



**Calculus of Variations** — *On a result by Boccardo-Ferone-Fusco-Orsina*, by MARCO SQUASSINA, communicated on 6 June 2011.

ABSTRACT. — Via a symmetric version of Ekeland’s principle recently obtained by the author we improve, in a ball or an annulus, a result of Boccardo-Ferone-Fusco-Orsina on the properties of minimizing sequences of functionals of calculus of variations in the non-convex setting.

KEY WORDS: Symmetry, non-convex problems, regularity of minimizing sequences.

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## 1. INTRODUCTION

In the study of non-convex minimization problems [4] of calculus of variations, the idea of selecting minimizing sequences with nice properties to guarantee the convergence towards a minimizer can be traced back to Hilbert and Lebesgue [6, 8]. In [10], the author has recently obtained an abstract symmetric version of the celebrated Ekeland’s variational principle [3] for lower semi-continuous functionals, which is probably one of the main tools to perform the selection procedure indicated above. More precisely, the new enhanced Ekeland type principle is able to select points which are not only almost critical, in a suitable sense, but also almost symmetric, provided that the functional does not increase under polarizations [1]. In turn, under rather mild assumptions, starting from a given minimizing sequence one can detect a new minimizing sequence enriched with very nice features. The additional symmetry characteristics play a rôle also in non-compact problems, providing compactifying effects. In 1999, Boccardo-Ferone-Fusco-Orsina [2] considered functionals  $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  of calculus of variations,

$$J(u) = \int_{\Omega} j(x, u, Du), \quad u \in W_0^{1,p}(\Omega),$$

with no convexity assumption on  $\xi \mapsto j(x, s, \xi)$  and showed that, by merely relying upon some classical [7] growth estimates on the integrand  $j(x, s, \xi)$ , the existence of minimizing sequences with enhanced smoothness can be obtained by combining the application of the classical Ekeland’s principle with a priori esti-

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mates (cf. [2, Lemmas 2.3 and 2.6]) based upon suitable Gehring-type lemmas [5]. We also refer the reader to [9] for other results in the same spirit.

The main goal of the present note is to highlight that, if we restrict the attention to the case where  $\Omega$  is either a ball or an annulus of  $\mathbb{R}^N$  and  $J$  decreases upon polarizations, then arguing as in [2] but using the Ekeland’s principle from [10], even more special minimizing sequences can be detected. More precisely, consider  $1 < p < N$ , let  $p^*$  denote the critical Sobolev exponent and let  $j : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  be a Carathéodory function such that

$$(1.1) \quad \alpha|\xi|^p - \varphi_2|s|^{\gamma_2} \leq j(x, s, \xi) \leq \beta|\xi|^p + \varphi_0 + \varphi_1|s|^{\gamma_1},$$

for a.e.  $x \in \Omega$  and every  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ , for some  $\alpha, \beta > 0$ ,

$$\varphi_0 \in L^{r_0}(\Omega), \quad r_0 > 1, \quad \varphi_1 \in L^{r_1}(\Omega), \quad r_1 > N/p, \quad \varphi_2 \in L^{r_2}(\Omega), \quad r_2 > N,$$

and  $\gamma_1, \gamma_2$  satisfying

$$0 \leq \gamma_1 < p^* \frac{r_1 - 1}{r_1}, \quad 0 \leq \gamma_2 < \min \left\{ p, \frac{N}{N-1} \frac{r_2 - 1}{r_2} \right\}.$$

We consider the following classes of half-spaces in  $\mathbb{R}^N$

$$\begin{aligned} \mathcal{H}_* &:= \{H \subset \mathbb{R}^N \text{ is a half-space with } 0 \in H\}, \quad \text{if } \Omega \text{ is a ball,} \\ \mathcal{H}_* &:= \{H \subset \mathbb{R}^N \text{ is a half-space with } \mathbb{R}^+ \times \{0\} \subset H \text{ and } 0 \in \partial H\}, \\ &\quad \text{if } \Omega \text{ is an annulus,} \end{aligned}$$

For any nonnegative measurable function  $u$  we define  $u^H$  to be the polarization of  $u$  with respect a half-space  $H \in \mathcal{H}_*$ . Moreover, we denote by  $u^*$  the Schwarz symmetrization (resp. the spherical cap symmetrization) if  $\Omega$  is a ball (resp. if  $\Omega$  is an annulus). For definitions and properties of these notions, we refer to [1] and to the references therein.

