation $\zeta^2 x_1 x_2 + \zeta x_1 x_3 + x_2 x_3 = 0$. By Proposition 6.1, we have $\theta : Bl_{p_1,\ldots,p_{12}}\mathbb{P}^2 \hookrightarrow \overline{M}_{0,12}$, embedding the proper transform $\tilde{\Gamma}$ of Γ. In [6], it is proved that $\tilde{\Gamma}$ is a rigid curve in $\overline{M}_{0,12}$.

By Theorem 5.1, this curve defines a primitive arrangement of 11 curves $\mathcal{A}$. To actually exhibit $\mathcal{A}$, we need to check all intersections between Γ and all the lines passing through pairs of points p_i (so, more than the ones in the dual Hesse arrangement). After that, it is easy to draw a picture of the arrangement. In Figure 3, we show all the singular fibers of the corresponding stable fibration. Notice that the arrangement belongs to $\mathscr{A}_t(\mathbb{P}^1, \mathcal{O}(3), 11)$.

For another model, we perform in $\mathbb{F}_3$ two elementary transformations on the fibers F_4 and F_5 by blowing up the corresponding singular points in $\mathcal{A}$. Then, we end up in $\mathbb{F}_1$ where the (-1)-curve is the proper transform of S_{12}. After blowing it down, we obtain a very special arrangement of 7 lines and 4 conics in $\mathbb{P}^2$.

7. Extended and partially extended arrangements

Fix the data $(C, \mathcal{L}, d)$ over $\mathbb{K} = \overline{\mathbb{K}}$ as in Section 2. We now study properties of certain log surfaces associated to arrangements of sections $\mathcal{A}$. First, we associate to $\mathcal{A}$ an extended arrangement $\mathcal{A}_\Delta$, and partially extended arrangements $\mathcal{A}_{p\Delta}$.

Definition 7.1. Consider the arrangement of sections $\mathcal{A}$ as a (reducible singular) curve, we denote its set of singular points by $\mathrm{sing}(\mathcal{A})$. Let $\{F_1, \dots, F_\delta\}$ be the fibers of $\pi : \mathbb{P}_C(\mathcal{L}) \to C$ which contain points in $\mathrm{sing}(\mathcal{A})$. Then, the *extended* arrangement $\mathcal{A}_\Delta$ associated to $\mathcal{A}$ is

$$\mathcal{A}_\Delta := \mathcal{A} \cup \{F_1, \dots, F_\delta\} \cup \{S_{d+1}\}.$$

Let $0 < \varepsilon \leq \delta - 2$ be an integer. Let $\{F_{i_1}, \dots, F_{i_\varepsilon}\}$ be a subset of $\{F_1, \dots, F_\delta\}$ such that for any $1 \leq j \leq \varepsilon$ and any point P in $\mathrm{sing}(\mathcal{A}) \cap F_{i_j}$, there are two sections in $\mathcal{A}$ intersecting at P with distinct tangent directions. Then, a *partially extended* arrangement $\mathcal{A}_{p\Delta}$ associated to $\mathcal{A}$ is

$$\mathcal{A}_{p\Delta} := \mathcal{A}_\Delta \backslash \{F_{i_1}, \dots, F_{i_\varepsilon}\}.$$

The numeration of the fibers will be irrelevant.

As before, we perform blow-ups at the points in $\mathrm{sing}(\mathcal{A})$ (and infinitely near points above them) to separate all sections S_i's in a minimal way (as in the proof of Theorem 5.1). This is described by a chain of blow-ups

$$\tilde{R} = R_s := Bl_{P_s} R_{s-1} \xrightarrow{\sigma_s} \cdots \xrightarrow{\sigma_3} R_2 := Bl_{P_2} R_1 \xrightarrow{\sigma_2} R_1 := Bl_{P_1} \mathbb{P}_C(\mathcal{L}) \xrightarrow{\sigma_1} R_0 := \mathbb{P}_C(\mathcal{L})$$

whose composition is denoted by $\sigma : \tilde{R} \to \mathbb{P}_C(\mathcal{L})$. The map σ gives the *minimal log resolution* of $\mathcal{A}$, and produces the semi-stable fibration of $(d+1)$-pointed genus zero curves $\tilde{\rho} : \tilde{R} \to C$. The $\sigma^*(\mathcal{A})_{\mathrm{red}}$ is a simple normal crossing divisor. Let $t(P_i)$ be the number of sections in the proper transform of $\mathcal{A}$ passing through P_i right before blowing up P_i (the center of the blowing up σ_i). We define

$$\tau(\mathcal{A}) := \sum_{i=1}^{s} (t(P_i) - 1).$$

The divisor $\sigma^*(\mathcal{A}_\Delta)_{\mathrm{red}}$ is the minimal log resolution of $\mathcal{A}_\Delta$, but $\sigma^*(\mathcal{A}_{p\Delta})_{\mathrm{red}}$ may not be minimal, since we may need to blow down (-1)-curves coming from some nodes in $\mathcal{A}$. The divisors $\sigma^*(\mathcal{A}_\Delta)_{\mathrm{red}}$ and $\sigma^*(\mathcal{A}_{p\Delta})_{\mathrm{red}}$ are denoted by $\overline{\mathcal{A}}_\Delta$ and $\overline{\mathcal{A}}_{p\Delta}$ respectively. The arrangement $\overline{\mathcal{A}}_\Delta$ may be seen as defined by the intersection of the boundary Δ in $\overline{M}_{0,d+2}$ with the surface R, where $\rho : R \to C$ is the stable fibration of $(d+1)$-pointed curves of genus zero induced by $\mathcal{A}$.

We now follow the exposition of log surfaces as in [22, Section 2], which is due to Iitaka, and the references given there. We are interested in the log surfaces $(\tilde{R}, \bar{\mathcal{A}}_\Delta)$ and $(\tilde{R}, \bar{\mathcal{A}}_{p\Delta})$, and their *log Chern classes*[2]

$$\bar{c}_i(\bar{\mathcal{A}}_\Delta) := c_i(\Omega^1_{\tilde{R}}(\log \bar{\mathcal{A}}_\Delta)^\vee), \quad \bar{c}_i(\bar{\mathcal{A}}_{p\Delta}) := c_i(\Omega^1_{\tilde{R}}(\log \bar{\mathcal{A}}_{p\Delta})^\vee)$$

where $\Omega^1_{\tilde{R}}(\log \bar{\mathcal{A}}_\Delta)^\vee$, $\Omega^1_{\tilde{R}}(\log \bar{\mathcal{A}}_{p\Delta})^\vee$ are the dual of the corresponding sheaves of log differentials (see [22, Def. 2.2]), and $i = 1, 2$.

Proposition 7.1. *Let $\mathcal{A}$ be an arrangement of sections with data $(C, \mathcal{L}, d)$, $\deg \mathcal{L} = e$ and $h^1(C, \mathcal{O}_C) = g$. Then,*

$$\bar{c}_1^2(\mathcal{A}_\Delta) = (d-1)(2\delta + 4(g-1) - e) + \tau(\mathcal{A}) \quad \bar{c}_2(\mathcal{A}_\Delta) = (d-1)(2(g-1) + \delta)$$

with δ as in Definition 7.1.

