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Mathematical Physics — *The 2-D constrained NS equation and stochastic vortex theory*, by M. Pulvirenti and E. Rossi.

Dedicato a Giovanni Prodi.

Abstract. — We consider a modified version of the two-dimensional NS equation introduced in [4], [5] preserving energy and momentum of inertia. The physical motivation of this approach is the occurrence of different dissipation time scales. The constraint we consider is related to the gradient flow structure of the 2-D Navier-Stokes equation with respect to the Wasserstein metric. In this paper we justify the choice of this metric from the point of view of the stochastic vortex theory.

Key words: Navier-Stokes equation, coherent structures, gradient flows, stochastic vortex theory.

2000 Mathematical Subject Classification: 35Q30, 60H30.

1. Introduction

Let us consider the Navier-Stokes equation (NS equation in the sequel) in the plane for the vorticity $\omega = \omega(x,t)$:

$$(\partial_t + u \cdot \nabla)\omega(x,t) = \Delta\omega(x,t). \tag{1.1}$$

Here $x \in \mathbb{R}^2$, $t \in \mathbb{R}^+$ and $u = u(x,t) \in \mathbb{R}^2$ is the velocity field defined as:

$$u = \nabla^{\perp}\psi, \quad \psi = -\Delta^{-1}\omega.$$

Explicitly:

$$u = K * \omega, \quad K(x) = \nabla^{\perp} g(x) = -\frac{1}{2\pi}\frac{x^{\perp}}{|x|^2}, \tag{1.2}$$

where

$$g(x) = -\frac{1}{2\pi}\log|x| \tag{1.3}$$

is the fundamental solution for the Poisson equation in the plane.

The present paper is dedicated to the memory of Professor Giovanni Prodi (28 luglio 1925–29 gennaio 2010). Also the first three issues of Rendiconti Lincei RLM 2011 were dedicated to the memory of Professor Prodi.

Due to the dissipation term in the right hand side of eqn. (1.1), the asymptotic behavior of the solutions is trivial, namely $\omega(x,t) \to 0$ pointwise and in the L^p sense for $p > 1$. However $\int \omega$ and $\int x\omega$ are conserved. In the sequel we shall assume ω to be a probability distribution so that ω is non negative and $\int \omega = 1$ constantly in time. We also fix the reference frame in such a way that $\int x\omega = 0$ constantly in time.

It is well known that for the Euler equation

$$(\partial_t + u \cdot \nabla)\omega(x,t) = 0, \tag{1.4}$$

we have many conserved quantities as the energy:

$$E(\omega) = \frac{1}{2}\int \psi\omega \, dx, \tag{1.5}$$

and the moment of inertia (related to the invariance of E with respect to the group of rotations):

$$I(\omega) = \frac{1}{2}\int x^2\omega \, dx. \tag{1.6}$$

Due to the condition $\operatorname{div} u = 0$ also all the integrals of the form

$$F_\phi(\omega) = \int \phi(\omega)\, dx \tag{1.7}$$

(some time called Casimirs) are equally well conserved.

Such Euler invariants vary for the NS evolution. More precisely I increases by a constant rate:

$$\dot{I}(\omega) = 2, \tag{1.8}$$

while E and F_ϕ, for a convex ϕ, are dissipated with rates

$$\dot{E} = -\int \omega^2, \quad \dot{F}_\phi = -\int \phi''(\omega)|\nabla\omega|^2. \tag{1.9}$$

Looking at eqns. (1.9) one realizes that the energy and the moment of inertia could vary on a different and longer scale of times with respect to F_ϕ (whenever the last term in (1.9) dominates on the first one).

This may suggest to consider, in the first approximation, E and I as constant, looking at a master equation which modifies the NS equation leaving constant both energy and moment of inertia, but retaining all the other features of the NS dynamics.

To implement this program it is convenient to consider the NS evolution as a gradient flow [19]. Namely one can write:

$$\partial_t \omega = -\mathrm{div}\left[\omega \nabla^{\perp} \frac{\delta E}{\delta \omega}\right] + \nu\, \mathrm{div}\left[\omega \nabla \frac{\delta S}{\delta \omega}\right], \tag{1.10}$$

where $S(\omega) = \int \omega \log \omega$ is the Entropy functional.

A way to modify eqn. (1.10) in order that the functional E and I are separately invariant, is to project the dissipation part on the manifold $E = \text{const}$ and $I = \text{const}$. This is equivalent to write:

$$\partial_t \omega = -\mathrm{div}\left[\omega \nabla^{\perp} \frac{\delta E}{\delta \omega}\right] + \nu\, \mathrm{div}\left[\omega \nabla \left(\frac{\delta S}{\delta \omega} - a \frac{\delta I}{\delta \omega} - b \frac{\delta E}{\delta \omega}\right)\right], \tag{1.11}$$

and determining the multipliers a and b to guarantee the constance of E and I.

A straightforward computation leads us to

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= \mathrm{div}(\nabla \omega - b \omega \nabla \psi - a \omega x) \\ &= \mathrm{div}\left[\omega \nabla \left(\log \omega - b \psi - a \frac{x^2}{2}\right)\right], \end{aligned} \tag{1.12}$$

where

$$b = \frac{2I \int \omega^2 + 2V}{2I \int \omega |\nabla \psi|^2 - V^2}; \quad a = -\frac{2 \int \omega |\nabla \psi|^2 + V \int \omega^2}{2I \int \omega |\nabla \psi|^2 - V^2}, \tag{1.13}$$

and

$$V = \int \omega x \cdot \nabla \psi = \int dx \int dy \omega(x) \omega(y) x \cdot \nabla g(x - y) = -\frac{1}{4\pi}. \tag{1.14}$$

Eqn. (1.12) has been introduced in [4]. In [5] the nature of the NS equation as a gradient flow on the manifold of the probability densities has been discussed in some more detail. We also remark that the procedure of projecting the NS equation on a given manifold is not unique. Here we made a choice suggested by the particular way of expressing the equation as a gradient flow. Indeed the NS equation can be written as

$$\partial_t \omega = -J \nabla_W E - \nabla_W S \tag{1.15}$$

where ∇_W is the gradient with respect to the Wasserstein metric and $J\nabla_W$ is the ant-gradient with respect to the same metric (see [19]).

For an analysis of the gradient flow theory in connection with the mass transport problem see also [1] and [19].

Note that if we want that the inertial (Euler) part of the NS equation is the anti-gradient of the energy functional, the metric must be the one given by the Wasserstein distance. On the other hand, fixed the metric, the functional for the dissipative part has to be the entropy. In other words, the energy fixes the metric and the metric fixes the entropy. The interest of the constrained NS equation is basically due to its asymptotic behavior. In facts it is proven in [4] that, as $t \to \infty$, the solutions of (1.12) tends to a particular stationary solution of the Euler equation which solves the so called Mean-Field equation

$$\omega = \frac{e^{b\psi + a(x^2/2)}}{Z} \tag{1.16}$$

where

$$Z = \int e^{b\psi + a(x^2/2)}, \tag{1.17}$$

introduced and studied in [2] and [3].

It is worth to underline that such solutions are really observed in numerical simulations (see [17]).

For fixed values of $a < 0$ and b in a suitable range, eqn. (1.16), has been introduced in connection with the approaches related to the Statistical Mechanics of point vortices (see [14], [10], [17]). For a mathematical study see [2], [3], [8], [9] and the Appendix of [4].

