



Calculus of Variations — $W_0^{1,1}$ minima of noncoercive functionals, by LUCIO BOCCARDO, GISELLA CROCE and LUIGI ORSINA, presented by Nicola Fusco on 16 June 2011.

ABSTRACT. — We study an integral non coercive functional defined on $H_0^1(\Omega)$, proving the existence of a minimum in $W_0^{1,1}(\Omega)$.

KEY WORDS: Integral functionals, direct methods of the calculus of variations, coercivity, lower order terms.

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In this paper we study a class of integral functionals defined on $H_0^1(\Omega)$, but non coercive on the same space, so that the standard approach of the Calculus of Variations does not work. However, the functionals are coercive on $W_0^{1,1}(\Omega)$ and we will prove the existence of minima, despite the non reflexivity of $W_0^{1,1}(\Omega)$, which implies that, in general, the Direct Methods fail due to lack of compactness.

Let J be the functional defined as

$$J(v) = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|]^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} fv, \quad v \in H_0^1(\Omega).$$

We assume that Ω is a bounded open set of \mathbb{R}^N , $N > 2$, that $j : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is such that $j(\cdot, \xi)$ is measurable on Ω for every ξ in \mathbb{R}^N , $j(x, \cdot)$ is convex and belongs to $C^1(\mathbb{R}^N)$ for almost every x in Ω , and

- (1) $\alpha|\xi|^2 \leq j(x, \xi) \leq \beta|\xi|^2$,
- (2) $|j_{\xi}(x, \xi)| \leq \gamma|\xi|$,

for some positive α , β and γ , for almost every x in Ω , and for every ξ in \mathbb{R}^N . We assume that b is a measurable function on Ω such that

- (3) $0 \leq b(x) \leq B$, for almost every x in Ω ,

where $B > 0$, while f belongs to some Lebesgue space. For $k > 0$ and $s \in \mathbb{R}$, we define the truncature function as $T_k(s) = \max(-k, \min(s, k))$.

In [3] the minimization in $H_0^1(\Omega)$ of the functional

$$I(v) = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + |v|]^\theta} - \int_{\Omega} f v, \quad 0 < \theta < 1, f \in L^m(\Omega),$$

was studied. It was proved that $I(v)$ is coercive on the Sobolev space $W_0^{1,q}(\Omega)$, for some $q = q(\theta, m)$ in (1, 2), and that $I(v)$ achieves its minimum on $W_0^{1,q}(\Omega)$. This approach does not work for $\theta > 1$ (see Remark 7 below). Here we will be able to overcome this difficulty thanks to the presence of the lower order term $\int_{\Omega} |v|^2$, which will yield the coercivity of J on $W_0^{1,1}(\Omega)$; then we will prove the existence of minima in $W_0^{1,1}(\Omega)$, even if it is a non reflexive space.

Integral functionals like J or I are studied in [1], in the context of the Thomas–Fermi–von Weizsäcker theory.

We are going to prove the following result.

THEOREM 1. *Let $f \in L^2(\Omega)$. Then there exists u in $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ minimum of J , that is,*

$$(4) \quad \int_{\Omega} \frac{j(x, \nabla u)}{[1 + b(x)|u|]^2} + \frac{1}{2} \int_{\Omega} |u|^2 - \int_{\Omega} f u \leq \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|]^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f v,$$

for every v in $H_0^1(\Omega)$. Moreover $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$.

In [2] we studied the following elliptic boundary problem:

$$(5) \quad \begin{cases} -\operatorname{div} \left(\frac{a(x) \nabla u}{(1 + b(x)|u|)^2} \right) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

under the same assumptions on Ω , b and f , with $0 < \alpha \leq a(x) \leq \beta$. It is easy to see that the Euler equation of J , with $j(x, \xi) = \frac{1}{2} a(x) |\xi|^2$, is not equation (5). Therefore Theorem 1 cannot be deduced from [2]. Nevertheless some technical steps of the two papers (for example, the a priori estimates) are similar.