In this framework, merely under assumption (1.1), we have the following

**THEOREM 1.1.** *Assume that  $\Omega$  is either a ball or an annulus in  $\mathbb{R}^N$  with  $N \geq 2$  and*

$$(1.2) \quad J(u^H) \leq J(u) \quad \text{for all } u \in W_{0+}^{1,p}(\Omega) \text{ and any } H \in \mathcal{H}_*.$$

*Then for an arbitrary minimizing sequence  $(u_h) \subset W_{0+}^{1,p}(\Omega)$  for  $J$  there exist  $q > p$ , a new minimizing sequence  $(v_h) \subset W_{0+}^{1,p}(\Omega)$  for  $J$  and continuous mappings  $\mathbb{T}_h : W_{0+}^{1,1}(\Omega) \rightarrow W_{0+}^{1,1}(\Omega)$  such that  $\mathbb{T}_h z$  is built from  $z$  via iterated polarizations by half-spaces in  $\mathcal{H}_*$ , such that*

$$\sup_{h \geq 1} \|v_h\|_{W_0^{1,q}(\Omega)} < +\infty, \quad \text{if } r_0 < \frac{N}{p}, \quad \sup_{h \geq 1} \|v_h\|_{L^\infty(\Omega)} < +\infty, \quad \text{if } r_0 > \frac{N}{p},$$

and, in addition,

$$\begin{aligned} \lim_h \|v_h - |v_h|^*\|_{L^{N/(N-1)}(\Omega)} &= 0, \\ \limsup_h \|v_h - u_h\|_{W_0^{1,1}(\Omega)} &\leq \limsup_h \|\mathbb{T}_h u_h - u_h\|_{W_0^{1,1}(\Omega)}. \end{aligned}$$

We stress that, under (1.1),  $J$  is bounded from below but, since we are not assuming the convexity of  $\xi \mapsto j(x, s, \xi)$ , we can by no means conclude that  $J$  has a minimum point. Nevertheless, a smooth minimizing sequence made by almost Schwarz symmetric (for the ball) or almost spherical cap symmetric (for the annulus) points can be constructed. As it can be readily checked by direct computation, a class of integrands which satisfy (1.2) (with the equality in place of the inequality) is, for instance,  $j(x, s, \xi) = j_0(s, |\xi|)$ , for some continuous function  $j_0 : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ . Observe also that, of course, from the last conclusion of Theorem 1.1, the limit  $v$  of  $(v_h)$  must be Schwarz symmetric, namely  $v = v^*$ .

REMARK 1.2. We conclude with an important remark, which is probably one of the main reasons why the conclusion of Theorem 1.1 is rather powerful in the non-convex framework. Should one *additionally* assume that  $\xi \mapsto j(x, s, \xi)$  is convex, it is then often the case that a functional which satisfies (1.2), fulfills in turn the corresponding symmetrization inequality  $J(u^*) \leq J(u)$ . In such a case, starting from a given minimizing sequence  $(u_h) \subset W_{0+}^{1,p}(\Omega)$  for  $J$  one has that  $(u_h^*) \subset W_{0+}^{1,p}(\Omega)$  is a minimizing sequence too and it is then immediate from [2] to find a further almost symmetric regular minimizing sequence  $(v_h)$ . On the other hand, without the convexity of  $j(x, s, \xi)$  in the gradient, to the author knowledge, no symmetrization inequality is available in the current literature. In some sense, while  $J(u^H) \leq J(u)$  is often an algebraic fact,  $J(u^*) \leq J(u)$  is rather a more geometrical fact.