Although the way to write the constrained NS equation in the form (1.15) is very natural and it is in agreement with the Statistical Mechanics of point vortices, it is also somehow arbitrary. The heat equation can be written as a gradient flow of many functionals (besides the entropy). For instance introducing the Enstrophy functional

$$\mathscr{E} = \frac{1}{2} \int \omega^2,$$

we have

$$\Delta\omega = \operatorname{div}\left[g(\omega) \nabla \frac{\delta \mathscr{E}}{\delta \omega} \right]$$

for the weight function $g(\omega) = 1$, which shows that the heat flow is also a gradient flow (for the metric H^{-1}) for the Enstrophy functional. Therefore projections on a given manifold depends on the metric we are considering and give different results.

In the present note we focus our attention in showing that eqn. (1.10) can indeed be justified from the point of view of the stochastic vortex theory for the NS equation which we are going to introduce. Actually the stochastic vortex model is a finite dimensional approximation for the NS equation so that, in this context, we can project without ambiguity.

Consider the diffusion process $X_N(t) = \{x_1(t) \dots x_N(t)\} \in R^{2N}$ solution of the following stochastic differential equation

$$dx_i(t) = \frac{1}{N} \sum_{j \neq i} K(x_i(t) - x_j(t))\, dt + \sqrt{2}\, dw_i(t) \tag{1.18}$$

where $w_i(t)$ are N independent Brownian motions. It is well known that the empirical measure in $\mathbb{R}^2$

$$\mu_N(t, dx) = \frac{1}{N} \sum_{i}^{N} \delta(x - x_i(t))\, dx \tag{1.19}$$

(actually a measure valued stochastic process) converge, in the limit $N \to \infty$ to the solution $\omega(t)$ of (1.1), provided that

$$\mu_N(0, dx) \to \omega_0(x)\, dx \tag{1.20}$$

weakly, in the sense of the convergence of probability measures.

There is a large literature on the argument. We quote [13] where a special class of diffusion processes yielding nonlinear diffusion equations in the limit $N \to \infty$ have been introduced. In [11] and [12] the problem relative to the NS equation has been approached, by regularizing the kernel K in (1.2) and removing the regularization suitably with N. Then in [15] and [16] the limit without regularization was directly exploited for a sufficiently large value of the viscosity coefficient.

Moreover it can be proven a convergence result looking at the whole process, rather than at the distribution at a fixed time, however, in the present context, we are interested only at some PDE aspects so that we do not look for the maximal generality.

Having at our disposal a finite dimensional approximation of the NS equation, it is natural to project this system on the constant energy and/or constant moment of inertia manifold. This procedure, due to the fact we are working in a finite dimensional setting, is not ambiguous. Then we take the limit $N \to \infty$ and interpret the result. We will find that this procedure will give us the same projection obtained accordingly to the Wasserstein metric, namely the one described in eqn. (1.12).

The plan of the paper is the following. In Section 2 we will constrain N independent Brownian motions to the manifold $I = \text{const}$ and find that the corresponding empirical measure $\mu_N(t, dx)$ approximate the solution of the constrained heat flow seen as a gradient flow for the Entropy functional.

We then extend those consideration to the interacting case, by using the Tanaka's map (introduced in [18]) which is a very efficient tool for this kind of limit whenever the diffusion coefficient is constant.

In Section 3 we consider (a regularized version of) the stochastic vortex system and constrain the motion to the constant energy manifold. Again we find the expected result namely eqn. (1.12) adapted to the present case.

Because the purpose of this paper is to show the correspondence between the projection made with the Wasserstein metric and the one obtained with the vortex approximation, we do not perform a simultaneous projection on both the manifolds $E = const.$ and $I = const.$, which is technically involved, but we only consider them separately. Moreover, as regards the projection on the constant energy manifold, we also introduce an extra cut-off preventing a denominator to become too small (see eqn. (3.21) below). This difficulty was also present in [5]. There it was overcome by means of a smallness assumption. Here our result, which is valid globally in time for the modified problem, can also be interpreted as holding for the original problem, but only for a short time.

This paper is suited for a special issue in memory of G. Prodi. He played a very important role in the scientific community, promoting the idea of approaching the mathematical theory of the NS equation from the point of view of its statistical properties. We hope that the present contribution is a small step in this direction.

2. Vortex theory for the constrained NS equation

According to the previous section, the NS equation constrained to the manifold $I = \text{const}$ reads as

$$(\partial_t + u \cdot \nabla)\omega = \Delta\omega + \frac{1}{I}\operatorname{div}(x\omega), \tag{2.1}$$

where $I = \dfrac{1}{2}\displaystyle\int \omega(t)x^2$ turns out to be a constant of motion. Note that eqn. (2.1) has been considered by Gallay and Wayne [7] by a different point of view, namely by suitably scaling the usual NS equation. The time asymptotics of eqn. (2.1) selects the so called Oseen's vortex, which is a gaussian function and hence a special solution to the Mean-Field equation for $\beta = 0$.

Now we want to show that eqn. (2.1) can be derived by projecting the stochastic vortex system into the manifold $I = \text{const}$ and then performing the mean-field limit exactly as for the unconstrained case. The starting point is the same study for the heat equation which clearly takes the form

$$\partial_t\omega = \Delta\omega + \frac{1}{I}\operatorname{div}(x\omega). \tag{2.2}$$

2.1. Constrained heat equation and particle approximation

Consider N independent standard Brownian motions $\{w_i\}_{i=1}^N$. Define $\{b_i\}_{i=1}^N$ by

$$b_i = x_i + w_i \tag{2.3}$$

where $\{x_i\}_{i=1}^N$ are N independent random variables in $\mathbb{R}^2$ with law ω_0. Then setting

$$\mu_N(t, dx) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - b_i(t))\, dx \tag{2.4}$$

the empirical measure (note that this is a measure valued stochastic process) we have, for any $t \geq 0$,

$$\lim_{N \to \infty} \mathbb{E}[\mu_N(t, dx)] = \omega(x, t)\, dx \tag{2.5}$$

in the sense of weak convergence of probability measures, where $\omega(t)$ solves the heat equation with initial datum ω_0.

Introducing

$$I = \frac{1}{2} \int dx x^2 \omega_0(x),$$

we want to project the process $B_N = \{b_i\}_{i=1}^N$ on the manifold

$$\left\{ Y_N = \{y_i\}_{i=1}^N \middle| \frac{1}{N} \sum_{i=1}^{N} y_i^2 = 2I \right\}.$$

For that we introduce the process $X_N(t)$ solution to the stochastic differential equation:

$$dX_N(t) = A(X_N(t))\, dt + \left(1 - \frac{X_N \otimes X_N}{2NI}\right) dB_N \tag{2.6}$$

where A is a drift term to be determined to assure the condition

$$I(X_N) = \frac{X_N^2}{2N} = \frac{1}{2N} \sum_{i=1}^{N} x_i^2(t) = I. \tag{2.7}$$

Note that, because of the identity

$$\nabla I(X_N) = \frac{1}{N} X_N \tag{2.8}$$

then

$$\frac{\nabla I(X_N) \otimes \nabla I(X_N)}{|\nabla I(X_N)|^2} = \frac{X_N \otimes X_N}{2NI} \tag{2.9}$$

thus the last term in the right hand side of (2.6) is nothing else than the projection on the manifold $I(X_N) = I$.

