



**Mathematical Analysis** — *A version of Gehring lemma in Orlicz spaces*, by LUIGI GRECO and GABRIELLA ZECCA.

*Dedicated to the memory of Professor Giovanni Prodi.*

ABSTRACT. — We present a version of the Gehring lemma, showing higher integrability in the scale of Orlicz spaces for a function  $g$  satisfying reverse Hölder's inequalities of the type

$$\left(\int_B g^m\right)^{1/m} \leq \int_{2B} fg + \left(\int_{2B} h^m\right)^{1/m},$$

under suitable integrability conditions on  $f$  which do not imply boundedness. We describe explicitly in the general case how the improved integrability of  $g$  depends on the assumptions on  $f$ , thus extending results of [4, 2] which deal with  $f$  exponentially integrable.

We also present some applications of our result to the theory of mappings of finite inner distortion.

KEY WORDS: Higher integrability, reverse Hölder's inequalities, mappings of finite distortion.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 26D15, 26B10, 30C65.

## 1. INTRODUCTION

In this paper we study higher integrability results which can be deduced as a consequence of reverse Hölder's inequalities of the type

$$(1.1) \quad \left(\int_B g^m\right)^{1/m} \leq \int_{2B} fg + \left(\int_{2B} h^m\right)^{1/m}.$$

Here,  $m > 1$ ,  $g$  and  $h$  are nonnegative functions in  $L^m_{\text{loc}}$  and (1.1) hold for all balls  $B$  with  $2B$  contained in a domain of  $\mathbb{R}^n$ . In the case that  $f$  is a bounded function, the celebrated Gehring Lemma ([5]) and its several extensions show that there exists a new exponent  $s > m$ , depending on  $n$ ,  $m$  and  $\|f\|_\infty$ , such that  $g \in L^s_{\text{loc}}$ , if the same is true for  $h$ . We refer to the survey paper [11] and references therein.

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The present paper is dedicated to the memory of Professor Giovanni Prodi (28 luglio 1925–29 gennaio 2010). Also the first three issues of Rendiconti Lincei RLM 2011 were dedicated to the memory of Professor Prodi.

On the other hand, studying problems with some kind of degeneracy naturally leads to consider the case of  $f$  unbounded. In this case, we do not have the same sort of improvement in the integrability properties of  $g$  as above. As far as we know, the first result for  $f$  unbounded appeared recently, in [4], where regularity of mappings of exponentially integrable distortion is studied. A key tool there was a result showing that, if reverse Hölder's inequalities (1.1) hold with  $m = 1 + 1/n$ ,  $\exp(\beta f) \in L^1_{\text{loc}}$  for some  $\beta > 0$ , and  $h \equiv 0$ , then  $g \in L^m \log^{c\beta} L_{\text{loc}}$ , with  $c = c(n) > 0$ . This result indicates precisely how the degree of the improved regularity depends on  $\beta$ , and is qualitatively sharp, in the sense that examples show that in general  $g \notin L^m \log^\alpha L_{\text{loc}}$ , for sufficiently large  $\alpha$ . The result has been extended in [2] allowing for general  $m > 1$  and the nonhomogeneous term involving  $h$ , and used again for applications to some regularity problems arising in the theory of generalized Newtonian fluids.

In [7, 8] we extended the results of [4] concerning mappings of finite distortion by considering more general conditions on the distortion. In those paper, higher integrability is not deduced from reverse Hölder's inequalities, but is proved directly by means of some estimates which appear to be stronger than reverse inequalities. In [9] similar arguments are used to deal with solutions to degenerate elliptic equations.

A generalization of the results on reverse Hölder's inequalities of [4, 2] appeared in a recent paper by Clop-Koskela on the regularity properties of mappings of finite distortion, see [3, Lemma 3.1]. They assume that  $f$  is subexponentially integrable, that is,

$$(1.2) \quad \exp(P(\beta f^m)) \in L^1_{\text{loc}},$$

for  $\beta > 0$  and an increasing function  $P$ , such that  $t \mapsto P(t)/t$  is decreasing and the *divergence condition* holds

$$(1.3) \quad \int_{t_0}^{\infty} \frac{P(t)}{t^2} dt = \infty,$$

for some  $t_0 > 0$ . Typical examples are  $P(t) = t$ ,

$$(1.4) \quad P(t) = \frac{t}{\log(e+t)}, \quad P(t) = \frac{t}{\log(e+t) \log \log(e^e+t)}, \dots$$

It is well known that (1.3) is a threshold condition in the theory of mappings of finite distortion, see e.g. [1, 12, 13]. However, for general  $P$ , conditions (1.2) and (1.3) do not even imply  $f^m$  locally integrable, so  $P$  should satisfy additional conditions. Under suitable assumptions, in [3] it is proved that, if reverse Hölder's inequalities (1.1) hold, then  $g^m E(g)^{c\beta}$  is locally integrable, where

$$(1.5) \quad E(t) = 1 + \int_1^t \frac{P(\tau)}{\tau^2} d\tau.$$

This expression  $E$  which governs the improved integrability of  $f$  in [3] is suggested by the following conjecture of Iwaniec and Martin.