Proof. In general, if $(Y, D = \sum_{i=1}^r D_i)$ is a log surface (as in [22, Section 2]), the log Chern numbers are (see [22, Proposition 2.4])

$$\bar{c}_1^2(Y, D) = c_1^2(Y) - \sum_{i=1}^r D_i^2 + 2t_2 + 4\sum_{i=1}^r (g(D_i) - 1),$$

and $\bar{c}_2(Y, D) = c_2(Y) + t_2 + 2\sum_{i=1}^r (g(D_i) - 1)$, where t_2 is the number of nodes of the curve D. In our case, $Y = \tilde{R}$ and $D = \bar{\mathcal{A}}_\Delta$. We will compute these numbers recursively. Let $\sigma_{i,1} : R_i \to \mathbb{P}_C(\mathcal{L})$ be the composition of the blow-ups $\sigma_1 \circ \cdots \circ \sigma_i$. Define

$$\bar{c}_1^2(i) := 8(1-g) - i - \sum_{j=1}^{s_i} D_{i,j}^2 + 2(i + (d+1)\delta) + 4(d+1)(g-1) - 4\delta - 4i$$

for $i = 0, 1, \ldots, s$, where $\sum_{j=1}^{r_i} D_{i,j}$ is the prime decomposition of $\sigma_{i,1}^*(\mathcal{A}_\Delta)_{\mathrm{red}}$. Then, one can check that

$$\bar{c}_1^2(0) = (d-1)(2\delta + 4(g-1) - e) \quad \bar{c}_1^2(s) = \bar{c}_1^2(\mathcal{A}_\Delta)$$

and $\bar{c}_1^2(i+1) = \bar{c}_1^2(i) + t(P_{i+1}) - 1$. Therefore, $\bar{c}_1^2(\mathcal{A}_\Delta) = (d-1) \cdot (2\delta + 4(g-1) - e) + \tau(\mathcal{A})$. On the other hand, by the formula for $\bar{c}_2(\tilde{R}, \bar{\mathcal{A}}_\Delta)$ above, we have

$$\bar{c}_2(\mathcal{A}_\Delta) = 4(1-g) + s + (s + (d+1)\delta) + 2(d+1)(g-1) - 2\delta - 2s$$

and so $\bar{c}_2(\mathcal{A}_\Delta) = (d-1)(2(g-1) + \delta)$. □

[2] The corresponding log Chern numbers for $\sigma^*(\mathcal{A}_{p\Delta})_{\mathrm{red}}$ and the minimal log resolution of $\mathcal{A}_{p\Delta}$ are the same (see for example [22, Proposition 2.4]).

COROLLARY 7.2. *Let* $\mathcal{A} = \{S_1, \ldots, S_d\}$ *as above. Then,* $\bar{c}_1^2(\mathcal{A}_\Delta) \geq 2d - 1$, $\bar{c}_2(\mathcal{A}_\Delta) \geq d - 1$, *and* $2 < \frac{\bar{c}_1^2(\mathcal{A}_\Delta)}{\bar{c}_2(\mathcal{A}_\Delta)}$. *The log canonical divisor* $K_{\tilde{R}} + \bar{\mathcal{A}}_\Delta$ *is big and nef, and so the surface* $\tilde{R} \backslash \bar{\mathcal{A}}_\Delta$ *is of log general type. When* $\mathbb{K} = \mathbb{C}$, *we have the (strict) log Miyaoka–Yau inequality*

$$\bar{c}_1^2(\mathcal{A}_\Delta) < 3\bar{c}_2(\mathcal{A}_\Delta),$$

and so $\tau(\mathcal{A}) < (d-1)(\delta + 2(g-1) + e)$.

PROOF. The second log Chern number $\bar{c}_2(\mathcal{A}_\Delta) = (d-1)(2(g-1)+\delta)$ is positive because $\delta \geq 3$. For the other one, take $S_1 \in \mathcal{A}$. Then, by looking at the intersections of S_1 with $S_2, \ldots, S_d$ in $\mathbb{P}_C(\mathcal{L})$, one sees that $\tau(\mathcal{A}) > e(d-1)$. The inequality is strict because $\bigcap_{i=1}^d S_i = \emptyset$. Therefore, by Proposition 7.1, one has $\bar{c}_1^2(\mathcal{A}_\Delta) > 0$. By the same reason,

$$2 < \frac{\bar{c}_1^2(\mathcal{A}_\Delta)}{\bar{c}_2(\mathcal{A}_\Delta)} = 2 + \frac{\tau(\mathcal{A}) - e(d-1)}{(d-1)(2(g-1)+\delta)}.$$

Let $D := \bar{\mathcal{A}}_\Delta$ and write its prime decomposition $D = \sum_{i=1}^{d+1} \tilde{S}_i + \sum_i E_i$ where $\tilde{S}_i$ is the proper transform of S_i under $\sigma : \tilde{R} \to \mathbb{P}_C(\mathcal{L})$, and $E_i \simeq \mathbb{P}^1$'s are the rest. Since $\tilde{S}_{d+1} = \sigma^* S_{d+1}$, we denote $S_{d+1} = \sigma^* S_{d+1}$ and $F = \sigma^* F$ where F is a general fiber of $\pi : \mathbb{P}_C(\mathcal{L}) \to C$. Then, $K_{\tilde{R}} + D \equiv -2S_{d+1} + (2g - 2 - e)F + \sum_i a_i E_i + D$ for some $a_i > 0$. But, for any fixed $j = 1, \ldots, d$, $\tilde{S}_j \equiv S_{d+1} + eF - \sum_i b_i E_i$ where $b_i \leq a_i$, and $\sum_i E_i \equiv \delta F$. Thus, say for $j = 1$, we have

$$K_{\tilde{R}} + D \equiv (2g - 2 + \delta)F + \sum_i (a_i - b_i)E_i + \sum_{i=2}^d \tilde{S}_i,$$

which means that the log canonical class is numerically equivalent to an effective divisor. Moreover, $\tilde{S}_i.(K_{\tilde{R}} + D) = 2g - 2 + \delta > 0$ for all i, and $E_i.(K_{\tilde{R}} + D) = -2 + E_i.(D - E_i) \geq 0$, and so $K_{\tilde{R}} + D$ is nef. This plus the fact $\bar{c}_1^2(\mathcal{A}_\Delta) = (K_{\tilde{R}} + D)^2 > 0$ implies that $K_{\tilde{R}} + D$ is big, and so $\tilde{R} \backslash \bar{\mathcal{A}}_\Delta$ is of log general type. When $\mathbb{K} = \mathbb{C}$, we use Sakai's theorem [18, Theorem 7.6] to conclude $\bar{c}_1^2(\mathcal{A}_\Delta) \leq 3\bar{c}_2(\mathcal{A}_\Delta)$. Notice that the curve D is semi-stable and has no exceptional curves with respect to D, as defined in [18, pp. 90–91]. For strictness, we apply Lemma 9.1. □

REMARK 7.1. From the previous proof, it is easy to see that the log canonical class is ample if and only if $\mathcal{A}$ is transversal (Definition 2.4). One just applies the Nakai-Moishezon criterion [8, p. 365].

PROPOSITION 7.3. *Let* $\mathcal{A}$ *be an arrangement of sections with data* $(C, \mathcal{L}, d)$, *and let* $\mathcal{A}_{p\Delta}$ *be a partially extended arrangement from* $\mathcal{A}$ *with* $\{F_{i_1}, \ldots, F_{i_\varepsilon}\}$ *for* *some* $0 < \varepsilon \leq \delta - 2$ *(see Definition* 7.1*). Let* $k_j^o := |\mathrm{sing}(\mathcal{A}) \cap F_{i_j}|$ *and* $k_j := |\mathcal{A} \cap F_{i_j}| + 1 \leq d$*. Then,*

$$\bar{c}_2(\mathcal{A}_{p\Delta}) = \bar{c}_2(\mathcal{A}_\Delta) - \sum_{j=1}^{\varepsilon} k_j + 2\varepsilon \quad \bar{c}_1^2(\mathcal{A}_{p\Delta}) = \bar{c}_1^2(\mathcal{A}_\Delta) - \sum_{j=1}^{\varepsilon} k_j^o - 2\sum_{j=1}^{\varepsilon} k_j + 4\varepsilon.$$

PROOF. The result follows directly from the formulas in [22, Proposition 2.4]. □

COROLLARY 7.4. *Let* $\mathcal{A}$ *be as above. Then,* $\bar{c}_1^2(\mathcal{A}_{p\Delta}) \geq 2$, $\bar{c}_2(\mathcal{A}_{p\Delta}) \geq 1$*. The corresponding log canonical divisor is big and nef, and so the log surface defined by* $\bar{\mathcal{A}}_{p\Delta}$ *is of log general type. When* $\mathbb{K} = \mathbb{C}$*, we again have (strict) log Miyaoka–Yau inequality* $\bar{c}_1^2(\mathcal{A}_{p\Delta}) < 3\bar{c}_2(\mathcal{A}_{p\Delta})$*.*

PROOF. Notice that $\sum_{j=1}^{\varepsilon} k_j - 2\varepsilon \leq d\varepsilon - 2\varepsilon \leq (d-2)(\delta-2)$, and so

$$\begin{aligned}\bar{c}_2(\mathcal{A}_{p\Delta}) &= \bar{c}_2(\mathcal{A}_\Delta) - \Big(\sum_{j=1}^{\varepsilon} k_j - 2\varepsilon\Big)\\ &\geq (d-1)(2(g-1)+\delta) - (d-2)(\delta-2) = 2g(d-1) + \delta - 2 \geq 1.\end{aligned}$$