A standard computation using the Ito's calculus yields

$$dI(X_N) = \frac{1}{N} X_N \cdot \Big(1 - \frac{X_N \otimes X_N}{2NI}\Big) dB_N + \frac{1}{N} X_N \cdot A\, dt + dt. \tag{2.10}$$

The last term is a consequence of the identity $\Delta I(X_N) = 2$. Setting

$$A(X_N) = -\frac{X_N}{2I} \tag{2.11}$$

we have

$$dI(X_N) = 0. \tag{2.12}$$

In conclusion the process we are going to consider is

$$dX_N(t) = -\frac{X_N}{2I} dt + \Big(1 - \frac{X_N \otimes X_N}{2NI}\Big) dB_N, \tag{2.13}$$

or, in terms of components,

$$dx_i(t) = -\frac{x_i}{2I} dt + db_i - \sum_j \frac{x_i \otimes x_j}{2NI} db_j. \tag{2.14}$$

Note that the interaction produced by the constraint, namely the last term in eqns. (2.14), is vanishing in the limit $N \to \infty$, as we shall see in a moment, thus the remaining effect of the constraint is the drift term, which, however, does not produce interactions. Indeed, by the Ito isometry, for all $i = 1 \dots N$

$$\mathbb{E}\Big[\Big(\sum_{j=1}^{N} \int_0^t \frac{x_i \otimes x_j}{2NI} db_j(s)\Big)^2\Big] = \frac{1}{2NI} \mathbb{E}\Big[\int_0^t ds x_i^2(s)\Big]. \tag{2.15}$$

On the other hand, by symmetry,

$$\mathbb{E}(x_i^2(s)) = \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}(x_j^2(s)) = 2I$$

and hence the right hand side of (2.15) is bounded by $\frac{t}{N}$.

Next we introduce N independent processes $Y_N = \{y_i\}_{i=1}^N$ defined as the solution of

$$dy_i(t) = -\frac{y_i}{2I} dt + db_i \tag{2.16}$$

with the same initial distribution ω_0.

Clearly

$$\mathbb{E}(|x_i - y_i(t)|^2) \to 0 \tag{2.17}$$

as $N \to \infty$. On the other hand Y_N is a Brownian motion on a different time scale. In facts defining:

$$h_i(t) = e^{(1/2I)t} y_i(t)$$

we have that

$$dh_i(t) = e^{(1/2I)t}\, db_i(t)$$

and hence $v_i(\tau(t)) = db_i(t)$ where $d\tau = e^{(1/I)t}\, dt$, is a Brownian motion. Finally we can apply (2.5) on the time scale τ to conclude

PROPOSITION 1. *Suppose that $\{x_i\}_{i=1}^N$ are N independent random variables with law ω_0, then for all $t \geq 0$:*

$$\lim_{N\to\infty} \mathbb{E}(\mu_N(t)) = \omega(t) \tag{2.18}$$

in the sense of weak convergence of measures, where ω solves eqn. (2.2) *with initial datum ω_0.*

In conclusion constraining the heat equation on the manifold $I = \text{const}$, produces only a time scaling.

Of course this could be seen from the very beginning by transforming eqn. (2.2) into the heat equation by a suitable change of coordinates.

2.2. Constrained (regularized) NS equation and particle approximation

When dealing with the stochastic particle approximation for the NS equation, the kernel K in (1.2) is often regularized by smearing the logarithmic divergence. The regularization can possibly removed when $N \to \infty$ (see e.g. [12]). The purpose of the present work is to understand the role of the projections on given manifolds so that we regularize the NS equation without removing the cutoff. As we shall see later on, the regularization is necessary in any case, to give sense to the stochastic process constrained to the manifold with fixed energy.

Therefore our starting point is the regularized NS equation

$$(\partial_t + u_\varepsilon \cdot \nabla)\omega(x,t) = \Delta\omega(x,t). \tag{2.19}$$

Here $x \in \mathbb{R}^2$, $t \in \mathbb{R}^+$ and $u_\varepsilon(x,t)$ is the regularized velocity field defined as:

$$u_\varepsilon = K_\varepsilon * \omega, \quad K_\varepsilon(x) = \nabla^\perp g_\varepsilon(x), \tag{2.20}$$

where g_ε is any regularization of the Poisson kernel g such that

$$|K_\varepsilon(x) - K_\varepsilon(y)| \leq L|x - y|. \tag{2.21}$$

The corresponding stochastic vortex approximation is

$$dx_i(t) = \frac{1}{N}\sum_{j\neq i} K_\varepsilon(x_i(t) - x_j(t))\,dt + \sqrt{2}\,dw_i(t). \tag{2.22}$$

To proceed as before for the heat equation, we project the right hand side on the manifold $I = \text{const}$. Since the inertial (Eulerian) part preserves the moment of inertia because

$$\sum_{j\neq i} x_i \cdot K_\varepsilon(x_i - x_j) = 0, \tag{2.23}$$

it is enough to project the Brownian part only so that the process we want to study is

$$dx_i(t) = \frac{1}{N}\sum_{j\neq i} K_\varepsilon(x_i(t) - x_j(t))\,dt - \frac{x_i}{2I}dt + db_i - \sum_j \frac{x_i \otimes x_j}{2NI} db_j. \tag{2.24}$$

We already showed that the last term in (2.24) is indeed negligible. Thus, introducing the process

$$dy_i(t) = \frac{1}{N}\sum_{j\neq i} K_\varepsilon(y_i(t) - y_j(t))\,dt - \frac{y_i}{2I}dt + db_i, \tag{2.25}$$

we realize that the two measures

$$\mu_N(t, dx) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i(t))\,dx, \quad \nu_N(t, dx) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - y_i(t))\,dx$$

have the same expected asymptotic behavior .

On the other hand we know that $\frac{y_i}{2I}dt + db_i$ is the differential of a Brownian motion so that, by a time scaling, we can reduce the present problem to the convergence problem of unconstrained stochastic vortex system which is very well understood (see e.g. [12]). In conclusion we have:

THEOREM 1. *Suppose that* $\{x_i\}_{i=1}^N$ *are* N *independent random variables with law* ω_0, *then for all* $t \geq 0$:

$$\lim_{N\to\infty} \mathbb{E}(\mu_N(t)) = \omega(t) \tag{2.26}$$

in the sense of weak convergence of measures, where ω *solves*

$$(\partial_t + u_\varepsilon \cdot \nabla)\omega = \Delta\omega + \frac{1}{I}\,\mathrm{div}(x\omega).$$

with initial datum ω_0.

3. Energy constraint for Navier-Stokes equation

Consider the regularized NS equation:

$$(\partial_t + u \cdot \nabla)\omega(x,t) = \Delta\omega(x,t), \tag{3.1}$$

where $u = u_\varepsilon$.

