



**Algebraic Geometry** — *Automorphisms of rational manifolds of positive entropy with Siegel disks*, by KEIJI OGUIISO and FABIO PERRONI, presented by Fabrizio Catanese on 16 June 2011.

ABSTRACT. — Using McMullen’s rational surface automorphisms, we construct projective rational manifolds of higher dimension admitting automorphisms of positive entropy with arbitrarily high number of Siegel disks and those with exactly one Siegel disk.

KEY WORDS: Rational manifolds, automorphisms of positive entropy, Siegel disks.

MATHEMATICS SUBJECT CLASSIFICATION 2000: 14J50, 14M20, 37B40, 37F50.

## 1. INTRODUCTION

In his beautiful paper [Mc07], McMullen constructed rational surface automorphisms of positive entropy with Siegel disks. They are the first examples among automorphisms of projective manifolds. (See also [BK06], [BK09].) From one side, positive entropy indicates that general orbits spread out vastly even though the initial points are very close, from which one might expect that the general orbit could be densely distributed. But on the other side, the existence of Siegel disks shows that there is no dense orbit and the orbit of any point in the disk never goes out of the disk. This contrast makes the study of automorphisms of manifolds of positive entropy with Siegel disks very attractive. In McMullen’s construction, the automorphism has exactly one Siegel disk and it is arithmetic (see also Section 2 for definitions and more details). It is then natural to ask:

- (i) How about in higher dimension?
- (ii) How many Siegel disks can an automorphism of positive entropy have?

The aim of this paper is to address these questions. More precisely we prove:

**THEOREM 1.1.**

1. *There is a pair  $(X, g)$  of a non-singular complex projective rational 3-fold  $X$  and an automorphism  $g \in \text{Aut}(X)$  such that:*
  - (1-i) *the entropy  $h(g)$  of  $g$  is positive; and*
  - (1-ii)  *$g$  admits exactly 2 Siegel disks and they are arithmetic.*
2. *Let  $n$  be any integer such that  $n \geq 4$  and let  $N$  be an arbitrary positive integer. Then, for each such  $n$  and  $N$ , there is a pair  $(X, g)$  of a non-singular complex projective rational variety  $X$  of dimension  $n$  and an automorphism  $g \in \text{Aut}(X)$  such that:*

- (2-i) *the entropy  $h(g)$  of  $g$  is positive; and*
- (2-ii)  *$g$  admits at least  $N$  Siegel disks and they are arithmetic.*
- 3. *Let  $n$  be any even integer such that  $n \geq 4$ . Then, for each such  $n$ , there is a pair  $(X, g)$  of a non-singular complex projective rational variety  $X$  of dimension  $n$  and an automorphism  $g \in \text{Aut}(X)$  such that:*
  - (3-i) *the entropy  $h(g)$  of  $g$  is positive; and*
  - (3-ii)  *$g$  admits exactly one Siegel disk and it is arithmetic.*

We believe that this theorem gives the first examples of automorphisms of projective manifolds of positive entropy with Siegel disks *in dimension  $\geq 3$* . Here, it is essential to make the entropy positive. Indeed, there are a lot of automorphisms of  $\mathbb{P}^n$ , of which the entropy is necessarily 0, having (arithmetic) Siegel disks. Our construction is the product construction made of McMullen's rational surfaces, projective toric manifolds and their automorphisms. In this sense, our construction of manifolds are rather easy modulo McMullen's deep construction. Nevertheless we should also note that for a manifold  $S$  and for an automorphism  $g$  with a fixed point  $P$ , the product automorphism  $g \times g$  of  $S \times S$  does not have a Siegel disk at  $(P, P)$ , even if  $g$  itself has a Siegel disk at  $P$ . So, the essential point in the product construction is to *choose* manifolds and automorphisms so that the eigenvalues of the product morphism at the fixed point are multiplicatively independent within the algebraic integers of absolute value 1. This turns out to be a kind of arithmetic problem which has its own interest. The precise formulation is given in Definition 4.1, and its solution is contained in Theorem 4.2.

A very interesting question is whether there are rational threefolds having automorphisms of positive entropy with Siegel disks that are not induced from automorphisms of lower dimensional pieces, such as for example the base or the fibers of a fibration. We could not yet give an answer to this important problem. One possible approach would be to consider blow-ups of  $\mathbb{P}^3$ , generalizing the construction of McMullen. In this approach [DO] Chapter V might be useful (see also [BK11]).

The structure of the paper is the following. In Section 2 we review McMullen's construction of automorphisms of rational surfaces of positive entropy with Siegel disks [Mc07]. We also recall some basic facts about the topological entropy and about Siegel disks. In Section 3 we present our construction of automorphisms of higher dimensional projective rational manifolds of positive entropy with Siegel disks and we prove that they satisfy all the properties stated in Theorem 1.1 except the fact that the Siegel disks are arithmetic. This fact is a consequence of Theorem 4.2 which is stated and proved in Section 4.

Throughout this note, we work over the field of complex numbers  $\mathbb{C}$ .

## 2. McMULLEN'S CONSTRUCTION OF AUTOMORPHISMS OF RATIONAL SURFACES OF POSITIVE ENTROPY WITH SIEGEL DISKS

In this section we review McMullen's construction of rational surface automorphisms together with some relevant notions.

(i) *Entropy.* Let  $X$  be a compact metric space with distance function  $d$ . Let  $g : X \rightarrow X$  be a continuous surjective map. Roughly speaking, the *entropy* of  $g$  is a measure of “how fast two orbits  $\{g^k(x)\}_{k \geq 0}, \{g^k(y)\}_{k \geq 0}$  spread out when  $k \rightarrow \infty$ ”. We recall here its definition and a characterization in cohomological terms that will be used later. For more details we refer to [KH95]. For any  $n \in \mathbb{Z}_{>0}$ , consider the metric

$$d_{g,n}(x, y) := \max\{d(g^k(x), g^k(y)) \mid 0 \leq k \leq n - 1\}.$$

The *entropy* of  $g$  is then defined as ([KH95], Page 108, formula (3.1.10)):

$$h(g) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log S(g, \varepsilon, n)}{n}.$$