Clearly, we have $k_i^o + 2k_i \leq 2d+1$, and so $\sum_{j=1}^{\varepsilon} k_j^o + 2\sum_{j=1}^{\varepsilon} k_j - 4\varepsilon \leq (2d-3)(\delta-2)$. Then, since $\tau(\mathcal{A}) - e(d-1) \geq 1$, we have

$$\begin{aligned}\bar{c}_1^2(\mathcal{A}_{p\Delta}) &= \bar{c}_1^2(\mathcal{A}_\Delta) - \sum_{j=1}^{\varepsilon} k_j^o - 2\sum_{j=1}^{\varepsilon} k_j + 4\varepsilon\\ &\geq 4(g-1)(d-1) + 2\delta(d-1) - e(d-1) + \tau(\mathcal{A}) - (2d-3)(\delta-2)\\ &\geq 1 + 4(g-1)(d-1) + 2(2d-3) + \delta \geq 1 - 4(d-1) + \delta + 2(2d-3)\\ &= \delta - 1 \geq 2.\end{aligned}$$

We prove nefness and bigness as we did in Corollary 7.2. It is enough to do it in $\tilde{R}$. Let $D := \sigma^*(\mathcal{A}_{p\Delta})_{\mathrm{red}}$ and write its prime decomposition $D = \sum_{i=1}^{d+1} \tilde{S}_i + \sum_i E_i$ where $\tilde{S}_i$ is the proper transform of S_i under $\sigma : \tilde{R} \to \mathbb{P}_C(\mathcal{L})$, and $E_i \simeq \mathbb{P}^1$'s are the rest. Since $\tilde{S}_{d+1} = \sigma^* S_{d+1}$, we denote $S_{d+1} = \sigma^* S_{d+1}$ and $F = \sigma^* F$ where F is a general fiber of $\pi : \mathbb{P}_C(\mathcal{L}) \to C$. Then, $K_{\tilde{R}} + D \equiv -2S_{d+1} + (2g-2-e)F + \sum_i a_i E_i + D$ for some $a_i > 0$. But, for any fixed $j = 1, \ldots, d$, $\tilde{S}_j \equiv S_{d+1} + eF - \sum_i b_i E_i$ where $b_i \leq a_i$, and $\sum_i E_i \equiv \delta F - \sum_{j=1}^{\varepsilon} \tilde{F}_{i_j}$ (see Definition 7.1). Thus, say for $j = 1$, we have

$$K_{\tilde{R}} + D \equiv (2g-2+\delta-\varepsilon)F + \sum_i (a_i - b_i)E_i + \sum_{i=2}^{d} \tilde{S}_i + \sum_{j=1}^{\varepsilon} (F - \tilde{F}_{i_j}),$$

which means that the log canonical class is numerically equivalent to an effective divisor. Moreover, $\tilde{S}_i.(K_{\tilde{R}} + D) \geq 2g - 2 + \delta - \varepsilon \geq 0$ for all i, and $E_i.(K_{\tilde{R}} + D) =$

$-2 + E_i.(D - E_i) \geq 0$ (this is by definition of partially extended arrangement), and so $K_{\tilde{R}} + D$ is nef. This plus the fact $(K_{\tilde{R}} + D)^2 > 0$ implies that $K_{\tilde{R}} + D$ is big, and so $\tilde{R} \backslash \bar{\mathcal{A}}_{p\Delta}$ is of log general type. When $\mathbb{K} = \mathbb{C}$ and by Sakai's theorem [18, Theorem 7.6], we have $\bar{c}_1^2(\mathcal{A}_{p\Delta}) \leq 3\bar{c}_2(\mathcal{A}_{p\Delta})$. For strictness, we apply again Lemma 9.1. □

EXAMPLE 7.1. We use Example 6.1 to show differences in $\frac{\bar{c}_1^2}{\bar{c}_2}$ between extended and partially extended arrangements. Let $\mathcal{A}$ be the arrangement in Example 6.1. For the partially extended arrangements, let Ξ be the set of fibers we take out from $\mathcal{A}_\Delta$ (see Definition 7.1). We label fibers according to Figure 3. Then we have the following table.

Ξ	$\{F_1, \ldots, F_8\}$	$\emptyset$	$\{F_9, \ldots, F_{20}\}$	$\{F_7, \ldots, F_{20}\}$	$\{F_6, \ldots, F_{20}\}$
$\bar{c}_1^2$	319	399	171	141	134
$\bar{c}_2$	147	180	72	58	55
$\frac{\bar{c}_1^2}{\bar{c}_2}$	2.170...	$2.21\bar{6}$	2.375	2.4310...	$2.4\overline{36}$

REMARK 7.2. In general, there are no inequalities between $\frac{\bar{c}_1^2(\mathcal{A}_{p\Delta})}{\bar{c}_2(\mathcal{A}_{p\Delta})}$ and $\frac{\bar{c}_1^2(\mathcal{A}_\Delta)}{\bar{c}_2(\mathcal{A}_\Delta)}$. Also, in general, we do not have $2 < \frac{\bar{c}_1^2(\mathcal{A}_{p\Delta})}{\bar{c}_2(\mathcal{A}_{p\Delta})}$. For instance take a general arrangement $\mathcal{A} \in \mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$ (a general line arrangement). We have $\delta = \binom{d}{2}$. Take $\varepsilon = \delta - 2$. Then, $\tau(\mathcal{A}) = \binom{d}{2}$ and $\frac{\bar{c}_1^2(\mathcal{A}_{p\Delta})}{\bar{c}_2(\mathcal{A}_{p\Delta})} = 2 - \frac{d-3}{\binom{d}{2}-2} < 2$.

REMARK 7.3. Almost any line arrangement is "log equivalent" to a $\mathcal{A}_{p\Delta} \in \mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$. More precisely, let $\mathscr{L}$ be a line arrangement in $\mathbb{P}^2$. Assume this line arrangement has at least two singular points so that each of them belongs to more than two lines (in particular, general line arrangements are not allowed). Take two distinct lines in $\mathscr{L}$, each of which contains exactly one of these two points. Now we blow up the intersection of these two lines. The total (reduced) transform of $\mathscr{L}$ is a $\mathcal{A}_{p\Delta}$ for some $\mathcal{A} \in \mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$.

Over $\mathbb{C}$ and by [22, Theorem 7.2], we have the Hirzebruch-Sakai inequality

$$\bar{c}_1^2(\mathcal{A}_{p\Delta}) \leq \frac{8}{3}\bar{c}_2(\mathcal{A}_{p\Delta}),$$

for any $\mathcal{A} \in \mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$, with equality if and only if $\mathcal{A}_{p\Delta}$ is the proper transform of the dual Hesse arrangement (see Example 6.1). Of course, the same holds true for $\mathcal{A}_\Delta$, but now equality is impossible. This shows that the log Miyaoka–Yau inequality in the previous corollaries may be far from optimal, and opens the interesting question:

QUESTION 7.5. *Find the number* $0 < \alpha(C, \mathcal{L}) < 3$ *which makes*

$$\bar{c}_1^2(\mathcal{A}_{p\Delta}) \leq \alpha(C, \mathcal{L})\, \bar{c}_2(\mathcal{A}_{p\Delta})$$

a sharp inequality, valid for any complex arrangement $\mathcal{A} \in \mathscr{A}(C, \mathcal{L}, d)$ *and any* $d \geq 3$.

Arrangements holding equality should be interesting. Proposition 7.6 will show that we indeed need to look only at arrangements in $\mathscr{A}(C, \mathcal{L}, d)$, i.e. primitive ones (Definition 2.3).