According to [4] and [5] we can constrain the NS equation on the manifold $E = const.$ in the following way:

$$(\partial_t + u \cdot \nabla)\omega = \Delta\omega - b(t)\,\mathrm{div}(\omega\nabla g * \omega), \tag{3.2}$$

where

$$b(t) = \frac{\int dx\omega\Delta g * \omega}{\int dx\omega|\nabla g * \omega|^2}. \tag{3.3}$$

We want to show that the same result can be obtained using the stochastic vortex theory. For this reason, as already done in section 2.1 for the moment of inertia I, we will consider a system of N stochastic vortices and constrain their equations on the manifold $E = const.$

3.1. *Constrained vortices: a formal analysis*

The starting system is (here $g = g_\varepsilon$):

$$dx_i = \frac{1}{N}\sum_{j=1}^{N} \nabla^{\perp} g(x_i - x_j)\,dt + \sqrt{2}\,dw_i \tag{3.4}$$

where $\{w_i\}_{i=1}^N$ are N independent Brownian motions. For a system of N stochastic vortices the mean field energy takes the form:

$$E = E(x_1, \dots, x_N) = \frac{1}{N}\sum_{i<j} g(x_i - x_j). \tag{3.5}$$

To assure that the process (3.4) is constrained on the manifold $E = const.$, we introduce a new process $Y_N = \{y_i\}_{i=1}^N$, which is a modified version of the processes (3.4), such that $dE(Y_N) = 0$.

Therefore we set:

$$dY_N = \nabla^{\perp} E\,dt + A(Y_N(t))\,dt + \sqrt{2}\Big(1 - \frac{\nabla E \otimes \nabla E}{|\nabla E|^2}\Big)\,dW_N \tag{3.6}$$

where the last term is the orthogonal projection of the N Brownian motions on the manifold $E = const.$, while A is a drift term to be determined in order that $dE = 0$.

The differential of the mean field energy is computed using the Ito formula:

$$dE(Y_N) = \nabla E \cdot dY_N + \frac{1}{2} dY_N D^2 E \, dY_N \tag{3.7}$$
$$= \sum_i \nabla_i E \cdot dy_i + \frac{1}{2} \sum_{i,j} dy_i D^2_{i,j} E \, dy_j.$$

Here:

$$\nabla_i E = \frac{1}{N} \sum_j \nabla_i g(y_i - y_j), \quad \nabla_i = \nabla_{y_i} \tag{3.8}$$

and

$$(D^2_{i,j} E)_{\alpha,\beta} = \frac{1}{N} \frac{\partial^2}{\partial y_i^\alpha \partial y_j^\beta} g(y_i - y_j). \tag{3.9}$$

Substituting (3.6) in (3.7) we obtain:

$$dE(Y_N) = \sqrt{2} \nabla E \cdot \left(1 - \frac{\nabla E \otimes \nabla E}{|\nabla E|^2}\right) dW_N + \nabla E \cdot A \, dt$$
$$+ \left(1 - \frac{\nabla E \otimes \nabla E}{|\nabla E|^2}\right) dW_N D^2 E \left(1 - \frac{\nabla E \otimes \nabla E}{|\nabla E|^2}\right) dW_N$$
$$+ dW_N D^2 E \, dW_N + 2 \left(1 - \frac{\nabla E \otimes \nabla E}{|\nabla E|^2}\right) dW_N D^2 E \, dW_N$$
$$= \nabla E \cdot A \, dt + \Delta E \, dt - \frac{\nabla E D^2 E \nabla E}{|\nabla E|^2} dt.$$

Setting now

$$A = -\frac{\Delta E}{|\nabla E|^2} \nabla E \, dt + \left(\frac{\nabla E D^2 E \nabla E}{|\nabla E|^4}\right) \nabla E \, dt \tag{3.10}$$

we have $dE(Y_N) = 0$, thus the process we are going to consider is:

$$dY_N = \left(\nabla^\perp E - \frac{\Delta E}{|\nabla E|^2} \nabla E\right) dt + \sqrt{2} \, dW_N \tag{3.11}$$
$$- \sqrt{2} \frac{\nabla E \otimes \nabla E}{|\nabla E|^2} dW_N + \frac{\nabla E D^2 E \nabla E}{|\nabla E|^4} \nabla E \, dt$$

or, in terms of components,

$$dy_i = \frac{1}{N}\Big(\sum_j \nabla^\perp g(y_i - y_j) - b_N(t)\nabla g(y_i - y_j)\Big)\,dt + \sqrt{2}\,dw_i \\ + \sum_{j,k} \frac{\nabla_j E D^2_{j,k} E \nabla_k E}{|\nabla E|^4} \nabla_i E\,dt - \frac{\sqrt{2}}{|\nabla E|^2}\sum_j \nabla_j E \cdot dw_j \nabla_i E, \tag{3.12}$$

where

$$b_N(t) = \frac{\Delta E}{|\nabla E|^2}. \tag{3.13}$$

Introducing

$$\mu_N(t, dx) = \frac{1}{N}\sum_i \delta(x - y_i(t))\,dx, \tag{3.14}$$

the empirical measure associated to the processes (3.12), remembering that:

$$\nabla_i E = \frac{1}{N}\sum_j \nabla_i g(x_i - x_j) = u^\perp(x_i), \tag{3.15}$$

we have:

$$|\nabla E|^2 = N\int |u(x)|^2 \mu_N(dx) \tag{3.16}$$

$$\Delta E = N\int d\mu_N \Delta g * \mu_N \tag{3.17}$$

and we can rewrite (3.12) in the following way:

$$dy_i = (u(y_i) - b_N(t) u^\perp(y_i))\,dt + \sqrt{2}\,dw_i \\ + \frac{1}{N^2}\sum_{j,k} \frac{u(y_j) D^2_{j,k} E u(y_k)}{|\int |u(x)|^2 \mu_N(dx)|^2} u(y_i)\,dt \\ - \frac{\sqrt{2}}{N\int |u(x)|^2 \mu_N(dx)}\sum_j u(y_j)\cdot dw_j u(y_i) \tag{3.18}$$

where

$$b_N(t) = \frac{\int d\mu_N \Delta g * \mu_N}{\int |u(x)|^2 \mu_n(dx)}. \tag{3.19}$$

3.2. *A further cutoff*

Equations (3.2) and (3.18) present denominators of the form:

$$\int |u(x)|^2 \omega(dx); \quad \int |u(x)|^2 \mu_N(dx). \tag{3.20}$$

We cannot a priori exclude that they are vanishing, so that for this reason, we introduce a further regularization.

The regularization we propose is the following. Consider

$$D^{\omega}(t) = \int |u(x)|^2 \omega(dx),$$

then we define

$$D_{\eta}^{\omega}(t) = \max\{\eta, D^{\omega}(t)\}$$

for some $\eta > 0$, and then we replace $D^{\omega}(t)$ by $D_{\eta}^{\omega}(t)$ in (3.2).

In a similar way, defining

$$D^{\mu_N}(t) = \int |u(x)|^2 \mu_N(dx)$$

we replace it by

$$D_{\eta}^{\mu_N}(t) = \max\{\eta, D_N^{\mu}(t)\}. \tag{3.21}$$

in (3.18).