CONJECTURE 1.1 ([12, pg. 267]). *Let  $F : \Omega \rightarrow \mathbb{C}$  be a planar mapping of finite distortion  $K$ , such that  $e^{P(K)} \in L^1$ . Then,  $|DF|^2$  belongs locally to the Orlicz space  $L^R(\Omega)$ , where*

$$R(t) = P(t) \left( 1 + \int_1^t \frac{P(\tau)}{\tau^2} d\tau \right)^{-1}.$$

The result of [3] would imply a positive answer to this conjecture. Unfortunately, the assumptions are rather involved and restrictive. So, while it is not clear whether the result of [3] applies to functions essentially different from those in (1.4), it is easy to find cases which cannot be handled by it. As an example, this happen for the function

$$(1.6) \quad P(t) = \frac{t}{\log^\vartheta(e+t)},$$

with  $0 < \vartheta < 1$ . This is not a technical point, because trying to use (1.5) with  $P$  given by (1.6) would give too high improvement, by a power of logarithm, which is the same as for  $P(t) = t$  and does not hold in this case, see Example 1.7 and Remark 1.3 below. Instead, this shows that  $E$  is not the correct expression to describe the improved integrability in the general case.

We shall also provide a more drastic example, showing that actually Conjecture 1.1 has a negative answer (see Section 5).

In this paper, using ideas of [7, 8, 9], we prove a general and sharp higher integrability result which extends those of [4, 2, 3], and describes exactly the improved integrability of  $g$  in terms of the integrability assumption on  $f$ .

For a given  $t_0 > 0$ , we consider a positive, continuous and strictly increasing function

$$(1.7) \quad P : [t_0, \infty[ \rightarrow [P(t_0), \infty[$$

diverging at  $\infty$  and verifying the divergence condition (1.3). We assume also that the inverse function  $P^{-1}$  satisfies the  $\Delta_2$ -condition: There exists a constant  $C_1 > 1$  such that

$$(1.8) \quad P^{-1}(2\sigma) \leq C_1 P^{-1}(\sigma), \quad \forall \sigma \geq P(t_0).$$

We define the function

$$(1.9) \quad \mathcal{A}(s) = \begin{cases} 1, & \text{for } 0 \leq s \leq \exp(P(t_0)) \\ \exp \left[ \int_{P(t_0)}^{\log s} \frac{d\sigma}{P^{-1}(\sigma)} \right], & \text{for } s \geq \exp(P(t_0)) \end{cases}$$

Notice that (1.3) implies

$$(1.10) \quad \int_{P(t_0)}^{\infty} \frac{d\sigma}{P^{-1}(\sigma)} = \infty,$$

see [8], therefore we have

$$(1.11) \quad \lim_{s \rightarrow \infty} \mathcal{A}(s) = \infty.$$

Moreover, since  $P^{-1}$  diverges at  $\infty$  the function  $\mathcal{A}$  increases at  $\infty$  more slowly than any power of  $s$  with positive exponent, see [9].

We state our main result.

**LEMMA 1.2.** *Let  $\Omega$  be a ball of  $\mathbb{R}^n$ . For given  $P$  such that (1.3), (1.8) hold, and a constant  $m > 1$ , there exists a positive exponent  $\varepsilon = \varepsilon(n, m, C_1)$  with the following property. Let  $f, g$  and  $h$  be nonnegative functions on  $\Omega$  verifying  $g, h \in L^m(\Omega)$ ,  $\exp(P(\beta f^m)) \in L^1(\Omega)$  for a constant  $\beta > 0$ , and (1.1) holds, for all balls  $B \subset 2B \subset \Omega$ . Then, we have  $g^m \mathcal{A}(g^m)^{\varepsilon\beta} \in L^1_{\text{loc}}(\Omega)$ , if the same is true for  $h$ . Moreover, for each  $0 < \sigma < 1$  we have*

$$(1.12) \quad \int_{\sigma\Omega} g^m \mathcal{A}\left(\frac{g^m}{g^m}\right)^{\varepsilon\beta} \leq C g^m \int_{\Omega} \exp(\beta f^m) + C \int_{\Omega} h^m \mathcal{A}\left(\frac{h^m}{g^m}\right)^{\varepsilon\beta}$$

where  $g^m_{\Omega} = \int_{\Omega} g^m$ , and  $C$  is a positive constant depending only on  $n, m, \Phi, \beta$  and  $\sigma$ .