Here  $S(g, \varepsilon, n)$  is the *minimal* number of  $\varepsilon$ -balls, with respect to  $d_{g,n}$ , that cover  $X$ . It is shown that  $h(g)$  does not depend on the choice of the distance  $d$  giving the same topology on  $X$  (see e.g. [KH95], Page 109, Proposition 3.1.2). From this definition it is easy to grasp the meaning of the entropy. However, for our computations, the following fundamental theorem, due to Gromov-Yomdin-Friedland ([Fr95], Theorem 2.1), will be more convenient:

**THEOREM 2.1.** *Let  $X$  be a compact Kähler manifold of dimension  $n$  and let  $g : X \rightarrow X$  be a holomorphic surjective map. Then*

$$h(g) = \log \rho \left( g^* \left| \bigoplus_{k=0}^n H^{2k}(X, \mathbb{Z}) \right. \right).$$

Here  $\rho(g^* | \bigoplus_{k=0}^n H^{2k}(X, \mathbb{Z}))$  is the spectral radius of the action of  $g^*$  on the total cohomology ring of even degree. In particular,  $h(g)$  is the logarithm of an algebraic integer.

We refer to [Zh09] for some results about the role of the topological entropy in the classification of higher dimensional varieties.

(ii) *Salem polynomials and Salem numbers.*

**DEFINITION 2.2.** *A Salem polynomial is a monic irreducible reciprocal polynomial  $\varphi(x)$  in  $\mathbb{Z}[x]$  such that*

$$\{x \in \mathbb{C} \mid \varphi(x) = 0\} = \left\{ \eta, \frac{1}{\eta}, \delta_1, \bar{\delta}_1, \dots, \delta_{n-1}, \bar{\delta}_{n-1} \right\},$$

where  $|\delta_i| = 1$  and  $\eta > 1$  is real. Notice that  $\varphi(x)$  is necessarily of even degree.

A Salem number is the unique real root  $\eta > 1$ . In other words, a Salem number of degree  $2n$  is a real algebraic integer  $\eta > 1$  whose Galois conjugates consist of  $1/\eta$  and  $2n - 2$  imaginary numbers on  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ .

Let  $\varphi_{2n}(x)$  be a Salem polynomial of degree  $2n$ . As  $\varphi_{2n}(x)$  is monic irreducible and reciprocal, there is a unique monic irreducible polynomial  $r_n(x) \in \mathbb{Z}[x]$  of degree  $n$  such that

$$\varphi_{2n}(x) = x^n \cdot r_n\left(x + \frac{1}{x}\right).$$

We call this polynomial  $r_n(x)$  the *Salem trace polynomial* of  $\varphi_{2n}(x)$ . If

$$\eta, \frac{1}{\eta}, \delta_i, \bar{\delta}_i = \frac{1}{\delta_i} \quad (1 \leq i \leq n - 1)$$

are the roots of  $\varphi_{2n}(x) = 0$ , then the roots of  $r_n(x) = 0$  are:

$$\eta + \frac{1}{\eta}, \delta_i + \frac{1}{\delta_i} = \delta_i + \bar{\delta}_i \quad (1 \leq i \leq n - 1).$$

(iii) *Coxeter element.* By  $E_n(-1)$  we denote the lattice represented by the Dynkin diagram with  $n$  vertices  $s_k$  ( $0 \leq k \leq n - 1$ ) of self-intersection  $-2$  such that  $n - 1$  vertices  $s_1, s_2, \dots, s_{n-1}$  form a Dynkin diagram of type  $A_{n-1}(-1)$  in this order and the remaining vertex  $s_0$  joins only to the vertex  $s_3$  by a simple line, as shown in Figure 1.

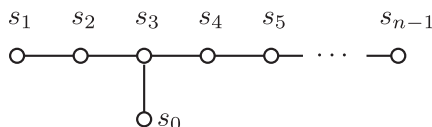


Figure 1. The  $E_n(-1)$  diagram.

The lattice  $E_n(-1)$  is of signature  $(1, n - 1)$ , when  $n \geq 10$ .

Let  $W(E_n(-1))$  be the Weyl group of  $E_n(-1)$ , i.e., the subgroup of  $O(E_n(-1))$  generated by the reflections

$$r_k(x) = x + (x, s_k)s_k.$$

The Weyl group  $W(E_n(-1))$  has a special conjugacy class called the *Coxeter class*. It is the conjugacy class of the product (in any order in our case) of the reflections

$$w_n := \prod_{k=0}^{n-1} r_k.$$

The following theorem follows from either [BK09], Theorem 3.3 or [GMH09], Theorem 1.1, Corollary 1.2. We follow the notation of [GMH09]:

**THEOREM 2.3.** *Let  $E_n(x)$  be the characteristic polynomial of the Coxeter element  $w_n$ . Then, for  $n \geq 10$ ,*

$$E_n(x) = C_n(x)\varphi(x)$$

where  $C_n(x)$  is the product of cyclotomic polynomials and  $\varphi(x)$  is a Salem polynomial. Moreover,  $C_n(x) = C_m(x)$  if  $n \equiv m \pmod{360}$ .

By [GMH09], Corollary 4.3, we have

$$(2.0.1) \quad E_n(x)(x - 1) = x^{n-2}(x^3 - x - 1) + (x^3 + x^2 - 1),$$

hence the formula in Theorem 2.3 can be used to determine both  $C_n(x)$  and  $\varphi(x)$ .