REMARK 7.4. The log Miyaoka–Yau inequalities in Corollaries 7.2 and 7.4 are not combinatorial, except in the case of $\mathscr{A}(\mathbb{P}^1, \mathcal{O}(1), d)$. As in the previous remark, let $\mathcal{A}_{p\Delta}$ be defined by an arrangement of lines $\mathscr{L} = \{L_1, \dots, L_s\}$ in $\mathbb{P}^2$. Let $\{p_1, \dots, p_r\}$ be the set of singular points of $\mathscr{L}$. The log Miyaoka–Yau inequality $\bar{c}_1^2(\mathscr{L}) \leq 3\bar{c}_2(\mathscr{L})$ becomes precisely $r \geq s$ [22]. This inequality was proved in a purely combinatorial manner by N. G. de Bruijn and P. Erdös in [5]. Moreover, they show that $s = r$ if and only if $\mathscr{L}$ has either $s - 1$ lines through a common point (in this case $\bar{c}_1^2 = \bar{c}_2 = 0$) or it is a finite projective plane. We proved this inequality in [22, Theorem 7.2] using some surface theory. The following is another combinatorial proof. Consider L_i as a vector in $\mathbb{Q}^r$ having a 1 in the j-th coordinate if $p_j \in L_i$, 0 otherwise. The assertion follows if $\mathscr{L}$ forms a linearly independent set. If not, say the line $L_1 = \sum_{i=2}^{s} a_i L_i$. Then, consider the inner product with $L_1 - L_j$, giving $a_j = \frac{L_1 \cdot L_1 - 1}{1 - L_j \cdot L_j} < 0$ for all j, a contradiction.

In general, one can exhibit "combinatorial arrangements" for which the inequality $\tau(\mathcal{A}) \leq (d-1)(\delta + 2(g-1) + e)$ does not hold. For example, one may take $\mathcal{A} \in \mathscr{A}(\mathbb{P}^1, \mathcal{O}(e), 4)$ with $e > 1$ such that any two sections are tangent of order e and $\delta = 3$ (when $e = 1$, $\mathcal{A}_\Delta$ is the Fano arrangement (with a point blown-up)). This combinatorial phenomena is produced by the freedom we have with respect to tangencies of higher order. See also Example 7.5, where positive characteristic is used.

PROPOSITION 7.6. *Let* $\mathcal{A}$, $\mathcal{A}'$ *be two arrangements of d sections so that* $\mathcal{A}$ *is a pull-back of* $\mathcal{A}'$, *as in Definition* 2.2. *Assume that the pull-back map g is separable. Then,*

$$\frac{\bar{c}_1^2(\mathcal{A}_\Delta)}{\bar{c}_2(\mathcal{A}_\Delta)} \leq \frac{\bar{c}_1^2(\mathcal{A}'_\Delta)}{\bar{c}_2(\mathcal{A}'_\Delta)}, \quad \frac{\bar{c}_1^2(\mathcal{A}_{p\Delta})}{\bar{c}_2(\mathcal{A}_{p\Delta})} \leq \frac{\bar{c}_1^2(\mathcal{A}'_{p\Delta})}{\bar{c}_2(\mathcal{A}'_{p\Delta})},$$

where $\mathcal{A}_{p\Delta}$ *is the pull-back of* $\mathcal{A}'_{p\Delta}$.

PROOF. By definition, we have the following working diagram

$$\begin{array}{ccc} \mathbb{P}_C(\mathcal{L}) & \xrightarrow{G} & \mathbb{P}_{C'}(\mathcal{L}') \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi'}\downarrow \\ C & \xrightarrow{g} & C' \end{array}$$

where $\mathcal{A}$ and $\mathcal{A}'$ are arrangements of d sections in $\mathbb{P}_C(\mathcal{L})$ and $\mathbb{P}_{C'}(\mathcal{L}')$ respectively. Since g is a finite separable morphism, we have the Hurwitz formula

$$2g - 2 = \deg(g)(2g' - 2) + \deg R$$

where $R = \sum_{c \in C} \text{length}(\Omega_{C/C'})_P P$, and so $\deg R \geq \sum_{c \in C}(e_c - 1)$ (see [8, p. 301]). Here e_c is the ramification index of g at c. As usual, $c' \in C'$ is a branch point of g if there is $c \in C$ such that $g(c) = c'$ and $e_c > 1$. We remak that for any $c' \in C'$ we have $g^*(c') = \sum_{g(c)=c'} e_c c$ and $\deg(g) = \sum_{g(c)=c'} e_c$ [8, p. 138].

It is not hard to see that $\delta + \deg R = \deg(g)\delta' + \alpha$, for some integer $\alpha \geq 0$. We also have $e = \deg(g)e'$ (Lemma 2.1, where $e = \deg \mathcal{L}$ and $e' = \deg \mathcal{L}'$). Notice that, by definition, the map g cannot be branched at any of the images of the special ε fibers (this is an empty statement when we consider the extended arrangement). This is because a pre-image of such a fiber would contain at least one point in $\text{sing}(\mathcal{A})$ where all sections in $\mathcal{A}$ through it have the same tangent direction. So, $\varepsilon = \deg(g)\varepsilon'$, $\sum_{i=1}^{\varepsilon} k_i = \deg(g) \sum_{i=1}^{\varepsilon'} k_i'$, and $\sum_{i=1}^{\varepsilon} k_i^o = \deg(g) \sum_{i=1}^{\varepsilon'} k_i^{o\prime}$. So we only need to compare $\tau(\mathcal{A})$ with $\tau(\mathcal{A}')$.

For any $P \in \text{sing}(\mathcal{A})$, define $t_P(\mathcal{A}) := \sum_{Q \in N(P)}(t(Q) - 1)$ where $N(P)$ is the set of points blown up by σ above P (so $N(P)$ contains P). Then, $t_P(\mathcal{A}) = e_{\pi(P)} t_{G(P)}(\mathcal{A}')$, and so $\tau(\mathcal{A}) = \deg(g)\tau(\mathcal{A}')$. Therefore,

$$\begin{aligned}
\frac{\bar{c}_1^2(\mathcal{A}_{p\Delta})}{\bar{c}_2(\mathcal{A}_{p\Delta})} &= 2 + \frac{\tau(\mathcal{A}) - e(d-1) - \sum_{i=1}^{\varepsilon} k_i^o}{(d-1)(\delta + 2(g-1)) - \sum_{j=1}^{\varepsilon} k_j + 2\varepsilon} \\
&= 2 + \frac{\deg(g)(\tau(\mathcal{A}') - e'(d-1) - \sum_{i=1}^{\varepsilon} k_i^{o\prime})}{(d-1)(\deg(g)(2g' - 2 + \delta') + \alpha) - \deg(g)(\sum_{j=1}^{\varepsilon'} k_j' - 2\varepsilon')} \\
&\leq 2 + \frac{\deg(g)(\tau(\mathcal{A}') - e'(d-1) - \sum_{i=1}^{\varepsilon} k_i^{o\prime})}{(d-1)\deg(g)(2g' - 2 + \delta') - \deg(g)(\sum_{j=1}^{\varepsilon'} k_j' - 2\varepsilon')} = \frac{\bar{c}_1^2(\mathcal{A}'_{p\Delta})}{\bar{c}_2(\mathcal{A}'_{p\Delta})}.
\end{aligned}$$

□

REMARK 7.5. The situation is different when the base change is not separable. Assume that $\mathbb{K}$ has positive characteristic p. Let $\mathcal{A}$, $\mathcal{A}'$ be two arrangements of d sections so that $\mathcal{A}$ is a pull-back of $\mathcal{A}'$, as in Proposition 7.6, but now let $g : C = C'_{p^r} \to C'$ be the composition of the $\mathbb{K}$-linear Frobenius morphism r times [8, p. 302]. Then,

$$\frac{\bar{c}_1^2(\mathcal{A}_\Delta)}{\bar{c}_2(\mathcal{A}_\Delta)} = 2 + p^r \frac{t(\mathcal{A}') - e'(d-1)}{(d-1)(2(g'-1) + \delta')} = 2 + p^r \left(\frac{\bar{c}_1^2(\mathcal{A}'_\Delta)}{\bar{c}_2(\mathcal{A}'_\Delta)} - 2 \right)$$

and so it becomes arbitrarily large when $r \gg 0$ (Corollary 7.2). In the next section, these examples will produce nonsingular projective surfaces violating any Miyaoka–Yau inequality.

8. RANDOM SURFACES ASSOCIATED TO ARRANGEMENTS

Fix the data $(C, \mathcal{L}, d)$ over $\mathbb{K} = \overline{\mathbb{K}}$ as always. We now associate to each arrangement of sections $\mathcal{A}$ of $\pi : \mathbb{P}_C(\mathcal{L}) \to C$ various collections of nonsingular projective surfaces. Each collection is produced by either $\mathcal{A}_\Delta$ or some $\mathcal{A}_{p\Delta}$. The construction is analogue to the one in [22, Theorem 6.1] but now we have more singular arrangements of curves. From now on, we consider $\mathcal{A}_\Delta$ as $\mathcal{A}_{p\Delta}$ with $\varepsilon = 0$, to save notation.