We will prove Theorem 1 by approximation. Therefore, we begin with the case of bounded data.

LEMMA 2. *If g belongs to $L^\infty(\Omega)$, then there exists a minimum w belonging to $H_0^1(\Omega) \cap L^\infty(\Omega)$ of the functional*

$$v \in H_0^1(\Omega) \mapsto \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|]^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} g v.$$

PROOF. Since the functional is not coercive on $H_0^1(\Omega)$, we cannot directly apply the standard techniques of the Calculus of Variations. Therefore, we

begin by approximating it. Let $M > 0$, and let J_M be the functional defined as

$$J_M(v) = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|T_M(v)]^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} gv, \quad v \in H_0^1(\Omega).$$

Since J_M is both weakly lower semicontinuous (due to the convexity of j and to De Giorgi's theorem, see [4]) and coercive on $H_0^1(\Omega)$, for every $M > 0$ there exists a minimum w_M of J_M on $H_0^1(\Omega)$. Let $A = \|g\|_{L^\infty(\Omega)}$, let $M > A$, and consider the inequality $J_M(w_M) \leq J_M(T_A(w_M))$, which holds true since w_M is a minimum of J_M . We have

$$\begin{aligned} & \int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)|T_M(w_M)]^2} + \frac{1}{2} \int_{\Omega} |w_M|^2 - \int_{\Omega} gw_M \\ & \leq \int_{\Omega} \frac{j(x, \nabla T_A(w_M))}{[1 + b(x)|T_M(T_A(w_M))]|^2} + \frac{1}{2} \int_{\Omega} |T_A(w_M)|^2 - \int_{\Omega} gT_A(w_M) \\ & = \int_{\{|w_M| \leq A\}} \frac{j(x, \nabla w_M)}{[1 + b(x)|T_M(w_M)]^2} + \frac{1}{2} \int_{\Omega} |T_A(w_M)|^2 - \int_{\Omega} gT_A(w_M), \end{aligned}$$

where, in the last passage, we have used that $T_M(T_A(w_M)) = T_M(w_M)$ on the set $\{|w_M| \leq A\}$, and that $j(x, 0) = 0$. Simplifying equal terms, we thus get

$$\begin{aligned} & \int_{\{|w_M| \geq M\}} \frac{j(x, \nabla w_M)}{[1 + b(x)|T_M(w_M)]^2} \\ & + \frac{1}{2} \int_{\Omega} [|w_M|^2 - |T_A(w_M)|^2] \leq \int_{\Omega} g[w_M - T_A(w_M)]. \end{aligned}$$

Dropping the first term, which is nonnegative, we obtain

$$\frac{1}{2} \int_{\Omega} [w_M - T_A(w_M)][w_M + T_A(w_M)] \leq \int_{\Omega} g[w_M - T_A(w_M)],$$

which can be rewritten as

$$\frac{1}{2} \int_{\Omega} [w_M - T_A(w_M)][w_M + T_A(w_M) - 2g] \leq 0.$$

We then have, since $w_M = T_A(w_M)$ on the set $\{|w_M| \leq A\}$,

$$\frac{1}{2} \int_{\{w_M > A\}} [w_M - A][w_M + A - 2g] + \frac{1}{2} \int_{\{w_M < -A\}} [w_M + A][w_M - A - 2g] \leq 0.$$

Since $|g| \leq A$, we have $A - 2g \geq -A$, and $-A - 2g < A$, so that

$$0 \leq \frac{1}{2} \int_{\{w_M > A\}} [w_M - A]^2 + \frac{1}{2} \int_{\{w_M < -A\}} [w_M + A]^2 \leq 0,$$

which then implies that $\text{meas}(\{|w_M| \geq A\}) = 0$, and so $|w_M| \leq A$ almost everywhere in Ω . Recalling the definition of A , we thus have

$$(6) \quad \|w_M\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}.$$

Since $M > \|g\|_{L^\infty(\Omega)}$, we thus have $T_M(w_M) = w_M$. Starting now from $J_M(w_M) \leq J_M(0) = 0$ we obtain, by (6),

$$\int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)|w_M|]^2} + \frac{1}{2} \int_{\Omega} |w_M|^2 \leq \int_{\Omega} g w_M \leq \text{meas}(\Omega) \|g\|_{L^\infty(\Omega)},$$

which then implies, by (1) and (3), and dropping the nonnegative second term,

$$\frac{\alpha}{[1 + B\|g\|_{L^\infty(\Omega)}]^2} \int_{\Omega} |\nabla w_M|^2 \leq \text{meas}(\Omega) \|g\|_{L^\infty(\Omega)}^2.$$