**REMARK 1.3.** The result of Lemma 1.2 is optimal in the following sense. Examples in [8, Section 6] show both that no improvement can be expected without the divergence condition (1.3), and that in general  $g^m \mathcal{A}(g^m)^{\alpha} \notin L^1_{\text{loc}}$ , for sufficiently large  $\alpha$ . In particular,  $\mathcal{A}$  given by formula (1.9) cannot be substituted by any function whose logarithm grows faster than  $\log \mathcal{A}$  at  $\infty$ . More details are given in Section 4.

**REMARK 1.4.** We stress that the exponent  $\varepsilon$  depends on  $P$  only through the constant  $C_1$  of the  $\Delta_2$ -condition (1.8). On the other hand, formula (1.9) with  $t \mapsto P(\beta t)$  instead of  $P$  yields  $\mathcal{A}^{\beta}$ . So in Lemma 1.2 it suffices to consider the case  $\beta = 1$ . We can also vary the parameter  $t_0$ ; this will affect only the constant  $C$  in estimate (1.12), and means that only sufficiently large values of  $f(x)$  are relevant.

**REMARK 1.5.** Often, we do not compute explicitly  $P^{-1}$ , but find an equivalent function. It will be clear from our proof of Lemma 1.2 that we can replace  $P^{-1}$  by  $Q$  in the definition (1.9) of  $\mathcal{A}$ , for any  $Q$  verifying

$$P^{-1}(\sigma) \leq \alpha Q(\sigma), \quad \forall \sigma \geq P(t_0),$$

for some constant  $\alpha > 0$ . More precisely, defining (for  $s \geq \exp(P(t_0))$ )

$$\mathcal{A}(s) = \exp \left[ \int_{P(t_0)}^{\log s} \frac{d\sigma}{Q(\sigma)} \right],$$

then under the assumption  $\exp(P(f^m)) \in L^1$ , we have

$$h^m \mathcal{A}(h^m)^{\varepsilon/\alpha} \in L^1 \quad \Rightarrow \quad g^m \mathcal{A}(g^m)^{\varepsilon/\alpha} \in L^1.$$

To illustrate better Lemma 1.2, we shall present several examples obtained particularizing the function  $P$ .

EXAMPLE 1.6. The result of Faraco-Koskela-Zhong [4] and that of Bildhauer-Fuchs-Zhong [2] are readily obtained in the case

$$(1.13) \quad P(t) = t.$$

EXAMPLE 1.7. Here we examine the function defined in (1.6), with  $0 < \vartheta < 1$ . We find (see Remark 1.5)

$$(1.14) \quad \mathcal{A}(s) = \exp[(\log \log s)^{1-\vartheta}].$$

More examples are contained in the following table, where we specify the asymptotic behavior of  $P(t)$  as  $t \rightarrow \infty$ , and that of  $\mathcal{A}(t)$  in the sense of Remark 1.5.

$P$	$\mathcal{A}$
$e^t$	$\exp\left[\frac{\log t}{\log \log t}\right]$
$t^\gamma, \gamma > 1$	$\exp[\log^{1-1/\gamma} t]$
$t$	$\log t$
$t(\log t)^{-\vartheta}, 0 < \vartheta < 1$	$\exp[(\log \log t)^{1-\vartheta}]$
$t(\log t)^{-1}$	$\log \log t$
$t(\log t)^{-1}(\log \log t)^{-1} \dots \overbrace{(\log \dots \log t)^{-\vartheta}}^k,$ $0 < \vartheta < 1$	$\exp[\overbrace{(\log \log \dots \log t)^{1-\vartheta}}^{k+1}]$
$t(\log t)^{-1}(\log \log t)^{-1} \dots \overbrace{(\log \dots \log t)^{-1}}^k$	$\overbrace{\log \log \dots \log t}^{k+1}$

We conclude this introduction mentioning that in Section 4 we shall give some applications to the study of mappings of finite distortion.

## 2. REVERSE HÖLDER'S INEQUALITIES IN $\mathbb{R}^n$

In this section, we prove a higher integrability result for functions defined in  $\mathbb{R}^n$ . From this, in Section 3 we will deduce Lemma 1.2. To shorten notation, we introduce the function

$$\Phi : [0, \infty[ \rightarrow [0, \infty[$$

by setting

$$(2.1) \quad \Phi(t) = \begin{cases} 0, & \text{for } 0 \leq t < t_0 \\ \exp(P(t)), & \text{for } t \geq t_0 \end{cases}$$