## 2. SYMMETRIC EKELAND’S PRINCIPLE

Let  $X$  and  $V$  be two Banach spaces and  $S \subseteq X$ . We shall consider two maps  $* : S \rightarrow V$ ,  $u \mapsto u^*$ , the symmetrization map, and  $h : S \times \mathcal{H}_* \rightarrow S$ ,  $(u, H) \mapsto u^H$ , the polarization map,  $\mathcal{H}_*$  being a path-connected topological space. As in [10], we assume the following:

- (1)  $X$  is continuously embedded in  $V$ ;
- (2)  $h$  is a continuous mapping;
- (3) for each  $u \in S$  and  $H \in \mathcal{H}_*$  it holds  $(u^*)^H = (u^H)^* = u^*$  and  $u^{HH} = u^H$ ;
- (4) there exists  $(H_m) \subset \mathcal{H}_*$  such that, for  $u \in S$ ,  $u^{H_1 \dots H_m}$  converges to  $u^*$  in  $V$ ;
- (5) for every  $u, v \in S$  and  $H \in \mathcal{H}_*$  it holds  $\|u^H - v^H\|_V \leq \|u - v\|_V$ .

Moreover, the mappings  $* : S \rightarrow V$  and  $h : S \times \mathcal{H}_* \rightarrow S$  can be extended to  $* : X \rightarrow V$  and  $h : X \times \mathcal{H}_* \rightarrow S$  respectively by setting  $u^* := (\Theta(u))^*$  and  $u^H := (\Theta(u))^H$  for all  $u \in X$ , where  $\Theta : (X, \|\cdot\|_X) \rightarrow (S, \|\cdot\|_V)$  is Lipschitz of constant  $C_\Theta$  and such that  $\Theta|_S = \text{Id}|_S$ . In the above framework, we recall the result from [10].

**THEOREM 2.1.** *Assume that  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper and lower semi-continuous functional bounded from below such that*

$$(2.1) \quad f(u^H) \leq f(u) \quad \text{for all } u \in S \text{ and } H \in \mathcal{H}_*.$$

Let  $u \in S$ ,  $\rho > 0$  and  $\sigma > 0$  with

$$f(u) \leq \inf_X f + \rho\sigma.$$

Then there exist  $v \in X$  and a continuous map  $\mathbb{T}_\rho : S \rightarrow S$  such that  $\mathbb{T}_\rho z$  is built via iterated polarizations of  $z$  by half-spaces in  $\mathcal{H}_*$  such that

- (a)  $\|v - v^*\|_V < C\rho$ ;
- (b)  $\|v - u\|_X \leq \rho + \|\mathbb{T}_\rho u - u\|_X$ ;
- (c)  $f(v) \leq f(u)$ ;
- (d)  $f(w) \geq f(v) - \sigma\|w - v\|_X$ , for all  $w \in X$ ,

for some positive constant  $C$  depending only upon  $V$ ,  $X$  and  $\Theta$ .

Let  $\Omega$  be either a ball or an annulus of  $\mathbb{R}^N$ ,  $N \geq 2$ . In particular, by choosing

$$X = (W_0^{1,1}(\Omega), \|\cdot\|_{W_0^{1,1}(\Omega)}), \quad \|u\|_{W_0^{1,1}(\Omega)} = \int_\Omega |Du|, \quad S = W_{0+}^{1,1}(\Omega),$$

as well as

$$V = (L^{N/(N-1)}(\Omega), \|\cdot\|_{L^{N/(N-1)}(\Omega)}), \quad \Theta(u) = |u|,$$

then (1)–(5) hold true. The following by product, adapted to our purposes, holds true.