The following example, for  $n = 19$ , will be used only in Section 4:

EXAMPLE 2.4.

$$E_{19}(x) = (x + 1)(x^4 + x^3 + x^2 + x + 1)\varphi_{14}(x).$$

Here  $\varphi_{14}(x)$  is a Salem polynomial of degree 14:

$$\varphi_{14}(x) = x^{14} - x^{13} - x^{11} + x^{10} - x^7 + x^4 - x^3 - x + 1.$$

(iv) *Siegel disks and arithmetic Siegel disks.*

DEFINITION 2.5. (1) Let  $\Delta^n$  be an  $n$ -dimensional unit disk with linear coordinates

$$(z_1, z_2, \dots, z_n).$$

A linear automorphism (written under the coordinate action)

$$f^*(z_1, z_2, \dots, z_n) = (\rho_1 z_1, \rho_2 z_2, \dots, \rho_n z_n)$$

is called an irrational rotation if

$$|\rho_1| = |\rho_2| = \dots = |\rho_n| = 1,$$

and  $\rho_1, \rho_2, \dots, \rho_n$  are multiplicatively independent, i.e.

$$(m_1, m_2, \dots, m_n) = (0, 0, \dots, 0)$$

is the only integer solution to

$$\rho_1^{m_1} \rho_2^{m_2} \dots \rho_n^{m_n} = 1.$$

(2) Let  $X$  be a complex manifold of dimension  $n$  and let  $g$  be an automorphism of  $X$ . A domain  $U \subset X$  is called a Siegel disk of  $(X, g)$  if  $g(U) = U$  and  $(U, g|U)$  is isomorphic to some irrational rotation  $(\Delta^n, f)$ . In other words,  $g$  has a Siegel disk if and only if there is a fixed point  $P$  at which  $g$  is locally analytically linearized as in the form of an irrational rotation. We call the Siegel disk arithmetic if in addition all  $\rho_i$  are algebraic integers.

The first examples of surface automorphisms with Siegel disks were discovered by McMullen ([Mc02], Theorem 1.1) within K3 surfaces. See also [Og10], Theo-

rem 1.1 for a similar example. The resultant K3 surfaces  $X$  are necessarily of algebraic dimension 0 ([Mc02], Theorem 3.5, see also [Og08], Theorem 2.4). Later, McMullen ([Mc07], Theorem 10.1) found rational surface automorphisms with arithmetic Siegel disks.

(v) *McMullen’s pair.* Let  $S$  be the blowup of  $\mathbb{P}^2$  at  $n$  distinct points. Then,  $H^2(S, \mathbb{Z})$  is isomorphic to the odd unimodular lattice of signature  $(1, n)$ . The orthogonal complement  $(-K_S)^\perp$  is then isomorphic to  $E_n(-1)$  and  $\text{Aut}(S)$  naturally acts on  $E_n(-1)$  (under a fixed marking). As a part of more general results, McMullen proves the following theorem (See [Mc07], Theorem 10.1, see also Theorem 10.3, proof of Theorem 10.4 and the formula 9.1):

**THEOREM 2.6.** *Let  $n$  be a sufficiently large integer such that  $n \equiv 1 \pmod 6$ . Then, for each such  $n$ , there are a rational surface  $S = S(n)$  which is the blow up of  $\mathbb{P}^2$  at  $n$  distinct points and an automorphism  $F = F(n)$  such that:*

1. *The characteristic polynomial of  $F^* \mid H^2(S, \mathbb{Z})$  is*

$$E_n(x)(x - 1) = (x - 1)C_n(x)\varphi(x)$$

*where  $E_n(x)$  is the characteristic polynomial of the Coxeter element of  $E_n(-1)$ ,  $C_n(x)$  is the product of cyclotomic polynomials and  $\varphi(x)$  is a Salem polynomial.*

2. *The fixed point set  $S^F$  consists of exactly 2 points, say,  $P$  and  $Q$ . Moreover,  $F$  has a Siegel disk at  $Q$  but  $F$  has no Siegel disk at  $P$  (in fact the eigenvalues of  $F^* \mid T_{S,P}^*$  are not multiplicatively independent).*
3. *Let*

$$F^*(z_1, z_2) = (\alpha(n)z_1, \beta(n)z_2)$$

*be the locally analytic linearization of  $F$  at  $Q$ . So,  $\alpha(n)$  and  $\beta(n)$  are multiplicatively independent and of absolute value 1. Then, there is a root  $\delta(n)$  of  $\varphi(x) = 0$  of absolute value 1 such that  $\alpha(n)$  and  $\beta(n)$  satisfy*

$$\alpha(n)\beta(n) = \delta(n), \quad 2 + \frac{\alpha(n)}{\beta(n)} + \frac{\beta(n)}{\alpha(n)} = \frac{\delta(n)(1 + \delta(n))^2}{(1 + \delta(n) + \delta(n)^2)^2}.$$

*In particular,  $\alpha(n)^2$  and  $\beta(n)^2$  are the roots of the quadratic equation of the form*

$$x^2 + a(\delta(n))x + \delta(n)^2 = 0$$

*where  $a(x) \in \mathbb{Q}(x)$ .*

4. *There are another root  $\delta'(n)$  of  $\varphi(x) = 0$  of absolute value 1 and complex numbers  $\alpha'(n), \beta'(n)$  such that*

$$|\alpha'(n)/\beta'(n)| \neq 1, \\ \alpha'(n)\beta'(n) = \delta'(n), \quad 2 + \frac{\alpha'(n)}{\beta'(n)} + \frac{\beta'(n)}{\alpha'(n)} = \frac{\delta'(n)(1 + \delta'(n))^2}{(1 + \delta'(n) + \delta'(n)^2)^2}.$$

In particular,  $\alpha'(n)^2$  and  $\beta'(n)^2$  are the roots of the quadratic equation

$$x^2 + a(\delta'(n))x + \delta'(n)^2 = 0.$$

We call any pair  $(S, F)$  as in Theorem 2.6 a *McMullen's pair*. Notice that, by Theorem 2.6 (1), any McMullen's pair is of positive entropy.

In the next proposition we slightly improve the previous result by showing that the Siegel disk of  $F$  at  $Q$  is arithmetic. This is only needed in Section 4.

**PROPOSITION 2.7.** *Let  $\alpha(n)$  and  $\beta(n)$  be as in Theorem 2.6. Then  $\alpha(n)$ ,  $\beta(n)$ ,  $\alpha(n)^{-1}$  and  $\beta(n)^{-1}$  are algebraic integers.*

**PROOF.** Set  $\alpha := \alpha(n)$ ,  $\beta := \beta(n)$  and  $\delta := \delta(n)$ . We first prove the Proposition for  $\alpha$  and  $\beta$ .