LEMMA 2.1. *Let  $f, g$  and  $h$  be nonnegative functions on  $\mathbb{R}^n$  such that  $g, h \in L^m(\mathbb{R}^n)$ ,  $\Phi(\beta f^m) \in L^1(\mathbb{R}^n)$  for a constant  $\beta > 0$ , and (1.1) holds, for all balls  $B$  of  $\mathbb{R}^n$ . Under this assumptions, we have  $g^m \mathcal{A}(g^m)^{\varepsilon\beta} \in L^1(\mathbb{R}^n)$  for some  $\varepsilon = \varepsilon(n, m, C_1) > 0$ , if the same is true for  $h$ . Moreover, we have the uniform estimate*

$$(2.2) \quad \int_{\mathbb{R}^n} g^m \mathcal{A}(g^m)^{\varepsilon\beta} \leq C \int_{\mathbb{R}^n} [\Phi(\beta f^m) + h^m \mathcal{A}(h^m)^{\varepsilon\beta}].$$

2.1. *Preliminary results.* We note explicitly some consequences of the  $\Delta_2$ -condition (1.8). First, we have

$$(2.3) \quad \Phi(t)^2 \leq \Phi(C_1 t), \quad \forall t \geq 0.$$

Similarly, for every  $\vartheta \in ]0, 1[$ , there exists a constant  $C_2 > 0$  such that

$$(2.4) \quad P^{-1}(\log s^m) \leq C_2 P^{-1}(\vartheta \log s^m), \quad \forall s \geq \exp(P(t_0)/(m\vartheta)).$$

We shall take

$$(2.5) \quad \vartheta = \frac{m-1}{2m}$$

so that  $C_2 = C_2(m, C_1)$ .

Moreover, (1.8) implies

$$\lim_{t \rightarrow \infty} \frac{P(t)}{\log t} = \infty.$$

Hence, without loss of generality, we may assume that

$$(2.6) \quad t \mapsto \frac{\Phi(t)}{t} \quad \text{is increasing and diverging at } \infty.$$

Let now

$$(2.7) \quad \Phi^*(s) = \sup_{t \geq 0} \{st - \Phi(t)\}, \quad s \geq 0,$$

be the conjugate function to  $\Phi$ . Condition (2.6) implies that  $\Phi^*$  is invertible. We denote by

$$\Psi = (\Phi^*)^{-1} : [0, \infty[ \rightarrow [0, \infty[$$

its inverse function, which is concave, strictly increasing and verifies  $\Psi(0) = 0$ . Since  $\Phi(t) = 0$  for  $0 \leq t < t_0$ , we easily find

$$(2.8) \quad \Psi(s) = \frac{s}{t_0}, \quad \forall s \in [0, \Phi(t_0)].$$

We need to recall more properties of  $\Psi$ , see [8, 9]. First, we have

$$(2.9) \quad \frac{s}{P^{-1}(\log s)} \leq \Psi(s) \leq 2 \frac{s}{P^{-1}(\log s)}, \quad \forall s \geq \Phi(t_0).$$

Obviously, (2.9) implies

$$(2.10) \quad \lim_{s \rightarrow \infty} \frac{\Psi(s)}{s} = 0,$$

and divergence condition (1.3) on  $P$  yields a similar condition for  $\Psi$ :

$$(2.11) \quad \int_1^\infty \frac{\Psi(s)}{s^2} ds = \infty.$$

Moreover, in view of  $\Delta_2$ -condition (1.8), (2.9) implies that for every  $\vartheta \in ]0, 1[$ , there exists a constant  $C = C(\vartheta, \Psi) > 0$  such that

$$(2.12) \quad \frac{\Psi(s^\vartheta)}{s^\vartheta} \leq C \frac{\Psi(s)}{s}, \quad \forall s > 0$$

and

$$(2.13) \quad s^\vartheta \leq C\Psi(s), \quad \forall s \geq 1.$$

We shall consider powers of the function  $\mathcal{A}$  defined by (1.9). To this aim, it will be easier to work with a modified version of the function. Given  $\varepsilon > 0$ , we define

$$(2.14) \quad \mathcal{A}_\varepsilon(s) = \begin{cases} 1, & \text{for } 0 \leq s \leq S \\ \exp \left[ \varepsilon \int_{\log S}^{\log s} \frac{d\sigma}{P^{-1}(\sigma)} \right], & \text{for } s \geq S \end{cases}$$

where  $S \geq \Phi(t_0)$ . Clearly, for every  $s \geq 0$ ,

$$\mathcal{A}_\varepsilon(s) \leq \mathcal{A}(s)^\varepsilon \leq \mathcal{A}_\varepsilon(s) \exp \left[ \varepsilon \int_{P(t_0)}^{\log S} \frac{d\sigma}{P^{-1}(\sigma)} \right].$$

The parameter  $S$  will be taken large enough depending on  $\varepsilon$ , as described in the following