The regularized equation we are now considering is:

$$(\partial_t + u \cdot \nabla)\omega = \Delta\omega - b^{\eta}(t)\, \mathrm{div}(\omega \nabla g * \omega) \tag{3.22}$$

where

$$b^{\eta}(t) = \frac{\int dx \omega \Delta g * \omega}{D_{\eta}^{\omega}(t)},$$

and the approximating process is defined accordingly

$$\begin{aligned} dy_i = {} & (u(y_i) - b_N^{\eta}(t) u^{\perp}(y_i))\, dt + \sqrt{2}\, dw_i \\ & + \frac{1}{N^2} \sum_{j,k} \frac{u(y_j) D_{j,k}^2 E u(y_k)}{(D_{N,\eta}^{\omega})^2} u(y_i)\, dt \\ & - \frac{\sqrt{2}}{N \int |u(x)|^2 \mu_N(dx)} \sum_j u(y_j) \cdot dw_j u(y_i) \end{aligned} \tag{3.23}$$

where

$$b_N^\eta(t) = \frac{\int d\mu_N \Delta g * \mu_N}{D_\eta^{\mu_N}(t)}.$$

We can now show that the expectation of the empiric measure related to the processes (3.23):

$$\mu_N(t, dx) = \frac{1}{N}\sum_i \delta(x - y_i(t))\, dx \tag{3.24}$$

converges to $\omega(x,t)$ solution of (3.22), in the limit $N \to \infty$.

Theorem 2. *Suppose that $\{x_i\}_{i=1}^N$ are N independent random variables with law ω_0, then for all $t \geq 0$:*

$$\lim_{N\to\infty} \mathbb{E}(\mu_N(t)) = \omega(t) \tag{3.25}$$

in the sense of weak convergence of measures, where ω solves

$$(\partial_t + u \cdot \nabla)\omega = \Delta\omega - b^\eta(t)\,\mathrm{div}(\omega\nabla g * \omega)$$

with initial datum ω_0.

Remark. Note that the above theorem can be interpreted as a convergence limit for the original uncutoffed problem, but for a short time T. Indeed suppose $D^\omega(0) \geq c > 0$ then choosing $\eta > 0$ sufficiently large, there exists $T > 0$ s.t. for $t < T$ the solution of the regularized equation (3.22) agrees with that of (3.2). We will not exploit further this point.

We shall prove Theorem 2 in two steps. We first show that the last two terms in (3.23) are small for large N, allowing us to consider the following essential process

$$dz_i = (u(z_i) - b_N(t)u^\perp(z_i))\, dt + \sqrt{2}\, dw_i. \tag{3.26}$$

Then we control this new process by using the Tanaka's map we shall introduce later on.

3.3. *A preliminary step*

Setting $dY_N = \{dy_i\}_{i=1}^N$ and $dW_N = \{dw_i\}_{i=1}^N$, process (3.23) can be written as:

$$dY_N = U(Y_N(t))\, dt + \sqrt{2}\, dW_N + R_1(Y_N(t))\, dW_N + R_2(Y_N(t))\, dt \tag{3.27}$$

where

$$U_i = u(y_i) - b_N^\eta(t)u^\perp(y_i) \tag{3.28}$$

and R_1 and R_2 denote the last two terms in the right and side of (3.23) respectively.

The expectation of R_1 and R_2 are negligible as can be seen by the use of the Ito isometry. Indeed, for any fixed $t > 0$:

$$\mathbb{E}\left[\left(\int_0^t R_1(Y_N(s))\,dW_N(s)\right)^2\right]$$
$$= \mathbb{E}\left[\left(\int_0^t \frac{u(y_i)\sum_j u(y_j)\cdot dw_j}{N\int |u(y)|^2 \mu_N(dy)}\right)^2\right]$$
$$= \sum_{j,k}\sum_{\alpha,\beta,\gamma}\mathbb{E}\left[\int_0^t\int_0^t \frac{(u^\alpha(y_i))^2 u^\beta(y_j) u^\gamma(y_k)}{(N\int |u(y)|^2\mu_N(dy))^2}\,dw_j^\beta\,dw_k^\gamma\right]$$
$$= \mathbb{E}\left[\int_0^t \frac{\sum_\alpha (u^\alpha(y_i))^2 \sum_{j,\beta}(u^\beta(y_j))^2}{(N\int |u(y)|^2\mu_N(dy))^2}\,ds\right]$$
$$= \frac{1}{N}\mathbb{E}\left[\int_0^t \frac{|u(y_i)|^2}{\int |u(y)|^2\mu_N(dy)}\,ds\right].$$

On the other hand, by symmetry:

$$\mathbb{E}\left[\frac{|u(y_i)|^2}{\int |u(y)|^2\mu_N(dy)}\right] = \frac{1}{N}\sum_j \mathbb{E}\left[\frac{|u(y_j)|^2}{\int |u(y)|^2\mu_N(dy)}\right] = 1.$$

Thus:

$$\mathbb{E}\left[\left(\int_0^t \frac{u(y_i(s))\sum_j u(y_j(s))\cdot dw_j(s)}{N\int |u(y)|^2\mu_N(dy)}\right)^2\right] \tag{3.29}$$
$$= \frac{1}{N}\mathbb{E}\left[\int_0^t \frac{|u(y_i(s))|^2}{\int |u(y)|^2\mu_N(dy)}\,ds\right] = \frac{t}{N}.$$

By the Cauchy-Schwarz inequality it follows that:

$$\frac{1}{N}\mathbb{E}\left[\int_0^t \left|\frac{\sum_j u(y_j(s))\cdot dw_j(s) u(y_i(s))}{\int |u(y)|^2\mu_n(dy)}\right|\right] \le \sqrt{\frac{t}{N}}. \tag{3.30}$$

On the other hand for R_2 we have:

$$\mathbb{E}\left[\left|\int_0^t ds R_2(Y_N(s))\right|\right] \le \frac{2ct}{\eta^2 N} \tag{3.31}$$

where we have used the boundedness of u, $D_\eta^{\mu_N}(s)$ and of $(D^2_{j,k}E)_{\alpha,\beta} = \frac{\partial^2}{\partial x_j^\alpha \partial x_k^\beta} g(x_j - x_k)$.

It is therefore natural to compare process (3.27) with the solution to:

$$dZ_N = U(Z_N(t))\,dt + \sqrt{2}\,dW_N. \tag{3.32}$$

Since the drift U is Lipschitz continuous, it follows easily that:

$$\frac{1}{N}\mathbb{E}\left[\sum_i |y_i(t) - z_i(t)|\right] \to 0 \tag{3.33}$$

as $N \to \infty$.

3.4. *Tanaka's map and constrained NS equation*

The proof of Theorem 2 makes use of the Kantorovich-Rubinstein distance (or Wasserstein distance) W which we are going to define.

Definition. *Let $\mathbb{X}$ be a metric space with metric function $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$. Let μ_1 and μ_2 be two Borel probability measure on $\mathbb{X}$. Denote by π a joint representation of μ_1 and μ_2, namely a Borel measure on $\mathbb{X} \times \mathbb{X}$ such that:*

$$\int \pi(dx_1, dx_2) f(x_i) = \int \mu_i(dx) f(x), \quad i = 1, 2, \tag{3.34}$$

for any measurable function f. Then:

$$W(\mu_1, \mu_2) = \inf_\pi \int \pi(dx_1, dx_2) d(x_1, x_2). \tag{3.35}$$

It is well known that W is a metric on the space $M(\mathbb{X})$ of all Borel probability measures on $\mathbb{X}$. Moreover the topology induced by the metric W is equivalent to the topology of the weak convergence of probability measure (see e.g. [6]).