Thus, $\{w_M\}$ is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and so, up to subsequences, it converges to some function w in $H_0^1(\Omega) \cap L^\infty(\Omega)$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω . We prove now that

$$(7) \quad \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w|]^2} \leq \liminf_{M \rightarrow +\infty} \int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)|w_M|]^2}.$$

Indeed, since j is convex, we have

$$\int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)|w_M|]^2} \geq \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w_M|]^2} - \int_{\Omega} \frac{j_\xi(x, \nabla w)}{[1 + b(x)|w_M|]^2} \cdot \nabla[w_M - w].$$

Using assumption (1), the fact that w belongs to $H_0^1(\Omega)$, the almost everywhere convergence of w_M to w and Lebesgue's theorem, we have

$$(8) \quad \lim_{M \rightarrow +\infty} \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w_M|]^2} = \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w|]^2}.$$

Using assumption (2), the fact that w belongs to $H_0^1(\Omega)$, and the almost everywhere convergence of w_M to w , we have by Lebesgue's theorem that

$$\lim_{M \rightarrow +\infty} \frac{j_\xi(x, \nabla w)}{[1 + b(x)|w_M|]^2} = \frac{j_\xi(x, \nabla w)}{[1 + b(x)|w|]^2}, \quad \text{strongly in } (L^2(\Omega))^N.$$

Since ∇w_M tends to ∇w weakly in the same space, we thus have that

$$(9) \quad \lim_{M \rightarrow +\infty} \int_{\Omega} \frac{j_{\xi}(x, \nabla w)}{[1 + b(x)|w_M|]^2} \cdot \nabla[w_M - w] = 0.$$

Using (8) and (9), we have that (7) holds true. On the other hand, using (1) and Lebesgue's theorem again, it is easy to see that

$$\lim_{M \rightarrow +\infty} \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|T_M(v)|]^2} = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|]^2}, \quad \forall v \in H_0^1(\Omega).$$

Thus, starting from $J_M(w_M) \leq J_M(v)$, we can pass to the limit as M tends to infinity (using also the strong convergence of w_M to w in $L^2(\Omega)$), to have that w is a minimum. \square

As stated before, we prove Theorem 1 by approximation. More in detail, if $f_n = T_n(f)$ then Lemma 2 with $g = f_n$ implies that there exists a minimum u_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of the functional

$$J_n(v) = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|]^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f_n v, \quad v \in H_0^1(\Omega).$$

In the following lemma we prove some uniform estimates on u_n .

LEMMA 3. *Let u_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ be a minimum of J_n . Then*

$$(10) \quad \int_{\Omega} \frac{|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} \leq \frac{1}{2\alpha} \int_{\Omega} |f|^2;$$

$$(11) \quad \int_{\Omega} |\nabla T_k(u_n)|^2 \leq \frac{(1 + Bk)^2}{2\alpha} \int_{\Omega} |f|^2;$$

$$(12) \quad \int_{\Omega} |u_n|^2 \leq 4 \int_{\Omega} |f|^2;$$

$$(13) \quad \int_{\Omega} |\nabla u_n| \leq \left[\frac{1}{2\alpha} \int_{\Omega} |f|^2 \right]^{1/2} \left(\text{meas}(\Omega)^{1/2} + 2B \left[\int_{\Omega} |f|^2 \right]^{1/2} \right);$$

$$(14) \quad \int_{\Omega} |G_k(u_n)|^2 \leq 4 \int_{\{|u_n| \geq k\}} |f|^2,$$

where $G_k(s) = s - T_k(s)$ for $k \geq 0$ and s in \mathbb{R} .