**LEMMA 2.2.** *For each  $\varepsilon > 0$  and  $k > 0$ , there exists  $S_0 \geq \Phi(t_0)^{2m/(m-1)}$  such that, if  $S \geq S_0$ , then*

$$(2.15) \quad s \mapsto \frac{\mathcal{A}_\varepsilon(s)}{s} \quad \text{is decreasing on } ]0, \infty[$$

and

$$(2.16) \quad \mathcal{A}_\varepsilon(\Phi^*(ks)) \leq 2\mathcal{A}_\varepsilon(s), \quad \forall s \geq 0.$$

PROOF. To show (2.15), for  $s \geq S$  we write

$$(2.17) \quad \frac{\mathcal{A}_\varepsilon(s)}{s} = \frac{1}{S} \exp \left[ \int_{\log S}^{\log s} \left( \frac{\varepsilon}{P^{-1}(\sigma)} - 1 \right) d\sigma \right],$$

hence it suffices to take  $S_0$  such that  $P^{-1}(\sigma) > \varepsilon$ ,  $\forall \sigma \geq \log S_0$ .

Inequality (2.16) will be deduced from

$$(2.18) \quad \lim_{s \rightarrow \infty} \frac{\mathcal{A}(\Phi^*(ks))}{\mathcal{A}(s)} = 1.$$

To prove (2.18) we first note that (2.10) implies

$$\lim_{s \rightarrow \infty} \frac{\Phi^*(ks)}{s} = \infty.$$

Hence, for  $s$  large enough we have  $\Phi^*(ks) > s$ , and by (1.9), (2.9)

$$\frac{\mathcal{A}(\Phi^*(ks))}{\mathcal{A}(s)} = \exp \left[ \int_s^{\Phi^*(ks)} \frac{d\sigma}{\sigma P^{-1}(\log \sigma)} \right] \leq \exp \left[ \int_s^{\Phi^*(ks)} \frac{\Psi(\sigma)}{\sigma^2} d\sigma \right]$$

therefore it suffices to show that the exponent in the last term converges to 0. To this end, integrating by parts and using (2.10) we can replace the integrand  $\Psi(\sigma)/\sigma^2$  by  $\Psi'(\sigma)/\sigma$ , and then by the change of variable  $\tau = \Psi(\sigma)$ , we arrive at the integral

$$(2.19) \quad \int_{\Psi(s)}^{ks} \frac{d\tau}{\Phi^*(\tau)} = \int_{\Psi(s)}^{ks} \frac{\tau}{\Phi^*(\tau)} \frac{d\tau}{\tau}.$$

Since  $\tau \mapsto \tau/\Phi^*(\tau)$  is decreasing, we see that the integral in (2.19) is controlled by

$$(2.20) \quad \frac{\Psi(s)}{s} \log \frac{ks}{\Psi(s)} = \frac{\Psi(s)}{s} \log k - \frac{\Psi(s)}{s} \log \frac{\Psi(s)}{s}$$

which clearly converges to 0 as desired.

Now, by (2.18), we find  $S_0 \geq \Phi(t_0)$  such that

$$(2.21) \quad s \geq S_0 \Rightarrow \left[ \frac{\mathcal{A}(\Phi^*(ks))}{\mathcal{A}(s)} \right]^\varepsilon < 2.$$



Let us check that (2.16) holds for  $S \geq S_0$ . Inequality (2.16) is trivial if  $\Phi^*(ks) \leq S$ , and reduces directly to (2.21) if  $s \geq S$ . Thus, we assume  $\Phi^*(ks) > S > s$ . In this case, we have

$$\frac{\mathcal{A}_\varepsilon(\Phi^*(ks))}{\mathcal{A}_\varepsilon(s)} = \left[ \frac{\mathcal{A}(\Phi^*(ks))}{\mathcal{A}(S)} \right]^\varepsilon \leq \left[ \frac{\mathcal{A}(\Phi^*(ks))}{\mathcal{A}(S)} \right]^\varepsilon$$

and we conclude using (2.21) again. □

*2.2. Proof of Lemma 2.1.* We may assume  $f \geq 1$ , and consider only the case  $\beta = 1$ . Our starting point is the following estimate for integrals on level sets of the functions involved, which is a consequence of reverse Hölder's inequalities:

$$(2.22) \quad \int_{g>t} g^m dx \leq C_3 t^{m-1} \int_{fg>t} fg dx + C_3 \int_{h>t} h^m dx,$$

with  $C_3$  depending only on  $n$  and  $m$ . This can be proved easily arguing as in Section 3 of [11], using Calderón-Zygmund decomposition and Vitali's Covering Lemma.