We shall use the above definition for $\mathbb{X} = \mathbb{R}^2$ with:

$$d(x, y) = \min\{1, |x - y|\} \tag{3.36}$$

and $\mathbb{X} = \Omega = C([0, T], \mathbb{R}^2)$ with:

$$d_T(x, y) = \sup_{t \in [0,T]} d(x(t), y(t)). \tag{3.37}$$

We denote by W the corresponding Wasserstein distance between measures for $\mathbb{X} = \mathbb{R}^2$ and by W_Ω when $\mathbb{X} = \Omega$.

Note that when $\mathbb{X} = \mathbb{R}^2$ if $\mu_i(dx) = \frac{1}{N}\sum_k \delta(x - x_k^i)\,dx$ with $i = 1, 2$ we have:

$$W(\mu_1, \mu_2) = \min_S \left\{ \frac{1}{N} \sum_i d(x_i^1, x_{S(i)}^2) \right\} \tag{3.38}$$

where the minimum is taken over all the permutations S of $1,\dots,N$. This formula is well known (see e.g. [11] for the proof). Therefore we can rephrase Theorem 2 in the following way:

THEOREM 3. *Suppose that $\{x_i\}_{i=1}^N$ are independent random variables with law ω_0, then for all $t \geq 0$:*

$$\lim_{N\to 0} \mathbb{E}[W(\mu_N(t),\omega(t))] = 0 \tag{3.39}$$

where $\mu_N(t,dx)$ is the empirical distribution (3.24) *and $\omega(t)$ solves* (3.22).

REMARK. It can be easily shown that Theorem 3 implies (3.25) at least for Lipschitz test functions. In fact let ϕ be a Lipschitz function then:

$$\begin{aligned}\mathbb{E}[\mu_N(\phi)] - \omega(\phi) &= \mathbb{E}\left[\int \phi(x)\mu_N(dx) - \int \phi(y)\omega(dy)\right]\\ &= \mathbb{E}\left[\int \pi(dx,dy)(\phi(x)-\phi(y))\right] \leq C\mathbb{E}\left[\int \pi(dx,dy)|x-y|\right]\end{aligned}$$

where $\pi \in M(\Omega)\times M(\Omega)$ is a joint representation of μ_N and ω and C is the Lipschitz constant of ϕ. Minimizing now on π we get:

$$\mathbb{E}[\mu_N(\phi)] - \omega(\phi) \leq C\mathbb{E}[W(\mu_N,\omega)]. \tag{3.40}$$

PROOF OF THEOREM 3. Consider the process introduced in eqn. (3.26)

$$dz_i = \frac{1}{N}\Big(\sum_j \nabla^\perp g(z_i - z_j) - b_N^\eta(t)\nabla g(z_i - z_j)\Big)\,dt + \sqrt{2}\,dw_i \tag{3.41}$$

or, in integral form,

$$z_i(t) = \frac{1}{N}\left[\sum_j \int_0^t (\nabla^\perp g(z_i - z_j) - b_N^\eta(s)\nabla g(z_i - z_j))\,ds\right] + \sqrt{2}w_i(t) \tag{3.42}$$

and the associated empirical measure:

$$\tilde{\mu}_N(t,dz) = \frac{1}{N}\sum_i \delta(z - z_i(t))\,dz. \tag{3.43}$$

By the use of triangular inequality we have:

$$W(\mu_N(t),\omega(t)) \leq W(\mu_N(t),\tilde{\mu}_N(t)) + W(\tilde{\mu}_N(t),\omega(t)). \tag{3.44}$$

We have already shown that the first term in the right hand side of (3.44) is vanishing (see (3.33)), thus it remains to control the last term only.

For any $\rho \in M(\Omega)$, define the map $y \in \Omega \to x_t^\rho(y) \in \Omega$ by the following implicit formula:

$$x_t^\rho(y) = y_t + \int_0^t ds \int K(x_s^\rho(y) - x_s^\rho(z))\rho(dz) - \int_0^t ds b_\rho^\eta(s) \int K^\perp(x_s^\rho(y) - x_s^\rho(z))\rho(dz), \tag{3.45}$$

where

$$b_\rho^\eta(s) = \frac{\int \Delta g(x_s^\rho(z) - x_s^\rho(y))\rho(dz)\rho(dy)}{D_\eta^\rho(s)} \tag{3.46}$$

and:

$$D_\eta^\rho(s) = \max\left\{\eta, \int |u(x_s^\rho(y))|^2 \rho(dy)\right\}. \tag{3.47}$$

The application $y \to x_t^\rho(y)$ is well defined by means of the classical iteration scheme, due to the Lipschitz continuity of K and Δg and the boundedness from below of $D_\eta^\rho(s)$.

It is now possible to define the Tanaka's map $\theta : M(\Omega) \to M(\Omega)$ in the following way:

$$\int F(y)(\theta\rho)(dy) = \int F(x^\rho(y))\rho(dy) \tag{3.48}$$

where $F \in C(\Omega)$.

The interest of the Tanaka's map is due to the following observations. Denoting by $P_x(dy)$ the conditional Wiener measure starting from x, by $\omega_0(dx)$ the distribution of x and by $\lambda(dy) \in M(\Omega)$ the measure:

$$\lambda(dy) = \int \omega_0(dx) P_x(dy) \tag{3.49}$$

we have that

$$\theta\lambda = \omega \tag{3.50}$$

where ω is the distribution of the process $x = \{x(t)\}_{t \in [0,T]}$ and $x(t)$ solves:

$$x(t) = \sqrt{2}w_t + \int_0^t ds \int [K(x(s) - y) - b^\eta(s) K^\perp(x(s) - y)]\omega_s(dy) \tag{3.51}$$

with $\omega_s(dy) = \omega_s(y)\, dy$ and $\omega_s(y)$ solution of (3.22).

Notice that if $F : \Omega \to \mathbb{R}^2$ is defined by:

$$F(w) = f(w(t)), \quad t \in [0, T] \tag{3.52}$$

for some measurable $f : \mathbb{R}^2 \to \mathbb{R}^2$, then $\lambda_t(dy) \in M(\mathbb{R}^2)$, where $M(\mathbb{R}^2)$ is the space of the Borel probability measures on $\mathbb{R}^2$, defined as:

$$\int \lambda_t(dy) f(y) = \int \lambda(dw) F(w) \tag{3.53}$$

is the distribution $x + w(t)$, where $w(t)$ is a standard Brownian motion and x is the random variable whose distribution is given by $\omega_0(dx)$.

In addition:

$$\theta \nu_N = \tilde{\mu}_N \tag{3.54}$$

where $\nu_N \in M(\Omega)$ is the empirical distribution of N Brownian motions w_i, i.e.:

$$\nu_N = \frac{1}{N} \sum_{i=1}^{N} \delta(x - \sqrt{2} w_i)\, dx, \quad w_i = \{w_i(t)\}_{t \in [0, T]} \tag{3.55}$$

and $\tilde{\mu}_N \in M(\Omega)$ is defined as:

$$\tilde{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \delta(z - z_i)\, dz, \quad z_i = \{z_i(t)\}_{t \in [0, T]}. \tag{3.56}$$

where $z_i(t)$ are the processes solutions of (3.42).

Relations (3.50) and (3.54) can be easily proven.