**COROLLARY 2.2.** *Let  $\Omega$  be either a ball or an annulus of  $\mathbb{R}^N$  and let  $J : W_0^{1,1}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semi-continuous functional bounded from below with*

$$(2.2) \quad J(u^H) \leq J(u) \quad \text{for all } u \in W_{0+}^{1,1}(\Omega) \text{ and } H \in \mathcal{H}_*.$$

Let  $u \in W_{0+}^{1,1}(\Omega)$  and  $\varepsilon > 0$  be such that

$$J(u) \leq \inf_{W_0^{1,1}(\Omega)} J + \varepsilon.$$

Then there exist  $v \in W_0^{1,1}(\Omega)$  and a continuous map  $\mathbb{T}_\varepsilon : W_{0+}^{1,1}(\Omega) \rightarrow W_{0+}^{1,1}(\Omega)$  such that  $\mathbb{T}_\varepsilon z$  is built via iterated polarizations of  $z$  by half-spaces in  $\mathcal{H}_*$  such that  $J(v) \leq J(u)$ ,

$$(2.3) \quad J(w) \geq J(v) - \sqrt{\varepsilon}\|w - v\|_{W_0^{1,1}(\Omega)}, \quad \text{for all } w \in W_{0+}^{1,1}(\Omega),$$

and

$$(2.4) \quad \|v - |v|^*\|_{L^{N/(N-1)}(\Omega)} \leq C\sqrt{\varepsilon}, \quad \|v - u\|_{W_0^{1,1}(\Omega)} \leq \sqrt{\varepsilon} + \|\mathbb{T}_{\varepsilon}u - u\|_{W_0^{1,1}(\Omega)},$$

for some positive constant  $C$ .

### 3. PROOF OF THEOREM 1.1

The argument closely follows [2, proof of Theorem 3.1, p. 128], aiming to apply Corollary 2.2 in place of the standard Ekeland’s variational principle [3]. We will denote by  $C$  a generic positive constant which may vary from line to line. Taking into account that  $\gamma_2 < p$  and  $\gamma_2 r'_2 < N/(N - 1)$ , it easily follows that  $\tilde{J} : W_0^{1,1}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ ,

$$\tilde{J}(u) = \begin{cases} J(u) & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty & \text{if } u \in W_0^{1,1}(\Omega) \setminus W_0^{1,p}(\Omega), \end{cases}$$

is a lower semi-continuous functional bounded from below. In light of assumption (1.2), we have

$$\tilde{J}(u^H) \leq \tilde{J}(u) \quad \text{for all } u \in W_{0+}^{1,1}(\Omega) \text{ and } H \in \mathcal{H}_*.$$

Hence, we are in the framework of Corollary 2.2. Given a minimizing sequence  $(u_h) \subset W_{0+}^{1,p}(\Omega)$  for  $J$ , let  $(\varepsilon_h) \subset (0, 1]$  be such that  $\varepsilon_h \rightarrow 0$  as  $h \rightarrow \infty$  and

$$(3.1) \quad J(u_h) \leq \inf_{W_0^{1,p}(\Omega)} J + \varepsilon_h, \quad \text{for any } h \geq 1.$$

Since the infimum of  $J$  over  $W_0^{1,p}(\Omega)$  equals the infimum of  $\tilde{J}$  over  $W_0^{1,1}(\Omega)$ , it holds

$$\tilde{J}(u_h) \leq \inf_{W_0^{1,1}(\Omega)} \tilde{J} + \varepsilon_h, \quad \text{for any } h \geq 1.$$

Then, by applying Corollary 2.2 to  $\tilde{J}$ ,  $u_h$  and  $\varepsilon_h$ , for any  $h \geq 1$ , there exists  $v_h \in W_0^{1,1}(\Omega)$  such that  $J(v_h) = \tilde{J}(v_h) \leq \tilde{J}(u_h) = J(u_h)$  and, for any  $h \geq 1$ ,

$$(3.2) \quad \begin{aligned} \|v_h - |v_h|^*\|_{L^{N/(N-1)}(\Omega)} &\leq C\sqrt{\varepsilon_h}, \\ \|v_h - u_h\|_{W_0^{1,1}(\Omega)} &\leq \sqrt{\varepsilon_h} + \|\mathbb{T}_{\varepsilon_h}u_h - u_h\|_{W_0^{1,1}(\Omega)}, \end{aligned}$$

for some continuous maps  $\mathbb{T}_{\varepsilon_h} : W_{0+}^{1,1}(\Omega) \rightarrow W_{0+}^{1,1}(\Omega)$  as well as, for any  $h \geq 1$ ,