Now we multiply both sides of (2.22) by  $(\mathcal{A}_\varepsilon(t^m))'$  and integrate w.r.t.  $t$  over  $(0, \infty)$ . Using Fubini theorem, we have

$$(2.23) \quad \int_0^\infty (\mathcal{A}_\varepsilon(t^m))' dt \int_{g>t} g^m dx = \int_{\mathbb{R}^n} g^m [\mathcal{A}_\varepsilon(g^m) - 1] dx$$

$$(2.24) \quad \int_0^\infty (\mathcal{A}_\varepsilon(t^m))' dt \int_{h>t} h^m dx = \int_{\mathbb{R}^n} h^m [\mathcal{A}_\varepsilon(h^m) - 1] dx.$$

On the other hand,

$$(2.25) \quad \int_0^\infty t^{m-1} (\mathcal{A}_\varepsilon(t^m))' dt \int_{fg>t} fg dx = \int_{\mathbb{R}^n} (fg) \mathcal{B}_\varepsilon(fg) dx,$$

where

$$(2.26) \quad \mathcal{B}_\varepsilon(s) = \int_0^s t^{m-1} (\mathcal{A}_\varepsilon(t^m))' dt.$$

Now we need the following Young-type inequality.

**LEMMA 2.3.** *We have*

$$(2.27) \quad fg \mathcal{B}_\varepsilon(fg) \leq \varepsilon [C_4 \Phi(f^m) + C_5 g^m \mathcal{A}_\varepsilon(g^m)],$$

for some positive constants  $C_4 = C_4(m, \Psi)$  and  $C_5 = C_5(m, C_1)$ .

PROOF. Clearly  $\mathcal{B}_\varepsilon(s) = 0$  for  $s \leq S^{1/m}$ . For  $s > S^{1/m}$ , we easily find

$$(2.28) \quad \mathcal{B}_\varepsilon(s) = \varepsilon \int_{S^{1/m}}^s \frac{m \mathcal{A}_\varepsilon(t^m) t^{m-2}}{P^{-1}(\log t^m)} dt \leq \varepsilon \mathcal{A}_\varepsilon(s^m) \int_{S^{1/m}}^s \frac{m t^{m-2}}{P^{-1}(\log t^m)} dt.$$

Moreover, as  $S \geq 1$ ,

$$(2.29) \quad \int_{S^{1/m}}^s \frac{t^{m-2}}{P^{-1}(\log t^m)} dt \leq \int_{S^{1/m}}^s \frac{t^{(m-1)/2} t^{(m-1)/2-1}}{P^{-1}(\log t^{(m-1)/2})} dt.$$

Since  $\sigma \mapsto \frac{\sigma}{P^{-1}(\log \sigma)}$  is increasing, we deduce

$$(2.30) \quad \int_{S^{1/m}}^s \frac{t^{m-2}}{P^{-1}(\log t^m)} dt \leq \frac{2}{m-1} \frac{s^{m-1}}{P^{-1}(\frac{m-1}{2m} \log s^m)}.$$

Hence, using (2.4) with (2.5), and (2.9), we find

$$(2.31) \quad s \mathcal{B}_\varepsilon(s) \leq \varepsilon \frac{2mC_2}{m-1} \frac{s^m \mathcal{A}_\varepsilon(s^m)}{P^{-1}(\log s^m)} \leq \varepsilon \frac{2mC_2}{m-1} \Psi(s^m) \mathcal{A}_\varepsilon(s^m).$$

We define for  $t \geq 0$  the function

$$(2.32) \quad F(t) = \Psi(t) \mathcal{A}_\varepsilon(t).$$

By the properties of  $\Psi$  and  $\mathcal{A}_\varepsilon$ , the function  $t \mapsto F(t)/t^2$  is decreasing, so we have the inequality

$$(2.33) \quad F(a+b) \leq 2[F(a) + F(b)],$$

for every  $a \geq 0$  and  $b \geq 0$ . Moreover, for  $t \geq 1$ ,

$$(2.34) \quad F(t) \leq F(1)t^2 = \Psi(1)t^2.$$

By Young inequality with the couple  $\Phi$  and  $\Phi^*$ , we get

$$(2.35) \quad f^m g^m \leq \Phi(f^m/C_1) + \Phi^*(C_1 g^m).$$

Applying  $F$  to both sides yields

$$(2.36) \quad \Psi((fg)^m) \mathcal{A}_\varepsilon((fg)^m) = F((fg)^m) \leq F[\Phi(f^m/C_1) + \Phi^*(C_1 g^m)] \\ \leq 2[F(\Phi(f^m/C_1)) + F(\Phi^*(C_1 g^m))].$$

We have also from (2.34) and (2.3)

$$(2.37) \quad F(\Phi(f^m/C_1)) \leq \Psi(1)\Phi(f^m/C_1)^2 \leq \Psi(1)\Phi(f^m).$$