In fact let ω_t be the solution of equation (3.22) and x_t the process solution of 3.51. Then, by the Lipschitz continuity of the drift, it is a well known fact that the pair (ω_t, x_t) is unique and that:

$$\int f(x) \omega_t(x)\, dx = \int \mathbb{E}_x[f(x_t)] \omega_0(x)\, dx \tag{3.57}$$

where $\omega_0(x)$ is the distribution of the initial datum x and $\mathbb{E}_x$ denotes the conditional expectation with respect to the event $x_0 = x$. Thus we can write for every $w \in \Omega$:

$$\begin{aligned} x(t) &= x_t(w) \\ &= \sqrt{2} w_t + \int_0^t ds \int [K(x_s(w) - y) - b^\eta(s) K^\perp(x_s(w) - y)] \omega_s(dy). \end{aligned} \tag{3.58}$$

Using relation (3.57) we have:

$$\begin{aligned}\int K(x_s(w)-y)\omega_s(dy) &= \int \mathbb{E}_x[K(x_s(w)-x(t))]\omega_0(x)\,dx \\ &= \int K(x_s(w)-x_s(b))\lambda(db)\end{aligned}$$

and an analogous result holds for $K^{\perp}(x-y)$. With similar calculations we obtain $b^{\eta}(t) = b^{\eta}_{\lambda}(t)$.

Using all these results in equation (3.58) gives:

$$x(t) = x_t^{\lambda}(w) \tag{3.59}$$

Thus by relations (3.57) and (3.59) we have

$$\begin{aligned}\int f(x)\omega_t(x)\,dx &= \int \mathbb{E}_x[f(x_t)]\omega_0(x)\,dx \\ &= \int f(x_t(y))\lambda_x(dy)\omega_0(x)\,dx = \int f(y)(\theta\lambda)(dy)\end{aligned}$$

proving (3.50).

Consider now equation (3.42) and note that we can solve it for every $w \in \Omega$:

$$\begin{aligned}z_i(t) &= z_t(w_i) \\ &= \frac{1}{N}\sum_j \int_0^t ds[\nabla^{\perp} g(z_s(w_i)-z_s(w_j)) - b_N^{\eta}(s)\nabla g(z_s(w_i)-z_s(w_j))] + \sqrt{2}w_i(t) \\ &= x_t^{\nu_N}(w_i).\end{aligned}$$

Using now this relation and the definition of the Tanaka's map we obtain:

$$\begin{aligned}\int F(y)(\theta\nu_N)(dy) &= \int F(x_t^{\nu_N}(y))\nu_N(dy) \\ &= \frac{1}{N}\sum_i F(x_t^{\nu_N}(b_i)) = \frac{1}{N}\sum_i \int F(y)\delta(y - x_t^{\nu_N}(b_i))\,dy \\ &= \frac{1}{N}\sum_i \int F(y)\delta(y-z_i(t))\,dy = \int F(y)\tilde{\mu}_N(dy)\end{aligned}$$

proving (3.54).

If we now take two distributions $\alpha, \beta \in M(\Omega)$, denoting by $\pi \in M(\Omega) \times M(\Omega)$ a joint representation of α and β, we define π_{θ} by:

$$\int F(z,y)\pi_{\theta}(dz,dy) = \int F(x^{\alpha}(z), x^{\beta}(y))\pi(dz,dy). \tag{3.60}$$

π_{θ} is obviously a joint representation of $\theta\alpha$ and $\theta\beta$.

Our aim is now to show that:

$$W((\theta\alpha)(t),(\theta\beta)(t)) \le CW_{\Omega}(\alpha,\beta) \tag{3.61}$$

for any pair $\alpha,\beta \in M(\Omega)$ (note that this inequality is purely deterministic). We will prove this inequality later on.

For the moment we apply (3.61) to distributions $\tilde{\mu}_N$ and ω:

$$\begin{aligned}\mathbb{E}[W(\tilde{\mu}_n(t),\omega(t))] &\le \mathbb{E}[W((\theta\nu_N)(t),(\theta\lambda)(t))]\\ &\le C\mathbb{E}[W_{\Omega}(\nu_N,\lambda)].\end{aligned} \tag{3.62}$$

where λ and ν_N are defined in (3.53) and (3.55) respectively.

The following result is well known:

THEOREM. *Suppose that $\{w_i\}_{i=1}^N$ are N independent Brownian motions. Then*

$$\lim_{N\to\infty}\mathbb{E}[W_{\Omega}(\nu_N,\lambda)] = 0 \tag{3.63}$$

where λ and ν_N are defined in (3.53) *and* (3.55) *respectively*. (See e.g. [11] for the proof).

Using now (3.63) in (3.62) we obtain the desired result.

Now it remains to show inequality (3.61). For this purpose we want to control the quantity $|x_t^{\alpha}(y) - x_t^{\beta}(h)|$.

We have:

$$\begin{aligned}
&|x_t^{\alpha}(y) - x_t^{\beta}(h)|\\
&\quad\le |y_t - h_t| + \int_0^t ds \int |K(x_s^{\alpha}(y) - x_s^{\alpha}(z)) - K(x_s^{\beta}(h) - x_s^{\alpha}(z))|\alpha(dz)\\
&\qquad + \int_0^t ds\left|\int K(x_s^{\beta}(h) - x_s^{\alpha}(z))\alpha(dz) - \int K(x_s^{\beta}(h) - x_s^{\beta}(v))\beta(dv)\right|\\
&\qquad + \int |b_{\beta}^{\eta}(s)| \int |K^{\perp}(x_s^{\beta}(h) - x_s^{\beta}(v)) - K^{\perp}(x_s^{\alpha}(y) - x_s^{\beta}(v))|\beta(dv)\\
&\qquad + \int_0^t ds\left|b_{\beta}^{\eta}(s)\left[\int K^{\perp}(x_s^{\alpha}(y) - x_s^{\beta}(v))\beta(dv) - \int K^{\perp}(x_s^{\alpha}(y) - x_s^{\alpha}(z))\alpha(dz)\right]\right|\\
&\qquad + \int_0^t ds|b_{\beta}^{\eta}(s) - b_{\alpha}^{\eta}(s)| \int |K(x_s^{\alpha}(y) - x_s^{\alpha}(z))|\alpha(dz).
\end{aligned}$$

Note that we can write:

$$\int K(x_s^\beta(h) - x_s^\alpha(z))\alpha(dz) - \int K(x_s^\beta(h) - x_s^\beta(v))\beta(dv)$$
$$= \int [K(x_s^\beta(h) - x_s^\alpha(z)) - K(x_s^\beta(h) - x_s^\beta(v))]\pi(dz, dv)$$

where π is a joint representation of α and β.