From the previous Theorem 2.6 (3), we know that  $\delta = \alpha\beta$  is an algebraic integer, therefore it is enough to show that  $\alpha + \beta$  is so. From the equation:

$$\frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{\delta(1 + \delta)^2}{(1 + \delta + \delta^2)^2},$$

we have that

$$\alpha + \beta = \pm \frac{\delta(1 + \delta)}{1 + \delta + \delta^2},$$

hence we only need to prove that  $\frac{1}{1 + \delta + \delta^2}$  is an algebraic integer. We use now formula (2.0.1) for the characteristic polynomial  $E_n(x)$  of the Coxeter element  $w_n$ . Since  $n = 6k + 1$ , it readily follows from (2.0.1) that there exists  $A(x) \in \mathbb{Z}[x]$  such that

$$(2.0.2) \quad E_n(x)(x - 1) = (x^2 + x + 1)A(x) - (x + 2).$$

Since  $E_n(\delta) = 0$ , we have:

$$(2.0.3) \quad \frac{A(\delta)}{\delta + 2} = \frac{1}{\delta^2 + \delta + 1}.$$

On the other hand, we can write

$$\frac{A(\delta)}{\delta + 2} = B(\delta) + \frac{A(-2)}{\delta + 2}, \quad \text{for some } B(x) \in \mathbb{Z}[x].$$

Hence, it is enough to prove that  $\frac{A(-2)}{\delta + 2}$  is an algebraic integer. From (2.0.2) it follows that  $E_n(-2) = -A(-2)$ , therefore, there exists  $C(x) \in \mathbb{Z}[x]$  such that

$$E_n(x) = (x + 2)C(x) - A(-2).$$

We conclude that

$$\frac{A(-2)}{\delta + 2} = C(\delta),$$

which is an algebraic integer, thus  $\alpha + \beta$  is an algebraic integer.

The statement for  $\alpha^{-1}$  and  $\beta^{-1}$  follows as before by replacing  $F$  with  $F^{-1}$  in Theorem 2.6 and from the fact that  $\frac{1}{1+\delta+\delta^2}$  is an algebraic number. □

### 3. AUTOMORPHISMS OF HIGHER DIMENSIONAL MANIFOLDS OF POSITIVE ENTROPY WITH SIEGEL DISKS

In this section we construct automorphisms of higher dimensional projective rational manifolds of positive entropy with Siegel disks. Thus proving Theorem 1.1 (1-i), (2-i) and (3-i). To complete the proof of Theorem 1.1 we need to show that our construction yields *arithmetic* Siegel disks. This follows from Theorem 4.2 which is stated and proved in Section 4.

*3.1. Proof of Theorem 1.1.1.* Let  $(S, F)$  be a McMullen’s pair as defined in Section 2. Let  $a \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$  and let  $f_a : \mathbb{P}^1 \rightarrow \mathbb{P}^1, f_a(y) = ay$ . Consider  $X := S \times \mathbb{P}^1$  and  $g := (F, f_a) \in \text{Aut}(X)$ .

The topological entropy of  $g$  coincides with that of  $F$ , as it is possible to deform continuously  $g$  to  $(F, id_{\mathbb{P}^1})$  and  $h(g)$  is determined by its representation on  $H^*(X, \mathbb{Z})$  by Theorem 2.1 (see also Proposition 3.2 below). Therefore Theorem 2.6 implies that (1-i) of Theorem 1.1 holds for the pair  $(X, g)$ .

The fixed point set of  $g$  is  $\{P, Q\} \times \{0, \infty\}$ . Since the eigenvalues of the linearization of  $F$  at  $P$  are not multiplicatively independent,  $g$  has neither a Siegel disk at  $(P, 0)$  nor at  $(P, \infty)$ , for any  $a \in S^1$ . On the other hand, for a generic choice of  $a \in S^1$ ,  $g$  will have two Siegel disks, one at  $(Q, 0)$  and another at  $(Q, \infty)$ . Indeed the set of  $a \in S^1$  such that the sequence  $(\alpha, \beta, a)$  is not multiplicatively independent is countable, where  $\alpha$  and  $\beta$  are as in Theorem 2.6.

To complete the proof of (1-ii) it remains to show that it is possible to choose  $a \in S^1$  such that these two Siegel disks are arithmetic. This fact relies on Theorem 4.2 which roughly speaking states that, it is always possible to extend any sequence of multiplicatively independent algebraic integers of absolute value 1 to a strictly bigger sequence satisfying the same properties. The statement and the proof of this result is postponed in Section 4. This concludes the proof of Theorem 1.1.1.

*3.2. Proof of Theorem 1.1.2.* The previous construction can be extended to any dimension by replacing  $\mathbb{P}^1$  with  $\mathbb{P}^d$  and by taking automorphisms  $f_a \in \text{Aut}(\mathbb{P}^d)$  of the form  $f_a(y_1, y_2, \dots, y_{d+1}) := (a_1y_1, a_2y_2, \dots, a_{d+1}y_{d+1})$ , where  $a := (a_1, \dots, a_{d+1}) \in (S^1)^{d+1}$ . Let  $g := (F, f_a) \in \text{Aut}(S \times \mathbb{P}^d)$ . The topological entropy of  $g$  satisfies  $h(g) = h(F)$  for the same reasons of the case  $d = 1$  (see also Proposition 3.2 below), moreover, for a generic choice of  $a \in (S^1)^{d+1}$ ,  $g$  will have



$d + 1$  Siegel disks. The fact that it is possible to choose such an  $a \in (S^1)^{d+1}$  so that these Siegel disks are arithmetic is a direct consequence of Theorem 4.2 of Section 4.

Let now  $n \geq 4$ , and let  $N \geq 3$ . In order to construct examples of automorphisms of smooth rational varieties of a given dimension  $n$  with at least  $N$  Siegel disks, we replace in the previous construction  $\mathbb{P}^d$  with a smooth projective toric variety of dimension  $d = n - 2$  and we use the automorphisms that are induced by the action of the torus. In the remaining part of this subsection we prove that this method works in this more general situation.

We recall that to any complete fan  $\Delta$  in a lattice  $L$  of rank  $d$  we can associate a  $d$ -dimensional complete toric variety  $Y_\Delta$ . The group  $(\mathbb{C}^*)^d$  acts on  $Y_\Delta$  and the morphism  $(\mathbb{C}^*)^d \rightarrow \text{Aut}(Y_\Delta)$  induced by this action is injective. We will denote by  $f_a$  the automorphism of  $Y_\Delta$  associated in this way to an element

$$a := (a_1, a_2, \dots, a_d) \in (\mathbb{C}^*)^d.$$

The results about toric varieties that will be used here can be found in [Oda88] or [CK99]. We will also denote  $Y_\Delta$  simply by  $Y$  when there is no danger of confusion.