We estimate  $F(\Phi^*(C_1 g^m))$  by (2.16). Accordingly,

$$(2.38) \quad F(\Phi^*(C_1 g^m)) = C_1 g^m \mathcal{A}_\varepsilon(\Phi^*(C_1 g^m)) \leq 2C_1 g^m \mathcal{A}_\varepsilon(g^m).$$

By (2.31), (2.36), (2.37) and (2.38), we see that (2.27) holds with

$$C_4 = 4C_2\Psi(1)\frac{m}{m-1} \quad \text{and} \quad C_5 = 8C_2C_1\frac{m}{m-1}. \quad \square$$

By (2.22), (2.23), (2.24), (2.25) and (2.27), we obtain

$$(2.39) \quad \int_{\mathbb{R}^n} g^m [\mathcal{A}_\varepsilon(g^m) - 1] \leq C_3 C_5 \varepsilon \int_{\mathbb{R}^n} g^m \mathcal{A}_\varepsilon(g^m) + C_3 C_4 \varepsilon \int_{\mathbb{R}^n} \Phi(f^m) \\ + C_3 \int_{\mathbb{R}^n} h^m [\mathcal{A}_\varepsilon(h^m) - 1].$$

Choosing  $\varepsilon$  so that  $C_3 C_5 \varepsilon < 1$ , the first integral in the right hand side of (2.39) can be absorbed in the left hand side and we get

$$(2.40) \quad \int_{\mathbb{R}^n} g^m \mathcal{A}_\varepsilon(g^m) \leq C \int_{\mathbb{R}^n} g^m + C \int_{\mathbb{R}^n} \Phi(f^m) \\ + C \int_{\mathbb{R}^n} h^m [\mathcal{A}_\varepsilon(h^m) - 1],$$

provided we already know that  $g^m \mathcal{A}_\varepsilon(g^m) \in L^1(\mathbb{R}^n)$ . To get rid of this condition, we apply the above argument with a truncation of  $\mathcal{A}_\varepsilon$ :

$$\mathcal{A}_{\varepsilon, T}(t) = \min\{\mathcal{A}_\varepsilon(t), \mathcal{A}_\varepsilon(T)\}$$

and then let  $T \rightarrow \infty$ . In fact, we immediately find properties (2.15) and (2.16) with  $\mathcal{A}_{\varepsilon, T}$  in place of  $\mathcal{A}_\varepsilon$ . Also, defining  $\mathcal{B}_{\varepsilon, T}$  using  $\mathcal{A}_{\varepsilon, T}$  in (2.26), we have

$$fg \mathcal{B}_{\varepsilon, T}(fg) \leq \varepsilon [C_4 \Phi(f^m) + C_5 g^m \mathcal{A}_{\varepsilon, T}(g^m)]$$

similarly to (2.27), with the same constants.

To conclude, we note that  $\Psi((fg)^m) \in L^1$ . Indeed, we find

$$\Psi((fg)^m) \leq \Phi(f^m) + g^m.$$

Moreover,  $\sigma \leq C\Psi(\sigma^m)$ ,  $\forall \sigma \geq 1$ , with a suitable constant  $C > 0$ , so that  $l = fg$  is integrable over the set  $\{x : l(x) \geq 1\}$ . On the other hand, we have  $\Psi(\sigma) = \sigma/t_0$ ,  $\forall \sigma \in [0, \Phi(t_0)]$ , so that  $l^m$  is integrable over the set  $\{x : l(x) \leq 1\}$ . Hence

$$\lim_{|B| \rightarrow \infty} |B|^{1/m-1} \int_{2B} l = 0,$$

and (1.1) implies

$$\int_{\mathbb{R}^n} g^m dx \leq 2^{-n} \int_{\mathbb{R}^n} h^m dx.$$

Inserting this into (2.40) yields (2.2), concluding the proof.

## 3. LOCAL INEQUALITIES

Here we prove Lemma 1.2. We shall modify the proof of Lemma 2.1 following [11, Section 6] and using some ideas of [4, 2]. First, we notice that we may assume

$$(3.1) \quad \int_{\Omega} g^m = 1.$$

Then, we introduce the functions defined in  $\mathbb{R}^n$

$$(3.2) \quad G = \rho g, \quad H = \rho h, \quad V = \chi_{\Omega},$$

where  $\rho(x) = \text{dist}^{n/m}(x, \mathbb{R}^n \setminus \Omega)$ . Arguing as in [11], see also [4, 2], it can be shown that the following reverse Hölder's inequalities

$$(3.3) \quad \left( \int_B G^m \right)^{1/m} \leq C \left\{ \int_{2B} fG + \left( \int_{2B} H^m \right)^{1/m} + \int_{2B} V \right\}$$

hold for all balls  $B$  of  $\mathbb{R}^n$ , with a constant  $C = C(n, m) > 0$ . As for (2.22), we can readily deduce from (3.3) the following

$$(3.4) \quad \int_{g>t} G^m \leq C \left\{ t^{m-1} \int_{fG>t} fG + \int_{H>t} H^m + t^{m-1} \int_{V>t} V \right\}.$$