Let $\mathcal{A} = \{S_1, \dots, S_d\}$. By definition,

$$S_i \sim S_{d+1} + \pi^*(\mathcal{L})$$

for $i = 1, \dots, d$, where $S_{d+1} = C_0$ with $C_0^2 = -e = -\deg \mathcal{L}$. Let $\{F_1, \dots, F_{\delta-\varepsilon}\}$ be the fibers which define $\mathcal{A}_{p\Delta} = \mathcal{A} \cup \{F_1, \dots, F_{\delta-\varepsilon}, S_{d+1}\}$. Let $\{0 < x_i\}_{i=1}^{d}$, $\{0 < y_i\}_{i=1}^{\delta-\varepsilon}$ be an integer solution of the equation

$$\mathbb{E} : \sum_{i=1}^{d} e x_i + \sum_{i=1}^{\delta-\varepsilon} y_i = p$$

for some prime number $p \neq \mathrm{char}(\mathbb{K})$, and let $x_{d+1} := p - \sum_{i=1}^{d} x_i$. When p is large enough, the equation $\mathbb{E}$ has nonnegative solutions, exactly (see [4])

$$\frac{p^{d+\delta-\varepsilon-1}}{(d+\delta-\varepsilon-1)!e^d} + O(p^{d+\delta-\varepsilon-2}).$$

In this way,

$$\sum_{i=1}^{d+1} x_i S_i + \sum_{i=1}^{\delta-\varepsilon} y_i F_i \sim p S_{d+1} + \pi^*\Big(\Big(\sum_{i=1}^{d} x_i\Big)\mathcal{L} + \mathcal{M}\Big),$$

for some line bundle $\mathcal{M}$ on C of degree $\sum_{i=1}^{\delta-\varepsilon} y_i$. Then, since

$$\Big(\sum_{i=1}^{d} x_i\Big) e + \deg \mathcal{M} = p,$$

there is an invertible sheaf $\mathcal{N}$ on C such that

$$\sum_{i=1}^{d+1} x_i S_i + \sum_{i=1}^{\delta-\varepsilon} y_i F_i \sim p(S_{d+1} + \pi^*\mathcal{N}).$$

The theorem below associates to each arrangement $\mathcal{A}$ various families of random smooth projective surfaces. We use the method in [22], with an extra care of the new singularities; in [22] we only had simple crossings (as in Definition 2.4). The randomness part relies on a large scale behavior of Dedekind sums and continued fractions (see [22, Apendix]). The proof will be based on the work done in [22].

THEOREM 8.1. *Let $\mathcal{A}$ be an arrangement of sections of $\pi : \mathbb{P}_C(\mathcal{L}) \to C$. Then, there exist nonsingular projective surfaces X of general type with*

$$\frac{c_1^2(X)}{c_2(X)} \text{ arbitrarily close to } \frac{\bar{c}_1^2(\mathcal{A}_{p\Delta})}{\bar{c}_2(\mathcal{A}_{p\Delta})},$$

for any $\mathcal{A}_{p\Delta}$.

PROOF. Let $Z := \mathbb{P}_C(\mathcal{L})$ and let Y be the surface which log resolves minimally the arrangement $\mathcal{A}_{p\Delta}$ (for example, $Y = \tilde{R}$ when $\varepsilon = 0$). Let $\sigma : Y \to Z$ be the minimal log resolution of $\mathcal{A}_{p\Delta}$. Choose a solution of $\mathbb{E}$, and define

$$D := \sigma^*\Big(\sum_{i=1}^{d+1} x_i S_i + \sum_{i=1}^{\delta-\varepsilon} y_i F_i\Big) \sim p\sigma^*(S_{d+1} + \pi^*\mathcal{N}).$$

This allows the construction of the p-th root cover $f : X \to Y$ along D, as in [22, Section 2]. Thus, X is a nonsingular projective surface. Let

$$D = \sum_{i=1}^{r} v_i D_i$$

be the decomposition of D into the sum of prime divisors. From $\mathbb{E}$ and the nature of $\mathcal{A}_{p\Delta}$, ones sees that $0 < v_i < p$.

As in [22, Apendix], for $0 < q < p$, we denote the corresponding Dedekind sum by $s(q,p)$ and the length of the corresponding negative continued fraction by $l(q,p)$. In [22, Proposition 3.4, 3.6, and 2.4], we computed the Chern numbers of X as functions of p, Chern numbers, and "error terms". Let $\bar{c}_1^2 := \bar{c}_1^2(\mathcal{A}_{p\Delta})$, $\bar{c}_2 := \bar{c}_2(\mathcal{A}_{p\Delta})$, $c_1^2 := c_1^2(Y)$, and $c_2 := c_2(Y)$. Then,

$$c_1^2(X) = \bar{c}_1^2 p + 2(c_2 - \bar{c}_2) + (c_1^2 - \bar{c}_1^2 + 2\bar{c}_2 - 2c_2)\frac{1}{p} - \sum_{i<j} c(p - v_i'v_j, p) D_i.D_j$$

$$c_2(X) = \bar{c}_2 p + (c_2 - \bar{c}_2) + \sum_{i<j} l(p - v_i'v_j, p) D_i.D_j$$

where $c(p - v_i'v_j, p) := 12s(p - v_i'v_j, p) + l(p - v_i'v_j, p)$. Let us denote the error terms by $CCF := \sum_{i<j} c(p - v_i'v_j, p)D_i.D_j$ and $LCF := \sum_{i<j} l(p - v_i'v_j, p)D_i.D_j$.

We prove the existence of "good" solutions of $\mathbb{E}$ for arbitrarily large primes p, which make $\frac{CCF}{p}$ and $\frac{LCF}{p}$ arbitrarily small. In addition, this will show that random partitions are "good", with probability approaching 1 as p becomes arbitrarily large. The key numbers to take care of are the $p - v_i'v_j$, which are defined for each node of D_{red}. The idea is to show that there are solutions of $\mathbb{E}$ for which all $p - v_i'v_j$ are outside of a certain *bad set* $\mathcal{F} \subset \{0, \ldots, p-1\}$ (defined in [22, Apendix], due to K. Girstmair).

We write down for each node in D_{red} the multiplicities v_a, v_b as functions on the numbers x_i, y_j. There are different cases, all described in the following table.

Type	I	II	III	IV	V
v_a	x_i	y_i	y_i	$\sum_{k=1}^{d} n_k x_k + y_j$ with $0 \leq n_k \leq e$	$n \sum_k x_k + z$, $z \neq 0$ has no x_k, $0 \leq n < e$
v_b	x_j	x_j	x_{d+1}	x_k with $n_k \neq 0$	$\sum_k x_k + v_a$

Notice that "$z \neq 0$ has no x_k" in case V because of our restriction on tangent directions at the singular points of $\mathcal{A}_{p\Delta}$ (Definition 7.1). Below we estimate for each type the number of solutions $b(v_a, v_b)$ of $\mathbb{E}$ producing a bad multiplicity $p - v_a' v_b \in \mathcal{F}$. We do it case by case.

(Type I) This is a node in $S_i \cap S_j$ (possible only when $\varepsilon > 0$). Since $\mathbb{E}$ is a weighted partition of p, we can use the estimate in [22, proof of Theorem 6.1 (1)], and so there exists a positive number M (independent of p) such that

$$|b(v_a, v_b)| < p \cdot |\mathcal{F}| \cdot Mp^{d+\delta-\varepsilon-3} = M|\mathcal{F}|p^{d+\delta-\varepsilon-2}.$$

(Type II) This is a node in $F_i \cap S_j$ with $j \neq d+1$. Then again, we apply what we did in [22, proof of Theorem 6.1], to obtain the same estimate as above.

(Type III) This is a node in $F_i \cap S_{d+1}$. Since $x_{d+1} = p - \sum_{k=1}^{d} x_k$, then we want $p - y_i'(p - \sum_{k=1}^{d} x_k) \in \mathcal{F} \bmod p$, so $y_i'(\sum_{k=1}^{d} x_k) \in \mathcal{F}$. But this is again as in [22, Theorem 6.1], and we have the same previous estimate. Notice that it works because $\delta - \varepsilon \geq 2$.