PROOF. The minimality of u_n implies that $J_n(u_n) \leq J_n(0)$, that is,

$$(15) \quad \int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_{\Omega} u_n^2 \leq \int_{\Omega} f_n u_n.$$

Using (1) on the left hand side, and Young's inequality on the right hand side gives

$$\alpha \int_{\Omega} \frac{|\nabla u_n|^2}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_{\Omega} u_n^2 \leq \frac{1}{2} \int_{\Omega} u_n^2 + \frac{1}{2} \int_{\Omega} f_n^2,$$

which then implies (10). Let now $k \geq 0$. The above estimate, and (3), give

$$\frac{1}{(1 + Bk)^2} \int_{\Omega} |\nabla T_k(u_n)|^2 \leq \int_{\{|u_n| \leq k\}} \frac{|\nabla u_n|^2}{[1 + b(x)|u_n|]^2} \leq \frac{1}{2\alpha} \int_{\Omega} |f|^2,$$

and therefore (11) is proved. On the other hand, dropping the first positive term in (15) and using Hölder's inequality on the right hand side, we have

$$\frac{1}{2} \int_{\Omega} |u_n|^2 \leq \int_{\Omega} |f_n u_n| \leq \left[\int_{\Omega} |f_n|^2 \right]^{1/2} \left[\int_{\Omega} |u_n|^2 \right]^{1/2},$$

that is, (12) holds. Hölder's inequality, assumption (3), and estimates (10) and (12) give (13):

$$\begin{aligned} (16) \quad \int_{\Omega} |\nabla u_n| &\leq \left[\int_{\Omega} \frac{|\nabla u_n|^2}{[1 + b(x)|u_n|]^2} \right]^{1/2} \left[\int_{\Omega} [1 + b(x)|u_n|]^2 \right]^{1/2} \\ &\leq \left[\frac{1}{2\alpha} \int_{\Omega} |f|^2 \right]^{1/2} \left(\text{meas}(\Omega)^{1/2} + 2B \left[\int_{\Omega} |f|^2 \right]^{1/2} \right). \end{aligned}$$

We are left with estimate (14). Since $J_n(u_n) \leq J_n(T_k(u_n))$ we have

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_{\Omega} |u_n|^2 - \int_{\Omega} f_n u_n \\ &\leq \frac{1}{2} \int_{\Omega} \frac{j(x, \nabla T_k(u_n))}{[1 + b(x)|T_k(u_n)|]^2} + \frac{1}{2} \int_{\Omega} |T_k(u_n)|^2 - \int_{\Omega} f_n T_k(u_n). \end{aligned}$$

Recalling the definition of $G_k(s)$, and using that $|s|^2 - |T_k(s)|^2 \geq |G_k(s)|^2$, the last inequality implies

$$\frac{1}{2} \int_{\Omega} \frac{j(x, \nabla G_k(u_n))}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_{\Omega} |G_k(u_n)|^2 \leq \int_{\Omega} f_n G_k(u_n).$$

Dropping the first term of the left hand side and using Hölder's inequality on the right one, we obtain

$$\frac{1}{2} \int_{\Omega} |G_k(u_n)|^2 \leq \left[\int_{\{|u_n| \geq k\}} |f|^2 \right]^{1/2} \left[\int_{\Omega} |G_k(u_n)|^2 \right]^{1/2},$$

that is, (14) holds. □

LEMMA 4. *Let u_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ be a minimum of J_n . Then there exists a subsequence, still denoted by $\{u_n\}$, and a function u in $W_0^{1,1}(\Omega) \cap L^2(\Omega)$, with $T_k(u)$ in $H_0^1(\Omega)$ for every $k > 0$, such that u_n converges to u almost everywhere in Ω , strongly in $L^2(\Omega)$ and weakly in $W_0^{1,1}(\Omega)$, and $T_k(u_n)$ converges to $T_k(u)$ weakly in $H_0^1(\Omega)$. Moreover,*

$$(17) \quad \lim_{n \rightarrow +\infty} \frac{\nabla u_n}{1 + b(x)|u_n|} = \frac{\nabla u}{1 + b(x)|u|} \quad \text{weakly in } (L^2(\Omega))^N.$$