Toric manifolds are always rational and they provide several interesting examples of rational manifolds. However, the following proposition (which is also used in the proof of Theorem 1.1) shows that they never admit automorphisms of positive entropy.

**PROPOSITION 3.1.** *Let  $Y$  be a non-singular projective toric variety. Then, for any  $f \in \text{Aut}(Y)$  we have:  $h(f) = 0$ .*

**PROOF.** Recall that the cone  $\overline{\text{NE}}(Y)$  of numerically effective curves of  $Y$  is finite, rational and polyhedral (see e.g. [Oda88], Page 107, Proposition 2.26). Thus, the nef cone  $\overline{A}(Y) \subset H^2(Y, \mathbb{R})$  is also finite, rational and polyhedral, as it is the dual cone of  $\overline{\text{NE}}(Y)$ . Moreover the ample cone  $A(Y)$  is open in  $\overline{A}(Y)$  by Kleiman’s criterion.

Let  $f \in \text{Aut}(Y)$  and let  $f^* : H^2(Y, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$  be the induced map. Let  $L_i$  ( $1 \leq i \leq \ell$ ) be the 1-dimensional edges of  $\overline{A}(Y)$  and  $v_i$  be the primitive vector of  $L_i$ . Since  $(f^*)^{\ell!}$  is the identity on  $\{L_1, \dots, L_\ell\}$  it follows that  $(f^*)^{\ell!}(v_i) = v_i$  ( $1 \leq i \leq \ell$ ). As  $A(Y)$  is open in  $H^2(Y, \mathbb{R})$  (because  $h^{2,0}(Y) = h^{0,2}(Y) = 0$ ),  $(f^*)^{\ell!}$  must be the identity on  $H^2(Y, \mathbb{R})$ , therefore  $\rho(f^* | H^2(Y, \mathbb{Z})) = 1$ . It follows from [DS04] Corollaire (2.2) that  $h(f) = 0$ . □

**PROPOSITION 3.2.** *Let  $Y$  be a non-singular projective toric variety,  $f \in \text{Aut}(Y)$  and  $F \in \text{Aut}(S)$  be an automorphism of a compact Kähler manifold  $S$ . Let*

$$g := (F, f) \in \text{Aut}(S \times Y).$$

*Then  $h(g) = h(F)$ . In particular,  $g$  is of positive entropy if and only if so is  $F$ .*

PROOF. Recall that  $H^i(Y, \mathbb{Z}) = 0$  for  $i$  odd by Jurkiewicz-Danilov’s Theorem (see e.g. [Oda88], Page 134). Then, by Künneth formula:

$$H^{2k}(S \times Y, \mathbb{Q}) = \bigoplus_{\ell=0}^k H^{2\ell}(S, \mathbb{Q}) \otimes H^{2k-2\ell}(Y, \mathbb{Q}).$$

Here  $g^* = F^* \otimes f^*$  on each direct summand. By Proposition 3.1, we have that  $h(f) = 0$ . Thus, the eigenvalues of  $f^*$  on  $H^{2k-2\ell}(Y, \mathbb{Q})$  are of absolute value 1. In fact, letting  $\varepsilon_i$  be the eigenvalues of  $f^* | H^{2k-2\ell}(Y, \mathbb{Q})$  counted with multiplicities, then  $|\varepsilon_i| \leq 1, \forall i$ . But  $\det f^* = \pm 1$  as  $f^*$  is an automorphism of  $H^{2k-2\ell}(Y, \mathbb{Z})$ , so

$$\left| \prod_i \varepsilon_i \right| = 1$$

hence  $|\varepsilon_i| = 1, \forall i$ . Thus

$$\rho(g^* | H^{2\ell}(S, \mathbb{Q}) \otimes H^{2k-2\ell}(Y, \mathbb{Q})) = \rho(F^* | H^{2\ell}(S, \mathbb{Q})).$$

From this follows the result. □

Let now  $Y$  be a smooth projective toric variety associated to the fan  $\Delta$  with  $d$ -dimensional cones  $\sigma_1, \dots, \sigma_\nu, \nu \geq N$ . Notice that such a variety always exists for  $d \geq 2$ . Then  $Y = \bigcup_{p=1}^\nu U_p$ , where  $U_p := \text{Spec } \mathbb{C}[\sigma_p^\vee \cap M]$ . As we assume  $Y$  to be non-singular, each  $\mathbb{C}[\sigma_p^\vee \cap M]$  can be written as

$$\mathbb{C}[\sigma_p^\vee \cap M] = \mathbb{C}[x^{K_1(p)}, x^{K_2(p)}, \dots, x^{K_d(p)}] \subset \mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_d^{\pm 1}]$$

and  $U_p \cong \mathbb{A}^d$  with the coordinates  $(x^{K_i(p)})$ . Here we use multi-index notation, namely

$$K_i(p) = (k_{i1}(p), k_{i2}(p), \dots, k_{id}(p)) \in \mathbb{Z}^d$$

and

$$x^{K_i(p)} = x_1^{k_{i1}(p)} x_2^{k_{i2}(p)} \dots x_d^{k_{id}(p)}.$$

For  $a := (a_1, a_2, \dots, a_d) \in (S^1)^d$  consider the corresponding automorphism  $f_a \in \text{Aut}(Y)$ . Each  $U_p$  is invariant under  $f_a$  and the action of  $f_a$  on  $U_p$  is given as follows (we use again multi-index notation)

$$(3.0.4) \quad \begin{aligned} f_a^*(x^{K_1(p)}, x^{K_2(p)}, \dots, x^{K_d(p)}) \\ = (a^{K_1(p)} x^{K_1(p)}, a^{K_2(p)} x^{K_2(p)}, \dots, a^{K_d(p)} x^{K_d(p)}). \end{aligned}$$

We have the following

LEMMA 3.3. *If  $a = (a_1, a_2, \dots, a_d) \in (S^1)^d$  is multiplicatively independent, then  $f_a \in \text{Aut}(Y)$  has exactly  $\nu$  fixed points on  $Y$ .*

PROOF. The set of  $d$ -dimensional cones  $\{\sigma_i\}_{i=1}^\nu$  bijectively corresponds to the set of 0-dimensional orbits of  $(\mathbb{C}^*)^d$  (each of which is clearly one point), say

$$Q_1, \dots, Q_\nu$$

(see e.g. [Oda88] Page 10, Proposition 1.6). As  $f_a \in \text{Aut}(Y)$  is associated to  $a \in (S^1)^d$ , it follows that  $f_a(Q_p) = Q_p$ , therefore  $f_a$  has at least  $\nu$  fixed points.