(Type IV) This is a node between S_k, $k \neq d+1$, and a exceptional divisor over the fiber F_j. Notice that $\mathcal{A}_{p\Delta}$ contains at least two fibers, so $\sum_{k=1}^{d} n_k x_k + y_j < p$. Hence we are as in case (2) in the proof of [22, Theorem 6.1].

(Type V) This is the new case, coming from nodes in the resolution of singularities of $\mathcal{A}$. It does not involve x_{d+1}. The idea is to analyze three equations $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$ from the equation $\mathbb{E}$, and estimate solutions for each.

Without loss of generality, we rearrange indices so that

$$v_a = n \sum_{i=1}^{\alpha} x_i + z,$$

for some α, and $v_b = \sum_{i=1}^{\alpha} x_i + v_a$, where $z = \sum_{i=\alpha+1}^{\beta} n_i x_i + \varsigma y_j$, for some β, with $\varsigma = 0$ or 1, and $0 < n_i < e$. Notice that $z \neq 0$ for any solution of $\mathbb{E}$. We define

$$\mathbb{E}_1 : \sum_{i=\beta+1}^{d} ex_i + \sum_{i \neq j} y_i = p_1,$$

equation with m_1 variables, $\mathbb{E}_2 : \sum_{i=1}^{\alpha} x_i = p_2$ with m_2 variables, and

$$\mathbb{E}_3 : \sum_{i=\alpha+1}^{\beta} n_i x_i + \varsigma y_j = p_3$$

with m_3 variables. So, $m_1 + m_2 + m_3 = d + \delta - \varepsilon$. Notice that p_i are numbers varying in the region $0 < p_i < p$, since we will look at solutions of $\mathbb{E}_i$ from solutions of $\mathbb{E}$.

Say that $p - v'_a v_b \in \mathcal{F}$, which means mod p, $(\sum_k x_k) v'_a \in -\mathcal{F} - 1$. Of course the set $-\mathcal{F} - 1$ has same size as $\mathcal{F}$. We now use repeatedly the fact that the number of nonnegative integer solutions of $a_1 z_1 + \cdots + a_m z_m = q$ for coprime a_i's is $\frac{q^{m-1}}{(m-1)! a_1 a_2 \cdots a_m} + O(q^{m-2})$ (see [4]). Let p be large enough. Given $0 < p_3 < p$, the number of solutions of $\mathbb{E}_3$ is $< M_3 p^{m_3 - 1}$.

Now, the key observation is that mod p we have $p_2(np_2 + p_3)' \equiv \bar{p}_2(n\bar{p}_2 + p_3)'$ and so

$$p_2(n\bar{p}_2 + p_3) \equiv \bar{p}_2(np_2 + p_3) \Rightarrow p_2 p_3 \equiv \bar{p}_2 p_3 \Rightarrow p_2 \equiv \bar{p}_2$$

because p_3 is not zero. In this way, we have to choose p_2 in a set of size $|\mathcal{F}|$. Now we fix p_2 and have at most $M_2 p^{m_2 - 1}$ solutions for $\mathbb{E}_2$. After we have solutions for $\mathbb{E}_3$ and $\mathbb{E}_2$, we have at most $M_1 p^{m_1 - 1}$ solutions for $\mathbb{E}_1$. Putting it all together,

$$b(v_a, v_b) < p \cdot M_3 p^{m_3 - 1} \cdot |\mathcal{F}| \cdot M_2 p^{m_2 - 1} \cdot M_1 p^{m_1 - 1} = M_1 M_2 M_3 |\mathcal{F}| p^{d+\delta-\varepsilon-2}.$$

But we know that $|\mathcal{F}| < \sqrt{p}(\log(p) + 2\log(2))$ [22, Apendix], and that the total number of solutions of $\mathbb{E}$ is $\frac{p^{d+\delta-\varepsilon-1}}{(d+\delta-\varepsilon-1)! e^d} + O(p^{d+\delta-\varepsilon-2})$. Then, since the number of nodes of D_{red} is of course independent of p, we have proved the existence of good solutions, and that a random one is good with probability tending to 1 as p becomes arbitrarily large.

Now, given good solutions with p large, we proceed as at the end of the proof of [22, Theorem 6.1], showing that[3]

$$LCF < \Big(\sum_{i<j} D_i.D_j\Big)(3\sqrt{p} + 2) \quad |CCF| < \Big(\sum_{i<j} D_i.D_j\Big)(6\sqrt{p} + 7).$$

This proves the asymptotic result. Finally, these surfaces are of general type because of the classification of algebraic surfaces (see [3] for any characteristic), since we know that $\bar{c}_1^2 > 0$ and $\bar{c}_2 > 0$ (Corollaries 7.2 and 7.4). □

REMARK 8.1. With this theorem, one recovers the log Miyaoka–Yau inequalities in Corollaries 7.2 and 7.4, when $\mathbb{K} = \mathbb{C}$. We just apply the (projective) Miyaoka–Yau inequality to the surfaces X for large primes p.

[3] In [22] there is a minor error for the estimate of $s(q, p)$. This is due to the usual normalization by 12 of a Dedekind sum. The correct estimate is $12|s(q, p)| \leq 3\sqrt{p} + 5$, which of course does not affect any asymptotic result.

A good looking corollary, consequence of Theorem 5.1, Section 7, and Theorem 8.1.

COROLLARY 8.2. *Let $d \geq 3$ be an integer, and let A be an irreducible projective curve in $\mathbb{P}^{d-2}(\mathbb{K})$ not contained in the hyperplane arrangement $\mathscr{H}_d$ (see above Example* 3.1). *Then, there exist nonsingular projective surfaces X associated to A such that X is of general type and* $0 < 2c_2(X) < c_1^2(X)$.

PROOF. Consider the arrangement $\mathcal{A}$ defined by A as in Corollary 5.2. Then use Theorem 8.1 for $\mathcal{A}_\Delta$, and Corollary 7.2. □

REMARK 8.2. In addition, one can prove that $\pi_1^{\text{ét}}(X) \simeq \pi_1^{\text{ét}}(\bar{A})$, where $\bar{A}$ is the normalization of the curve A, and $\pi_1^{\text{ét}}$ denotes the étale fundamental group [23].

COROLLARY 8.3. *Assume $\mathbb{K} = \mathbb{C}$. Let $\mathcal{A}$ be an arrangement of sections of $\pi : \mathbb{P}_C(\mathcal{L}) \to C$. Then, there exist nonsingular projective surfaces X of general type such that*

$$2 < \frac{c_1^2(X)}{c_2(X)} < 3,$$

(and so of positive index) having $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to $\frac{\bar{c}_1^2(\mathcal{A}_\Delta)}{\bar{c}_2(\mathcal{A}_\Delta)}$. In addition, there is an induced connected fibration $\pi' : X \to C$ which gives an isomorphism: $\pi_1(X) \simeq \pi_1(C)$. In this way, $Alb(X) \simeq Jac(C)$ and π' is the Albanese fibration of X.

PROOF. The first part is implied by Theorem 8.1 for $\mathcal{A}_\Delta$ and Corollary 7.2. The map $f : X \to Y$ in the proof of Theorem 8.1 is totally ramified along $\bar{\mathcal{A}}_\Delta$, and $\pi \circ \sigma : Y \to C$ is a connected fibration with at least one simply connected fiber and one section in $\bar{\mathcal{A}}_\Delta$. Therefore, the construction induces a connected fibration $\pi' : X \to C$, and by [22, Proposition 8.3] we have $\pi_1(X) \simeq \pi_1(C)$. The last part is a simple consequence of Albanese maps which applies to any such fibration (see [3] for example). □

REMARK 8.3. Corollary 8.3 is also valid for any $\mathcal{A}_{p\Delta}$ except for $2 < \frac{c_1^2(X)}{c_2(X)}$. If one thinks that the closest $\frac{c_1^2(X)}{c_2(X)}$ is to 3, the more interesting are the surfaces X, then one may consider the construction starting with some $\mathcal{A}_{p\Delta}$ (this is with $\varepsilon > 0$). For line arrangements, this is indeed the case (see Remark 7.3). By Proposition 7.6, we only need to consider primitive arrangements in order to find upper bounds for Chern ratios.

EXAMPLE 8.1. The conic in Example 6.1 produces an arrangement $\mathcal{A} \in \mathscr{A}(\mathbb{P}^1, \mathcal{O}(3), 11)$. In the table of Example 7.1, we computed log Chern ratios for the extended and some partially extend arrangements induced by $\mathcal{A}$. Then, by Corollary 8.3, there are simply connected nonsingular projective surfaces of general type with Chern ratios arbitrarily close to the ones in that table. Notice that the highest is attained by a partially extended arrangement, which avoids "too many" double points.