PROOF. By (13), the sequence u_n is bounded in $W_0^{1,1}(\Omega)$. Therefore, it is relatively compact in $L^1(\Omega)$. Hence, up to subsequences still denoted by u_n , there exists u in $L^1(\Omega)$ such that u_n almost everywhere converges to u . From Fatou's lemma applied to (12) we then deduce that u belongs to $L^2(\Omega)$.

We are going to prove that u_n strongly converges to u in $L^2(\Omega)$. Let E be a measurable subset of Ω ; then by (14) we have

$$\begin{aligned} \int_E |u_n|^2 &\leq 2 \int_E |T_k(u_n)|^2 + 2 \int_E |G_k(u_n)|^2 \\ &\leq 2k^2 \text{meas}(E) + 2 \int_\Omega |G_k(u_n)|^2 \\ &\leq 2k^2 \text{meas}(E) + 8 \int_{\{|u_n| \geq k\}} |f|^2. \end{aligned}$$

Since u_n is bounded in $L^2(\Omega)$ by (12), we can choose k large enough so that the second integral is small, uniformly with respect to n ; once k is chosen, we can choose the measure of E small enough such that the first term is small. Thus, the sequence $\{u_n^2\}$ is equiintegrable and so, by Vitali's theorem, u_n strongly converges to u in $L^2(\Omega)$.

Now we to prove that u_n weakly converges to u in $W_0^{1,1}(\Omega)$. Let E be a measurable subset of Ω . By Hölder's inequality, assumption (3), and (10), one has, for $i \in \{1, \dots, N\}$,

$$\begin{aligned} \int_E \left| \frac{\partial u_n}{\partial x_i} \right| &\leq \int_E |\nabla u_n| \leq \left[\int_E \frac{|\nabla u_n|^2}{[1 + b(x)|u_n|]^2} \right]^{1/2} \left[\int_E [1 + b(x)|u_n|]^2 \right]^{1/2} \\ &\leq \left[\frac{1}{2\alpha} \int_\Omega |f|^2 \right]^{1/2} \left[\int_E [1 + B|u_n|]^2 \right]^{1/2}. \end{aligned}$$

Since the sequence $\{u_n\}$ is compact in $L^2(\Omega)$, this estimate implies that the sequence $\left\{ \frac{\partial u_n}{\partial x_i} \right\}$ is equiintegrable. Thus, by Dunford–Pettis theorem, and up to subsequences, there exists Y_i in $L^1(\Omega)$ such that $\frac{\partial u_n}{\partial x_i}$ weakly converges to Y_i in $L^1(\Omega)$. Since $\frac{\partial u_n}{\partial x_i}$ is the distributional partial derivative of u_n , we have, for every n in \mathbb{N} ,

$$\int_\Omega \frac{\partial u_n}{\partial x_i} \varphi = - \int_\Omega u_n \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).$$

We now pass to the limit in the above identities, using that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$, and that u_n strongly converges to u in $L^2(\Omega)$: we obtain

$$\int_{\Omega} Y_i \varphi = - \int_{\Omega} u \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).$$

This implies that $Y_i = \frac{\partial u}{\partial x_i}$, and this result is true for every i . Since Y_i belongs to $L^1(\Omega)$ for every i , u belongs to $W_0^{1,1}(\Omega)$, as desired.

Since by (11) it follows that the sequence $\{T_k(u_n)\}$ is bounded in $H_0^1(\Omega)$, and since u_n tends to u almost everywhere in Ω , then $T_k(u_n)$ weakly converges to $T_k(u)$ in $H_0^1(\Omega)$, and $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k \geq 0$.