On the other hand, as  $a$  is multiplicatively independent,  $f_a$  has exactly one fixed point on each  $U_p$  ( $1 \leq p \leq \nu$ ), namely the point with coordinates  $x^{K_i(p)} = 0$ ,  $\forall i$  (see formula (3.0.4)). Hence  $f_a$  has exactly  $\nu$  fixed points  $Q_1, \dots, Q_\nu$ .  $\square$

Let now  $(S, F)$  be a McMullen’s pair. Recall that  $S^F$ , the set of fixed points of  $F$ , consists of two points  $P, Q$  and that  $F$  admits a Siegel disk only at  $Q$ . This means that there are analytic coordinates  $(z_1, z_2)$  at  $Q$  such that

$$F^*(z_1, z_2) = (\alpha z_1, \beta z_2),$$

where  $(\alpha, \beta) \in (S^1)^2$  is a multiplicatively independent sequence. Now we extend this sequence to a multiplicatively independent sequence

$$(\alpha, \beta, a_1, a_2, \dots, a_d) \in (S^1)^{d+2}$$

(see the proof of Theorem 1.1.1). Set

$$g := (F, f_a) \in \text{Aut}(S \times Y).$$

It follows from Lemma 3.3 that  $g$  has exactly  $2\nu$  fixed points

$$(P, Q_1), \dots, (P, Q_\nu), (Q, Q_1), \dots, (Q, Q_\nu).$$

However, the first  $\nu$  of these have no Siegel disk. Let us show that  $g$  has a Siegel disk at each of the last  $\nu$  points, namely

$$(Q, Q_1), \dots, (Q, Q_\nu).$$

Using the same notation as in Formula (3.0.4), we have local coordinates  $(y_i)_{i=1}^d := (x^{K_i(p)})_{i=1}^d$  at  $Q_p \in Y$  such that

$$f_a^*(y_1, y_2, \dots, y_d) = (a^{K_1(p)}y_1, a^{K_2(p)}y_2, \dots, a^{K_d(p)}y_d).$$

So, with local coordinates  $(z_1, z_2, y_1, y_2, \dots, y_d)$  on  $S \times Y$ , the action of  $(F, f_a)$  at  $(Q, Q_p)$  is linearized as

$$g^*(z_1, z_2, y_1, y_2, \dots, y_d) = (\alpha z_1, \beta z_2, a^{K_1(p)}y_1, a^{K_2(p)}y_2, \dots, a^{K_d(p)}y_d).$$

It remains to prove that

$$\alpha, \beta, a^{K_1(p)}, a^{K_2(p)}, \dots, a^{K_d(p)}$$

form a multiplicatively independent sequence. For this we take

$$(\ell_1, \ell_2, r_1, r_2, \dots, r_d) \in \mathbb{Z}^{d+2}$$

such that

$$(3.0.5) \quad \alpha^{\ell_1} \beta^{\ell_2} (a^{K_1(p)})^{r_1} (a^{K_2(p)})^{r_2} \dots (a^{K_d(p)})^{r_d} = 1.$$

If  $(r_1, r_2, \dots, r_d) = (0, 0, \dots, 0)$ , then  $\ell_1 = \ell_2 = 0$  since  $(\alpha, \beta)$  is multiplicatively independent. Therefore we can assume  $(r_1, r_2, \dots, r_d) \neq (0, 0, \dots, 0)$ . Then

$$\alpha^{\ell_1} \beta^{\ell_2} a_1^{s_1} a_2^{s_2} \dots a_d^{s_d} = 1,$$

where

$$(s_1, s_2, \dots, s_d) = (r_1, r_2, \dots, r_d) \cdot \begin{pmatrix} k_{11}(p) & k_{12}(p) & \dots & k_{1d}(p) \\ k_{21}(p) & k_{22}(p) & \dots & k_{2d}(p) \\ \dots & \dots & \dots & \dots \\ k_{d1}(p) & k_{d2}(p) & \dots & k_{dd}(p) \end{pmatrix}.$$

Since  $Y$  is complete and non-singular, the primitive vectors of the 1-dimensional rays of  $\sigma_p$  form a  $\mathbb{Z}$ -basis of the lattice  $L \cong \mathbb{Z}^d$  (see e.g. [Oda88] Page 15, Theorem 1.10). Thus the row vectors of the previous matrix, which generate the dual cone  $\sigma_p^\vee \subset M_{\mathbb{R}}$ , form a  $\mathbb{Z}$ -basis of  $M$  as well. We conclude that  $(s_1, s_2, \dots, s_d) \neq (0, 0, \dots, 0)$  and hence we get a contradiction since the sequence

$$\alpha, \beta, a_1, a_2, \dots, a_d$$

is multiplicative independent by construction.

We have proved that for  $X := S \times Y$  and  $g := (F, f_a)$ , with  $a \in (S^1)^d$  generic, the topological entropy of  $g$  is positive (Proposition 3.2) and that  $g$  has exactly  $\nu \geq N$  Siegel disks. To complete the proof of Theorem 1.1.2 we need to show that  $a \in (S^1)^d$  can be chosen such that its components  $a_i$  are algebraic integers of absolute value 1 and the sequence  $(\alpha, \beta, a_1, \dots, a_d)$  is multiplicatively independent. This follows from Theorem 4.2 and hence Theorem 1.1.2 is proved.