EXAMPLE 8.2. Assume $\mathbb{K}$ has positive characteristic p. Take any $\mathcal{A}' \in \mathscr{A}(C, \mathcal{L}, d)$ for some curve C and line bundle $\mathcal{L}$. Consider the $\mathbb{K}$-linear Frobenius pull-back of $\mathcal{A}'$ composed r times, as in Remark 7.5. Denote the resulting arrangement by $\mathcal{A}$. Then, by Remark 7.5 and Theorem 8.1, there are nonsingular projective surfaces of general type X with $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to $\frac{\bar{c}_1^2(\mathcal{A}_\Delta)}{\bar{c}_2(\mathcal{A}_\Delta)} = 2 + p^r\left(\frac{\bar{c}_1^2(\mathcal{A}'_\Delta)}{\bar{c}_2(\mathcal{A}'_\Delta)} - 2\right)$, and so arbitrarily large. We can prove that $\pi_1^{\text{ét}}(X) \simeq \pi_1^{\text{ét}}(C)$ (see [23]). Therefore, for any given positive characteristic and nonsingular projective curve C, there are nonsingular projective surfaces of general type X with $\pi_1^{\text{ét}}(X) \simeq \pi_1^{\text{ét}}(C)$ and violating any sort of Miyaoka–Yau inequality.

9. APPENDIX: LOG INEQUALITIES

In this section, the ground field is $\mathbb{C}$. After fixing $\mathscr{A}(C, \mathcal{L}, d)$, it is clearly of our interest to find optimal upper bounds for $\frac{\bar{c}_1^2}{\bar{c}_2}$ for extended and partially extended arrangements (see Remark 7.3). Arrangements attaining upper bounds should be very special, and they would produce interesting surfaces via Theorem 8.1.

In this appendix, we show through Theorems 9.2 and 9.3 how this question about sharp upper bounds is connected to old questions by Lang and others on effective height inequalities [15, pp. 149–153], via an inequality of Liu [16, Theorem 0.1]. Also, in a more general setting, we show a way to obtain strictness for the log inequalities in Corollaries 7.2 and 7.4, and Theorems 9.2 and 9.3. The next lemma follows from Kobayashi [14] and Mok [17].

LEMMA 9.1. *Let Y be a smooth projective surface, and let D be a simple normal crossing divisor in Y. Assume $K_Y + D$ is big and nef, and $\bar{c}_1^2(Y, D) = 3\bar{c}_2(Y, D)$.*

Then, D is a disjoint union of smooth elliptic curves.

PROOF. By [14, p. 46], $K_Y + D$ big and nef and $\bar{c}_1^2(Y, D) = 3\bar{c}_2(Y, D)$ imply that the universal covering of $Y \backslash D$ is the complex two dimensional ball $\mathbb{B} = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$. Hence, there exist a discrete group Γ in Aut($\mathbb{B}$) such that $\mathbb{B}/\Gamma \simeq Y \backslash D$. In particular, $\mathbb{B}/\Gamma$ has finite volume. Notice that Γ is torsion-free since it acts freely on $\mathbb{B}$. Therefore, by [17, Main Theorem], there exists a smooth projective Mumford compactification W of $\mathbb{B}/\Gamma$ such that $W \backslash (\mathbb{B}/\Gamma)$ is a disjoint union of smooth elliptic curves E_i. In this way, we obtain a birational map $W \dashrightarrow Y$. We now resolve this map and get a birational morphism $\sigma : \tilde{W} \to Y$. Then, the inverse image $\tilde{E}_i$ of each E_i under σ dominates D_i, after reordering indices. It is easy to see that $\tilde{E}_i = D_i$. But $\tilde{E}_i$ is a smooth elliptic curve with some finite trees of $\mathbb{P}^1$'s attached. Given one of these trees, one has a smooth rational curve F intersecting $\tilde{E}_i - F$ at one point. But then $0 \le (K_Y + D).F = -2 + 1 = -1$. So, there are no trees, and $E_i = D_i$ for all i. □

Let $f : Y \to C$ be a fibration of a smooth projective surface over a smooth projective curve C, denote by g the genus of the generic (connected) fiber of f and by q the genus of C. Let $\omega_{Y|C} := K_Y - f^*(\omega_C)$ be the relative dualizing sheaf.

Let $S_1, \dots, S_n$ be n mutually disjoint sections of f. Assume f is a semi-stable fibration of n-pointed curves of genus g, marked by these sections. Let

$$D = S_1 + \cdots + S_n + f^*(c_1 + \cdots + c_\delta)$$

where $c_1, \dots, c_\delta$ are the images of the singular fibers of f.

Theorem 9.2. *Let f be not isotrivial, i.e., the moduli of its fibers varies as n-pointed semi-stable curves. Assume $D \neq \emptyset$, and $n \geq 1$ when $g = 1$. Then*

$$0 < \bar{c}_1^2(Y, D) < 3\bar{c}_2(Y, D).$$

Proof. The generic fiber has $\bar{\kappa}(f^{-1}(c), (S_1 + \cdots + S_n)|_{f^{-1}(c)}) = 1$ ($\bar{\kappa}$ denotes the log Kodaira dimension): $\mathbb{P}^1$ minus at least four points or elliptic curve minus at least one point or the rest. Now, since f is not isotrivial, it has at least 3 singular fibers when $C = \mathbb{P}^1$ (see [2]), or at least one when C is an elliptic curve (see [1, p. 127]). So, in any case, the base is of log general type (on the base we take the log curve $(C, c_1 + \cdots + c_\delta)$). By a theorem of Kawamata [10, Theorem 11.15], we have for a general $c \in C$

$$\bar{\kappa}(Y, D) \geq \bar{\kappa}(f^{-1}(c), (S_1 + \cdots + S_n)|_{f^{-1}(c)}) + \bar{\kappa}(C, c_1 + \cdots + c_\delta),$$

and so (Y, D) is of log general type. Notice that D is a semi-stable curve, just because the fibration is semi-stable, and when $C = \mathbb{P}^1$ we have at least 3 singular fibres. Therefore, Sakai's theorem [18] applies, and so $\bar{c}_1^2(Y, D) \leq 3\bar{c}_2(Y, D)$. Below we show that $K_Y + D$ is nef to obtain the strict inequality.

(case $g = 0$): We can explicitly show that $K_Y + D$ is nef (Corollary 7.2).

(case $g \geq 2$): Let $g : Y \to Y'$ be the relative minimal model of f. Let E_i be the exceptional divisors. We can write

$$K_Y + D \equiv g^*(\omega_{Y'|C}) + \sum_k n_k E_k + f^*(\omega_C) + D$$

where for some positive integers n_i's. Since $g \geq 2$, the dualizing sheaf $\omega_{Y'|C}$ is nef (due to Arakelov). If $q > 0$, then $f^*(\omega_C)$ is nef as well, and so we check that $K_Y + D$ is nef by intersecting it with components of D (notice that D includes E_k's). If $q = 0$, we have at least 3 singular fibers, and so we delete $f^*(\omega_C)$ using D, and again to check nef we intersect $K_Y + D$ with the components of D.

(case $g = 1$): As in the previous case, we go to Y'. Notice that Y' has no multiple fiber. By the canonical bundle formula we have [1, p. 214]

$$K_Y + D \equiv (\chi(Y) + 2(q-1))F + \sum_k n_k E_k + D$$

where F is a general fiber of f. So, if $q > 0$, we are done by the previous argument. If $q = 0$, we are done by the same argument, since there are at least 3 singular fibers by [2].

Therefore, $K_Y + D$ is nef, and strict inequality follows from Lemma 9.1. □

Theorem 9.2 is also valid when $D = \emptyset$ (i.e., a Kodaira fibration). It follows from [16, Theorem 0.1]. Our argument does not prove it. Actually, up to the case $D = \emptyset$, Theorem 9.2 is just a small extension of [16, Theorem 0.1] (since we also consider the cases $g = 0, 1$), as we now see. We have $\bar{c}_2(Y, D) = e(X) - e(D)$ ($e(A)$ is the Euler topological characteristic of A) and the usual formula [3, Lemma VI.4]

$$e(Y) = 4(g-1)(q-1) + \sum_{i=1}^{\delta}(e(f^{-1}(c_i)) - e(F))$$

where F is a generic fiber of f. One sees that $e(D) = \sum_{i=1}^{\delta} e(f^{-1}(c_i)) + n(2-2q) - \delta n$. So, $\bar{c}_2(Y, D) = (2g - 2 + n)(2q - 2 + \delta)$. Obviously $\omega_{Y|C} + D = K_Y + D - f^*(\omega_C + \sum_{i=1}^{\delta} c_i)$. We then square it and see the inequality in [16, Theorem 0.1].