It is now possible to use the following estimates:

$$|K(x-y) - K(z-y)| \le L|x-z|, \tag{3.64}$$

$$|b_\alpha^\eta(s)| = \left|\frac{\int \Delta g(x_s^\rho(z) - x_s^\rho(y))\alpha(dz)\alpha(dy)}{D_\eta^\alpha(s)}\right| \le \frac{C}{\eta} \tag{3.65}$$

and

$$\int |K(x_s^\alpha(y) - x_s^\alpha(z))|\alpha(dz) \le C_1 \tag{3.66}$$

to obtain

$$|x_t^\alpha(y) - x_t^\beta(h)| \le |y_t - h_t| + L\int_0^t ds|x_s^\alpha(y) - x_s^\beta(h)|$$
$$+ L\int_0^t ds \int |x_s^\alpha(y) - x_s^\beta(h)|\pi(dy, dh)$$
$$+ \frac{CL}{\eta}\int_0^t ds \int |x_s^\alpha(y) - x_s^\beta(h)|\pi(dy, dh)$$
$$+ \frac{CL}{\eta}\int_0^t ds|x_s^\alpha(y) - x_s^\beta(h)| + C_1\int_0^t ds|b_\beta^\eta(s) - b_\alpha^\eta(s)|.$$

For the last term in the above inequality we have:

$$|b_\beta^\eta(s) - b_\alpha^\eta(s)|$$
$$= \left|\frac{\int \Delta g(x_s^\beta(h) - x_s^\beta(y))\beta(dh)\beta(dy)}{D_\eta^\beta(s)} - \frac{\int \Delta g(x_s^\alpha(z) - x_s^\alpha(v))\alpha(dz)\alpha(dv)}{D_\eta^\alpha(s)}\right|$$
$$\le \frac{1}{\eta^2}\left|(D_\eta^\alpha(s) - D_\eta^\beta(s))\int \Delta g(x_s^\alpha(z) - x_s^\alpha(v))\alpha(dz)\alpha(dv)\right|$$
$$+ \frac{|D_\eta^\alpha(s)|}{\eta^2}\left|\int \Delta g(x_s^\beta(h) - x_s^\beta(y))\beta(dh)\beta(dy)\right.$$
$$\left. - \int \Delta g(x_s^\alpha(z) - x_s^\alpha(v))\alpha(dz)\alpha(dv)\right|.$$

Consider now:

$$\begin{aligned}
&\left|\int \Delta g(x_s^\beta(h) - x_s^\beta(y))\beta(dh)\beta(dy) - \int \Delta g(x_s^\alpha(z) - x_s^\alpha(v))\alpha(dz)\alpha(dv)\right| \\
&\quad \le \left|\int \Delta g(x_s^\beta(h) - x_s^\beta(y))\beta(dh)\beta(dy) - \int \Delta g(x_s^a(z) - x_s^\beta(y))\alpha(dz)\beta(dy)\right| \\
&\qquad + \left|\int \Delta g(x_s^\alpha(z) - x_s^\beta(y))\alpha(dz)\beta(dy) - \int \Delta g(x_s^a(z) - x_s^\alpha(v))\alpha(dz)\alpha(dv)\right| \\
&\quad \le 2L' \int |x_s^\beta(y) - x_s^\alpha(z)|\pi(dy, dz)
\end{aligned}$$

where L' is the Lipschitz constant of Δg and π is a joint representation of the distributions α and β defined in (3.34).

Using now the boundedness of Δg and $D_\eta^\alpha(s)$ we have:

$$|b_\beta^\eta(s) - b_\alpha^\eta(s)| \le \frac{C}{\eta^2}|D_\eta^\alpha(s) - D_\eta^\beta(s)| + \frac{2L'C}{\eta^2}\int |x_s^\beta(y) - x_s^\alpha(z)|\pi(dy, dz).$$

Because $D_\eta^\alpha(s) = \max\left\{\eta, \int |u^\alpha(x_s^\alpha(z))|^2\alpha(dz)\right\}$ we must consider three different cases:

- $D_\eta^\alpha(s) = D_\eta^\beta(s) = \eta$, that implies $|D_\eta^\alpha - D_\eta^\beta| = 0$,
- $D_\eta^\alpha(s) = \int |u^\alpha(x_s^\alpha(z))|^2\alpha(dz)$, $D_\eta^\beta(s) = \int |u^\beta(x_s^\beta(z))|^2\beta(dz)$, which implies

$$\begin{aligned}
|D_\eta^\alpha(s) - D_\eta^\beta(s)| &= \left|\int |u^\alpha(x_s^\alpha(z))|^2\alpha(dz) - \int |u^\beta(x_s^\beta(y))|^2\beta(dy)\right| \\
&= \int [|u^\alpha(x_s^\alpha(z))| - |u^\beta(x_s^\beta(y))|] \\
&\qquad \times [|u^\alpha(x_s^\alpha(z))| + |u^\beta(x_s^\beta(y))|]\pi(dz, dy) \\
&\le 4C_1 L \int |x_s^\alpha(z) - x_s^\beta(y)|\pi(dz, dy);
\end{aligned}$$

  where L' is the Lipschitz constant on Δg, π is a joint representation of α and β defined in (3.34).
- $D_\eta^\alpha(s) = \int |u^\alpha(x_s^\alpha(z))|^2\alpha(dz)$, $D_\eta^\beta(s) = \eta$ that implies:

$$\begin{aligned}
|D_\eta^\alpha - D_\eta^\beta| &= \int |u^\alpha(x_s^\alpha(z))|^2\alpha(dz) - \eta \\
&\le \int |u^\alpha(x_s^\alpha(z))|^2\alpha(dz) - \int |u^\beta(x_s^\beta(z))|^2\beta(dz) \\
&\le 4C_1 L \int |x_s^\alpha(z) - x_s^\beta(y)|\pi(dz, dy).
\end{aligned} \tag{3.67}$$

Thus we have

$$\int_0^t ds|D_\eta^\alpha - D_\eta^\beta| \le 4C_1 L \int_0^t ds \int |x_s^\alpha(z) - x_s^\beta(y)|\pi(dz, dy). \tag{3.68}$$

Putting all these inequality together we have

$$|x_t^\alpha(y) - x_t^\beta(h)| \le |y_t - h_t| + L_1 \int_0^t ds|x_s^\alpha(y) - x_s^\beta(h)|$$
$$+ L_2 \int_0^t ds \int |x_s^\alpha(y) - x_s^\beta(h)|\pi(dy, dh)$$

where

$$L_1 = L\Big(1 + \frac{C}{\eta}\Big)$$
$$L_2 = L_1 + \frac{2L'C_1}{\eta^2} + \frac{8CC_1^2}{\eta^2}L.$$

Using now the Gronwall Lemma and definition (3.36) we have:

$$d(x_t^\alpha(y), x_t^\beta(h)) \le \left[d(y_t, h_t) + L_2 \int_0^t ds \int d(x_s^\alpha(y), x_s^\beta(h))\pi(dy, dh)\right] e^{L_1 T}.$$

Defining:

$$\gamma(y, h, t) = \sup_{s \le t} |x_s^\alpha(y) - x_s^\beta(h)| \tag{3.69}$$

and

$$\chi(t) = \int \pi(dy, dh)\gamma(y, h, t), \tag{3.70}$$

we have for $r \le \tau$

$$d(x_r^\alpha(y), x_r^\alpha(h)) \le \left[\sup_{r \le \tau} |y_r - h_r| + \int_0^r ds\chi(s)\right] e^{L_1 T} \tag{3.71}$$

and hence

$$\gamma(y, h, t) \le \left[d_T(y, h) + L_2 \int_0^t ds\chi(s)\right] e^{L_1 T}. \tag{3.72}$$

Integrating now by $\pi(dy, dh)$ and using again the Gronwall Lemma we have

$$\chi(t) \le \left[\int \pi(dy, dh)\, d_T(y, h)\right] e^{L_2 T e^{L_1 T}} e^{L_1 T}. \tag{3.73}$$

Minimizing now on π we obtain

$$W((\theta\alpha)(t), (\theta\beta)(t)) \leq W_{\Omega}(\alpha, \beta) e^{L_2 T e^{L_1 T}} e^{L_1 T}, \tag{3.74}$$

so that (3.61) is proved.

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