3.3. *Proof of Theorem 1.1.3.* Let  $d = n/2$  and let

$$(3.0.6) \quad (S_1, F_1), \dots, (S_d, F_d)$$

be a sequence of McMullen’s pairs. Set

$$g := (F_1, \dots, F_d) \in \text{Aut}(S_1 \times \dots \times S_d).$$

As in the proof of Proposition 3.2, by Künneth decomposition, we obtain that

$$h(g) = h(F_1) + h(F_2) + \dots + h(F_d) > 0.$$

Hence (3-i) of Theorem 1.1 is verified for the pair  $(X, g)$ .

The fixed point set of  $g$  is

$$(S_1 \times \dots \times S_d)^g = \{R = (R_1, \dots, R_d) \mid R_i \in \{P_i, Q_i\}, 1 \leq i \leq d\}.$$

By construction, we have a Siegel disk at

$$(Q_1, Q_2, \dots, Q_d)$$

but, if  $R_i = P_i$  for some  $i$ , then we have no Siegel disk at  $R$  by Theorem 2.6(3). Therefore  $g$  has only one Siegel disk. Moreover, from Proposition 2.7 and from the proof of Theorem 4.2 (of Section 4), it follows that we can choose the pairs  $(S_i, F_i)$  in (3.0.6) such that the eigenvalues of the linearization of  $g$  at  $(Q_1, Q_2, \dots, Q_d)$  are multiplicatively independent algebraic integers of absolute value 1. This completes the proof of Theorem 1.1.3.

#### 4. SALEM POLYNOMIALS AND MULTIPLICATIVELY INDEPENDENT SEQUENCES

In this section we introduce the notion of “multiplicatively independent sequence of algebraic integers on the unit circle (MAU) of length  $2m$ ” and show its existence for any  $m \geq 0$ . The existence of a MAU is crucial in our product construction.

DEFINITION 4.1. *Let*

$$\alpha_1, \beta_1, \dots, \alpha_m, \beta_m$$

*be a sequence of complex numbers of length  $2m$ . We call this sequence a “multiplicatively independent sequence of algebraic integers on the unit circle of length  $2m$ ” (MAU of length  $2m$  for short) if the following (i), (ii) and (iii) are satisfied:*

- (i)  $\alpha_i, \beta_i$  ( $1 \leq i \leq m$ ) *are algebraic integers;*
- (ii)  $\alpha_i, \beta_i$  ( $1 \leq i \leq m$ ) *are of absolute value 1, i.e., they are on the unit circle; and*
- (iii)  $(\alpha_1, \beta_1, \dots, \alpha_m, \beta_m)$  *is multiplicatively independent.*

*By abuse of language, we call the following subsequence of a MAU of length  $2m$ ,*

$$\alpha_1, \beta_1, \dots, \alpha_{m-1}, \beta_{m-1}, \alpha_m$$

*a MAU of length  $2m - 1$ .*

THEOREM 4.2. *Any MAU of length  $2m$  can be extended to a MAU of length  $2(m + 1)$ . In particular, there is an infinite sequence*

$$(4.0.7) \quad \alpha_1, \beta_1, \dots, \alpha_m, \beta_m, \alpha_{m+1}, \beta_{m+1}, \dots$$

*such that for any given integer  $n > 0$ , the first  $n$  terms of this sequence form a MAU of length  $n$ .*

REMARK 4.3. In the proof, we shall give an explicit construction of the sequence (4.0.7). This explicit construction is essential in our proof of Theorem 1.1. In fact in our construction of the sequence (4.0.7) there is a McMullen's pair  $(S_m, F_m)$  for each  $m$  such that  $(S_m)^{F_m} = \{P_m, Q_m\}$  and  $F_m$  has an arithmetic Siegel disk at  $Q_m$  with

$$F_m^*(z_1, z_2) = (\alpha_m z_1, \beta_m z_2),$$

for appropriate local coordinates  $(z_1, z_2)$  at  $Q_m$  (but no Siegel disk at  $P_m$ ).

PROOF. We construct  $\alpha_m, \beta_m$  inductively.

For each positive integer  $k$ , we set

$$n(k) := 360k + 19, \quad d(k) := 180k + 7.$$

Then  $n(k) \equiv 1 \pmod{6}$ . Note that 180 and 7 are coprime, hence by Dirichlet's Theorem (see e.g. [Se73], Page 25, Lemma 3) there are infinitely many prime numbers in the sequence

$$d(1), d(2), \dots, d(k), \dots$$

First we construct a MAU  $\alpha_1, \beta_1$  of length 2. Choose a sufficiently large prime number  $p_1 = d(k_1)$  and set  $n_1 := n(k_1)$ . As  $n_1 \equiv 1 \pmod{6}$  and  $n_1$  is also sufficiently large, we can apply Theorem 2.6 for this  $n_1$ . Hence we obtain a McMullen's pair

$$(S_1, F_1) := (S(n_1), F(n_1))$$

with a Siegel disk at  $Q_1$  such that

$$(F_1)^*(z_1, z_2) = (\alpha(n_1)z_1, \beta(n_1)z_2)$$

at  $Q_1$ . This Siegel disk is arithmetic by Proposition 2.7. Here and hereafter, to describe McMullen's pairs, we adopt the same notation as in Theorem 2.6. Set

$$\alpha_1 := \alpha(n_1), \quad \beta_1 := \beta(n_1).$$

Then,  $\alpha_1$  and  $\beta_1$  form a MAU of length 2.

Next, assuming that we have constructed a MAU of length  $2m$

$$\alpha_1, \beta_1, \dots, \alpha_m, \beta_m,$$

we shall extend this sequence to a MAU of length  $2(m+1)$ .

Let us consider the field extension

$$K := \mathbb{Q}(\alpha_1, \beta_1, \dots, \alpha_m, \beta_m).$$

We put  $\ell := [K : \mathbb{Q}]$ . Then, choose sufficiently large  $k$  such that  $q := d(k)$  is a prime number with  $q > \ell$ . Set  $n := n(k)$ . As  $n \equiv 1 \pmod 6$ , we can apply Theorem 2.6 for this  $n$ . Then, we obtain a McMullen’s pair  $(S(n), F(n))$  with important values  $\delta(n), \alpha(n), \beta(n), \delta'(n), \alpha'(n), \beta'(n)$  and the Salem polynomial  $\varphi(x)$  as described in Theorem 2.6. We set:

$$\delta := \delta(n), \quad \delta' := \delta'(n), \quad \alpha := \alpha(n), \quad \beta := \beta(n), \quad \alpha' := \alpha'(n), \quad \beta' := \beta'(n).$$

We shall show that

$$(4.0.8) \quad \alpha_1, \beta_1, \dots, \alpha_m, \beta_m, \alpha, \beta$$

is a MAU of length  $2(m + 1)$ . We then define  $\alpha_{m+1} := \alpha$  and  $\beta_{m+1} := \beta$ , and continue in this way.