Remark 9.1. In [16, Theorem 0.1], where $g \geq 2$ is assumed, we have that f is isotrivial if and only if $\bar{c}_1^2(Y, D) = 3\bar{c}_2(Y, D)$. But we now show that this corresponds to uninteresting situations. First notice that $K_Y + D$ is nef by the same argument used in (case $g \geq 2$) of Theorem 9.2. If $\bar{c}_1^2(Y, D) > 0$, then $K_Y + D$ becomes big and nef, and we apply Lemma 9.1 to obtain a contradiction, unless $q = 1$ and $\delta = 0$. But then $\bar{c}_2(Y, D) = 0$, which is a contradiction to our assumption $\bar{c}_1^2(Y, D) > 0$. Therefore, we are in the trivial case $\bar{c}_1^2(Y, D) = \bar{c}_2(Y, D) = 0$.

For completeness' sake, we explicitly show the connection with height inequalities of algebraic points on curves over function fields. This is another proof of Tan's height inequality [19, Theorem A]. Let $f : Y \to C$ be a connected fibration as before, denoting by g the genus of the generic fiber of f and by q the genus of C. Assume that f is semi-stable. Let $\mathrm{K}(C)$ be the function field of C. For an algebraic point $P \in Y(\overline{\mathrm{K}(C)})$, let C_P be the corresponding horizontal curve (i.e. multisection) in Y. As usual, let

$$h_K(P) = \frac{\omega_{Y|C}.C_P}{F.C_P} \quad d(P) = \frac{2g(\overline{C_P}) - 2}{F.C_P}$$

be the geometric height and the geometric logarithmic discriminant respectively. The curve $\overline{C_P}$ is the normalization of C_P, and F is a general fiber of f.

Theorem 9.3. *Assume $g \geq 2$, and that f is not isotrivial. Let δ be the number of singular fibers of f. Then, for any algebraic point P, we have*

$$h_K(P) < (2g-1)(d(P) + \delta) - \omega_{Y|C}^2.$$

Proof. Let C_P be the horizontal curve in Y defined by P, and let $g : \overline{C_P} \to C$ be the composition of the normalization of C_P with f, so $d := \deg(g) = F.C_P$. Then, we have

$$\begin{array}{ccccc} Y_P & \longrightarrow & \overline{Y} & \xrightarrow{G} & Y \\ & \searrow^{f_P} & \downarrow \bar{f} & & \downarrow f \\ & & \overline{C_P} & \xrightarrow{g} & C \end{array}$$

where $\bar{f}$ is the unique semi-stable fibration induced by g. Notice that $G^*(C_P)$ contains a section S of $\bar{f}$, by construction. The map f_P is the induced semi-stable fibration with a marked point (marked by S). Let δ_P be the number of singular fibers of f_P. Notice that δ_P is at most $d\delta$. Consider $D = S' + f_P^*(c_1 + \cdots + c_{\delta_P})$ where $c_1, \ldots, c_{\delta_P}$ are the images of the singular fibers of f_P in $\overline{C_P}$, and S' is the strict transform of S. We now apply Theorem 9.2 to have $(K_{Y_P} + D)^2 < 3(2g - 2 + 1)(2g(\overline{C_P}) - 2 + \delta_P)$. But, one checks that $(K_{Y_P} + D)^2 = (K_{\overline{Y}} + S + \bar{f}^*(c_1 + \cdots + c_{\delta_P}))^2 = (\omega_{\overline{Y}|\overline{C_P}} + S + (\delta_P + 2g(\overline{C_P}) - 2)F)^2$. Also, since $\bar{f}$ is semi-stable, we know that $G^*(\omega_{Y|C}) = \omega_{\overline{Y}|\overline{C_P}}$, and by the projection formula $S.\omega_{\overline{Y}|\overline{C_P}} = C_P.\omega_{Y|C}$. So, the log inequality above becomes

$$d\omega_{Y|C}^2 + \omega_{Y|C}.C_P + 2(2g-1)(2g(\overline{C_P}) - 2 + \delta_P) < 3(2g-1)(2g(\overline{C_P}) - 2 + \delta_P),$$

and so we rearrange to obtain the claimed height inequality (also use $\delta_P \leq d\delta$). □

References

[1] W. P. Barth - K. Hulek - C. A. M. Peters - A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., second edition, vol. 4, Springer-Verlag, Berlin, 2004.

[2] A. Beauville, *Le nombre minimum de fibres singulières d'une courbe sur* $\mathbb{P}^1$, Astérisque 86 (1981), 97–108.

[3] A. Beauville, *Complex algebraic surfaces*, London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996.

[4] M. Beck - I. M. Gessel - T. Komatsu, *The polynomial part of a restricted partition function related to the Frobenius problem*, Electron. J. Combin. 8 (2001), no. 1, Note 7, 5 pp. (electronic).

[5] N. G. de Bruijn - P. Erdös, *On a combinatorial problem*, Nederl. Akad. Wetensch., Proc. 51 (1948), 1277–1279 = Indagationes Math. 10 (1948), 421–423.

[6] A.-M. Castravet - J. Tevelev, *Exceptional loci on* $\overline{M}_{0,n}$ *and hypergraph curves*, arXiv:0809.1699v1.

[7] P. Deligne - D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.

[8] R. Hartshorne, *Algebraic geometry*, Graduate Text in Mathematics v. 52, Springer, 1977.

[9] F. Hirzebruch, *Arrangements of lines and algebraic surfaces*, Arithmetic and geometry, Vol. II, Progr. Math. 36, Birkhäuser, Boston, Mass., 1983, 113–140.

[10] S. Iitaka, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 76, Springer-Verlag, New York, 1982.
[11] M. M. Kapranov, *Veronese curves and Grothendieck-Knudsen moduli space* $\overline{M_{0,n}}$, J. Algebraic Geom. 2 (1993), 239–262.
[12] M. M. Kapranov, *Chow quotients of Grassmannians I*, Adv. Soviet Math. 16, part 2, A.M.S. (1993), 29–110.
[13] F. F. Knudsen, *The projectivety of moduli spaces of stable curves II: the stacks* $M_{g,n}$, Math. Scand. 52 (1983), 161–199.
[14] R. Kobayashi, *Einstein-Kaehler metrics on open algebraic surfaces of general type*, Tohoku Math. J. (2), Vol. 37, No. 1 (1985), 43–77.
[15] S. Lang, *Number Theory III: diophantine geometry*, Encyclopaedia of Mathematical Sciences: V. 60 (1991).
[16] K. Liu, *Geometric height inequalities*, Math. Res. Lett. 3 (1996), 693–702.
[17] N. Mok, *Projective-algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume*, pre-print in http://hkumath.hku.hk/~imr/IMRPreprintSeries/2009/IMR2099-4.pdf (2009).
[18] F. Sakai, *Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps*, Math. Ann. 254 (1980), no. 2, 89–120.
[19] S.-L. Tan, *Height inequality of algebraic points on curves over functional fields*, J. Reine Angew. Math. 461 (1995), 123–135.
[20] G. Urzúa, *Arrangements of curves and algebraic surfaces*, Ph.D. Thesis, University of Michigan (2008).
[21] G. Urzúa, *On line arrangements with applications to* 3*-nets*, Adv. Geom. 10 (2010), 287–310.
[22] G. Urzúa, *Arrangements of curves and algebraic surfaces*, J. of Algebraic Geom. 19 (2010), 335–365.
[23] G. Urzúa, *Simply connected surfaces in positive characteristic*, in preparation (2009).

Received 29 March 2011.

Current address
Department of Mathematics and Statistics
University of Massachusetts at Amherst, USA
urzua@math.umass.edu

Future Address (from June 2011)
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Campus San Joaquín
Avenida Vicuña Mackenna 4860 Santiago
Chile
urzua@mat.puc.cl