Finally, we prove (17). Let Φ be a fixed function in $(L^\infty(\Omega))^N$. Since u_n almost everywhere converges to u in Ω , we have

$$\lim_{n \rightarrow +\infty} \frac{\Phi}{1 + b(x)|u_n|} = \frac{\Phi}{1 + b(x)|u|} \quad \text{almost everywhere in } \Omega.$$

By Egorov’s theorem, the convergence is therefore quasi uniform; i.e., for every $\delta > 0$ there exists a subset E_δ of Ω , with $\text{meas}(E_\delta) < \delta$, such that

$$(18) \quad \lim_{n \rightarrow +\infty} \frac{\Phi}{1 + b(x)|u_n|} = \frac{\Phi}{1 + b(x)|u|} \quad \text{uniformly in } \Omega \setminus E_\delta.$$

We now have

$$\begin{aligned} & \left| \int_{\Omega} \frac{\nabla u_n}{1 + b(x)|u_n|} \cdot \Phi - \int_{\Omega} \frac{\nabla u}{1 + b(x)|u|} \cdot \Phi \right| \\ & \leq \left| \int_{\Omega \setminus E_\delta} \nabla u_n \cdot \frac{\Phi}{1 + b(x)|u_n|} - \int_{\Omega \setminus E_\delta} \nabla u \cdot \frac{\Phi}{1 + b(x)|u|} \right| \\ & \quad + \|\Phi\|_{L^\infty(\Omega)} \int_{E_\delta} [|\nabla u_n| + |\nabla u|]. \end{aligned}$$

Using the equiintegrability of $|\nabla u_n|$ proved above, and the fact that $|\nabla u|$ belongs to $L^1(\Omega)$, we can choose δ such that the second term of the right hand side is arbitrarily small, uniformly with respect to n , and then use (18) to choose n large enough so that the first term is arbitrarily small. Hence, we have proved that

$$(19) \quad \lim_{n \rightarrow +\infty} \frac{\nabla u_n}{1 + b(x)|u_n|} = \frac{\nabla u}{1 + b(x)|u|} \quad \text{weakly in } (L^1(\Omega))^N.$$

On the other hand, from (10) it follows that the sequence $\frac{\nabla u_n}{1 + b(x)|u_n|}$ is bounded in $(L^2(\Omega))^N$, so that it weakly converges to some function σ in the same space. Since (19) holds, we have that $\sigma = \frac{\nabla u}{1 + b(x)|u|}$, and (17) is proved. \square

REMARK 5. The fact that we need to prove (17) is one of the main differences with the paper [2].

PROOF OF THEOREM 1. Let u_n be as in Lemma 4. The minimality of u_n implies that

$$(20) \quad \int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_{\Omega} |u_n|^2 - \int_{\Omega} f_n u_n \leq \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|]^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f_n v$$

for every v in $H_0^1(\Omega)$. The result will then follow by passing to the limit in the previous inequality. The right hand side of (20) is easy to handle since f_n converges to f in $L^2(\Omega)$. Let us study the limit of the left hand side of (20). The convexity of j implies that

$$\int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2} \geq \int_{\Omega} \frac{j(x, \nabla T_k(u))}{[1 + b(x)|u_n|]^2} - \int_{\Omega} \frac{j_{\xi}(x, \nabla T_k(u))}{[1 + b(x)|u_n|]} \cdot \left(\frac{\nabla u_n}{[1 + b(x)|u_n|]} - \frac{\nabla T_k(u)}{[1 + b(x)|u_n|]} \right).$$

By (17), assumptions (1) and (2), and Lebesgue's theorem, we have

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2} \geq \int_{\Omega} \frac{j(x, \nabla T_k(u))}{[1 + b(x)|u|]^2} - \int_{\Omega} \frac{j_{\xi}(x, \nabla T_k(u))}{[1 + b(x)|u|]} \cdot \frac{\nabla[u - T_k(u)]}{[1 + b(x)|u|]},$$

that is, since $j_{\xi}(x, \nabla T_k(u)) \cdot \nabla(u - T_k(u)) = 0$,

$$\int_{\Omega} \frac{j(x, \nabla T_k(u))}{[1 + b(x)|u|]^2} \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2}.$$