By the assumption (on the first  $2m$  terms) and by Theorem 2.6(3) and Proposition 2.7, we already know that each term of (4.0.8) is an algebraic integer of absolute value 1. Thus, it suffices to show that they are multiplicatively independent. We shall prove this from now.

First we compute the degree of the Salem polynomial  $\varphi(x)$  and its Salem trace polynomial  $r(x)$ . Since

$$n = 360k + 19 \equiv 19 \pmod{360},$$

we have  $C_n(x) = C_{19}(x)$  in Theorem 2.6(1) (cf. Theorem 2.3). On the other hand, by Example 2.4, we have  $\deg C_{19}(x) = 5$ . Thus

$$\deg \varphi(x) = 360k + 19 - 5 = 360k + 14.$$

Hence

$$\deg r(x) = \frac{\deg \varphi(x)}{2} = 180k + 7 = d(k) = q.$$

As  $r(x)$  is irreducible over  $\mathbb{Z}$ , it follows that

$$\left[ \mathbb{Q} \left( \delta + \frac{1}{\delta} \right) : \mathbb{Q} \right] = q.$$

As  $q > \ell$  and  $q$  is a prime number, we have then that

$$\left[ K \left( \delta + \frac{1}{\delta} \right) : K \right] = q.$$

So,  $r(x)$  is also irreducible over  $K$ . Let  $L$  be the Galois closure of  $K(\delta + \frac{1}{\delta})$  in the algebraic closure  $\bar{K}$ . As

$$\delta + \frac{1}{\delta}, \delta' + \frac{1}{\delta'}, \eta + \frac{1}{\eta}$$

are roots of  $r(x) = 0$ , there are  $\sigma \in \text{Gal}(L/K)$  and  $\tau \in \text{Gal}(L/K)$  such that

$$\sigma\left(\delta + \frac{1}{\delta}\right) = \delta' + \frac{1}{\delta'}, \quad \tau\left(\delta + \frac{1}{\delta}\right) = \eta + \frac{1}{\eta}.$$

Here  $\eta$  is the Salem number of  $\varphi(x)$ . Extending  $\sigma$  and  $\tau$  to  $\text{Gal}(\bar{K}/K)$ , we have

$$\begin{aligned} \sigma(\delta) = \delta' \quad \text{or} \quad \sigma(\delta) = \frac{1}{\delta'} = \bar{\delta}', \\ \tau(\delta) = \eta \quad \text{or} \quad \tau(\delta) = \frac{1}{\eta}. \end{aligned}$$

Let

$$(4.0.9) \quad \alpha_1^{\ell_1} \beta_1^{k_1} \dots \alpha_m^{\ell_m} \beta_m^{k_m} \alpha^{m+1} \beta^{k_{m+1}} = 1$$

where  $\ell_i, m_i$  are integers. Transforming (4.0.9) by  $\sigma$  (and switching  $\alpha'$  and  $\beta'$  if necessary), we obtain either

$$\begin{aligned} \alpha_1^{\ell_1} \beta_1^{k_1} \dots \alpha_m^{\ell_m} \beta_m^{k_m} (\alpha')^{\ell_{m+1}} (\beta')^{k_{m+1}} = \pm 1 \quad \text{or} \\ \alpha_1^{\ell_1} \beta_1^{k_1} \dots \alpha_m^{\ell_m} \beta_m^{k_m} (\bar{\alpha}')^{\ell_{m+1}} (\bar{\beta}')^{k_{m+1}} = \pm 1. \end{aligned}$$

Here we use Theorem 2.6(3) and (4). In the first case, taking (the square of) the norm, we get

$$1 = |(\alpha')^{2\ell_{m+1}} (\beta')^{2k_{m+1}}| = |(\alpha'/\beta')^{\ell_{m+1}-k_{m+1}} (\alpha'\beta')^{\ell_{m+1}+k_{m+1}}| = |\alpha'/\beta'|^{\ell_{m+1}-k_{m+1}}.$$

Here we use  $|\alpha'\beta'| = |\delta'| = 1$  (Theorem 2.6(4)). As  $|\alpha'/\beta'| \neq 1$  (Theorem 2.6(4)), it follows that

$$\ell_{m+1} - k_{m+1} = 0.$$

For the same reason, this is true also for the second case. Substituting this into (4.0.9), and using  $\alpha\beta = \delta$ , we obtain

$$\alpha_1^{\ell_1} \beta_1^{k_1} \dots \alpha_m^{\ell_m} \beta_m^{k_m} \delta^{k_{m+1}} = 1.$$

Transforming this equality by  $\tau$ , we get either

$$\alpha_1^{\ell_1} \beta_1^{k_1} \dots \alpha_m^{\ell_m} \beta_m^{k_m} \eta^{k_{m+1}} = 1 \quad \text{or} \quad \alpha_1^{\ell_1} \beta_1^{k_1} \dots \alpha_m^{\ell_m} \beta_m^{k_m} \left(\frac{1}{\eta}\right)^{k_{m+1}} = 1.$$

Taking the norm, we get

$$\eta^{k_{m+1}} = 1.$$



As  $\eta > 1$ , this implies  $k_{m+1} = 0$ . Thus

$$\ell_{m+1} = k_{m+1} = 0.$$

Substituting this into (4.0.9), we obtain

$$\alpha_1^{\ell_1} \beta_1^{k_1} \dots \alpha_m^{\ell_m} \beta_m^{k_m} = 1.$$

As  $\alpha_1, \beta_1, \dots, \alpha_m, \beta_m$  is a MAU of length  $2m$ , it follows that

$$\ell_1 = k_1 = \dots = \ell_m = k_m = 0.$$

This completes the proof. □

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