$$\tilde{J}(v_h) \leq \tilde{J}(w) + \sqrt{\varepsilon_h} \int_{\Omega} |Dw - Dv_h|, \quad \text{for all } w \in W_0^{1,1}(\Omega),$$

that is, being  $\tilde{J}(v_h) < +\infty$ ,

$$(3.3) \quad \int_{\Omega} j(x, v_h, Dv_h) \leq \int_{\Omega} j(x, w, Dw) + \sqrt{\varepsilon_h} \int_{\Omega} |Dw - Dv_h|, \quad \text{for all } w \in W_0^{1,p}(\Omega).$$

Observe that  $(u_h)$  is bounded in  $W_0^{1,p}(\Omega)$  since (3.1) and (1.1) yield

$$(3.4) \quad \alpha \|Du_h\|_{L^p(\Omega)}^p \leq C + C \|\varphi_2\|_{L^2(\Omega)} \|Du_h\|_{L^p(\Omega)}^{\gamma_2}, \quad (\gamma_2 < p).$$

In turn,  $(v_h)$  is bounded in  $W_0^{1,1}(\Omega)$ , since by the second inequality of (3.2), it holds

$$\begin{aligned} \|v_h\|_{W_0^{1,1}(\Omega)} &\leq \|v_h - u_h\|_{W_0^{1,1}(\Omega)} + \|u_h\|_{W_0^{1,1}(\Omega)} \\ &\leq \sqrt{\varepsilon_h} + \|\mathbb{T}_{\varepsilon_h} u_h - u_h\|_{W_0^{1,1}(\Omega)} + \|u_h\|_{W_0^{1,1}(\Omega)} \\ &\leq \sqrt{\varepsilon_h} + 3\|u_h\|_{W_0^{1,1}(\Omega)} \leq 1 + C\|u_h\|_{W_0^{1,p}(\Omega)} \leq C. \end{aligned}$$

In the last line, we exploited the fact that, by construction of  $\mathbb{T}_{\varepsilon_h}$ , for any  $h \geq 1$ ,

$$\|\mathbb{T}_{\varepsilon_h} u_h\|_{W_0^{1,1}(\Omega)} = \int_{\Omega} |Du_h^{H_0 \dots H_{m_{\varepsilon_h}}}| = \int_{\Omega} |Du_h^{H_0 \dots H_{m_{\varepsilon_h}-1}}| = \dots = \int_{\Omega} |Du_h|.$$

In conclusion,  $(v_h)$  is bounded in  $W_0^{1,p}(\Omega)$  since by  $J(v_h) \leq C$ , (1.1) and  $\gamma_2 r'_2 < N/(N-1)$

$$(3.5) \quad \alpha \|Dv_h\|_{L^p(\Omega)}^p \leq C + C \|\varphi_2\|_{L^2(\Omega)} \|v_h\|_{W_0^{1,1}(\Omega)}^{\gamma_2} \leq C,$$

and the variational inequality (3.3), choosing  $w = v_h + \varphi$  for a  $\varphi \in W_0^{1,p}(\Omega)$ , yields

$$(3.6) \quad \int_{\text{supt}(\varphi)} j(x, v_h, Dv_h) \leq \int_{\text{supt}(\varphi)} j(x, v_h + \varphi, Dv_h + D\varphi) + \sqrt{\varepsilon_h} \int_{\text{supt}(\varphi)} |D\varphi|,$$

for all  $h \geq 1$  and any  $\varphi \in W_0^{1,p}(\Omega)$ . Once these facts hold, the boundedness of  $(v_h)$  in  $W_0^{1,q}(\Omega)$  (case  $r_0 < N/p$ ) for some  $q > p$  follows as in [2, proof of Theorem 3.4] using (1.1) in (3.6). The boundedness in  $L^\infty(\Omega)$  (case  $r_0 > N/p$ ), follows by choosing  $w = \max(-k, \min(k, v_h)) \in W_0^{1,p}(\Omega)$  in (3.3) and arguing as in [2, proof of Theorems 3.5]. Recalling (3.2), the proof is complete.

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