Letting k tend to infinity, and using Levi's theorem, we obtain

$$(21) \quad \int_{\Omega} \frac{j(x, \nabla u)}{[1 + b(x)|u|]^2} \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2}.$$

Inequality (21) and Lemma 4 imply that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_{\Omega} |u_n|^2 - \int_{\Omega} f_n u_n \\ \geq \int_{\Omega} \frac{j(x, \nabla u)}{[1 + b(x)|u|]^2} + \frac{1}{2} \int_{\Omega} |u|^2 - \int_{\Omega} f u. \end{aligned}$$

Thus, for every v in $H_0^1(\Omega)$,

$$\int_{\Omega} \frac{j(x, \nabla u)}{[1 + b(x)|u|]^2} + \frac{1}{2} \int_{\Omega} |u|^2 - \int_{\Omega} fu \leq \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|]^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} fv,$$

so that u is a minimum of J ; its regularity has been proved in Lemma 4. □

REMARK 6. If we suppose that the coefficient $b(x)$ satisfies the stronger assumption

$$0 < A \leq b(x) \leq B, \quad \text{almost everywhere in } \Omega,$$

it is possible to prove that $J(u) \leq J(w)$ not only for every w in $H_0^1(\Omega)$, but also for the test functions w such that

$$(22) \quad \begin{cases} T_k(w) \text{ belongs to } H_0^1(\Omega) \text{ for every } k > 0, \\ \log(1 + A|w|) \text{ belongs to } H_0^1(\Omega), \\ w \text{ belongs to } L^2(\Omega). \end{cases}$$

Indeed, if w is as in (22), we can use $T_k(w)$ as test function in (4) and we have

$$J(u) \leq J(T_k(w)) = \int_{\Omega} \frac{j(x, \nabla T_k(w))}{[1 + b(x)|T_k(w)|]^2} + \frac{1}{2} \int_{\Omega} |T_k(w)|^2 - \int_{\Omega} fT_k(w).$$

In the right hand side is possible to pass to the limit, as k tends to infinity, so that we have $J(u) \leq J(w)$, for every test function w as in (22).

REMARK 7. We explicitly point out the differences, concerning the coercivity, between the functionals studied in [3] and the functionals studied in this paper. Indeed, let $0 < \rho < \frac{N-2}{2}$, and consider the sequence of functions

$$v_n = \exp \left[T_n \left(\frac{1}{|x|^\rho} - 1 \right) \right] - 1,$$

defined in $\Omega = B_1(0)$. Then

$$\log(1 + |v_n|) = T_n \left(\frac{1}{|x|^\rho} - 1 \right),$$

is bounded in $H_0^1(\Omega)$ (since the function $v(x) = \frac{1}{|x|^\rho} - 1$ belongs to $H_0^1(\Omega)$ by the assumptions on ρ), but, by Levi's theorem,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla v_n| = \rho \int_{\Omega} \frac{\exp \left[\frac{1}{|x|^\rho} - 1 \right]}{|x|^{\rho+1}} = +\infty.$$

Hence, the functional

$$v \in H_0^1(\Omega) \mapsto \int_{\Omega} \frac{|\nabla v|^2}{(1 + |v|)^2} = \int_{\Omega} |\nabla \log(1 + |v|)|^2,$$

which is of the type studied in [3], is non coercive on $W_0^{1,1}(\Omega)$. On the other hand, recalling (16), we have

$$\int_{\Omega} |\nabla v| = \int_{\Omega} \frac{|\nabla v|}{1+|v|} (1+|v|) \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(1+|v|)^2} + \frac{1}{2} \int_{\Omega} (1+|v|)^2.$$

Thus, the functional

$$v \in H_0^1(\Omega) \mapsto \int_{\Omega} \frac{|\nabla v|^2}{(1+|v|)^2} + \int_{\Omega} |v|^2,$$

which is of the type studied here, is coercive on $W_0^{1,1}(\Omega)$.

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