Now we can repeat the argument of the proof of Lemma 2.1, starting with (3.4) instead of (2.22), using that  $V \leq 1$  and  $(\mathcal{A}_\varepsilon(t^m))' = 0$  for  $t \leq 1$ . In this way, we end up with the following inequality

$$(3.5) \quad \int_{\Omega} G^{m, \mathcal{A}_\varepsilon}(G^m) \leq C \left\{ \int_{\Omega} \Phi(f^m) + \int_{\Omega} H^{m, \mathcal{A}_\varepsilon}(H^m) \right\}$$

replacing (2.40).

To conclude, we recall the normalization (3.1) and use that

$$\inf\{\rho(x)^m : x \in \sigma\Omega\} = \frac{(1-\sigma)^n}{\omega_n} |\Omega|, \quad \sup\{\rho(x)^m : x \in \Omega\} = \frac{|\Omega|}{\omega_n},$$

and the  $\Delta_2$ -property of  $\mathcal{A}_\varepsilon$ .

## 4. SOME APPLICATIONS

Here we study regularity of mappings of finite distortion. We shall recall only a few fundamental concepts, referring the reader to [12] for a comprehensive treatment. Given a domain  $\Omega$  of  $\mathbb{R}^n$ ,  $n \geq 2$ , we consider mappings  $F$  in the Sobolev class  $W_{\text{loc}}^{1, n-1}(\Omega; \mathbb{R}^n)$  whose Jacobian  $J_F = \det DF$  is nonnegative and locally integrable in  $\Omega$ . A mapping  $F$  of this type is said to have finite inner

distortion if the cofactor matrix  $D^\#F(x)$  of  $DF(x)$  vanishes a.e. in the zero set of the Jacobian:

$$J_F(x) = 0 \Rightarrow D^\#F(x) = 0 \quad \text{a.e. in } \Omega.$$

Under this condition, we can define the inner distortion by

$$(4.1) \quad K_I(x) = K_I(x, F) = \begin{cases} \frac{|D^\#F(x)|^n}{J_F(x)^{n-1}}, & \text{if } J_F(x) > 0, \\ 1, & \text{otherwise.} \end{cases}$$

Hereafter, we use the operator norm for matrices. The following distortion inequality obviously holds

$$(4.2) \quad |D^\#F(x)|^n \leq K_I(x)J_F(x)^{n-1}.$$

Similarly,  $F$  is said to have finite outer distortion if  $DF(x)$  vanishes a.e. in the zero set of the Jacobian:

$$J_F(x) = 0 \Rightarrow DF(x) = 0 \quad \text{a.e. in } \Omega.$$

We can define the outer distortion by

$$(4.3) \quad K_O(x) = K_O(x, F) = \begin{cases} \frac{|DF(x)|^n}{J_F(x)}, & \text{if } J_F(x) > 0, \\ 1, & \text{otherwise} \end{cases}$$

and have the distortion inequality

$$(4.4) \quad |DF(x)|^n \leq K_O(x)J_F(x).$$

In dimension  $n = 2$  the two notions coincide. In any dimension, a mapping of finite outer distortion has clearly also finite inner distortion. On the contrary, if  $n > 2$ , there exist mappings of finite inner distortion which do not have finite outer distortion. In view of Hadamard's inequality

$$J_F \leq |D^\#F|^{n/(n-1)} \leq |DF|^n$$

we have  $K_I \geq 1$ ,  $K_O \geq 1$ . Moreover, if  $F$  has finite outer distortion,

$$(4.5) \quad K_I^{1/(n-1)} \leq K_O \leq K_I^{n-1}.$$

In [4, 7, 8, 3] the integrability properties of  $DF$  and  $J_F$  are studied for a mapping  $F$  of finite outer distortion, under suitable integrability assumptions on  $K_O$ . In particular, [8, Theorem 4.1] ensures local integrability of  $J_F \mathcal{A}(J_F)^{\epsilon\beta}$  under the condition

$$(4.6) \quad \Phi(\beta K_O) \in L^1_{\text{loc}}(\Omega),$$

where  $\beta > 0$ , the notation and the assumptions of Introduction and Section 2 being in force. Here we give a higher integrability result for the Jacobian  $J_F$  of a mapping of finite inner distortion  $F$ , under the hypothesis

$$(4.7) \quad \Phi(\beta K_I^{1/(n-1)}) \in L^1_{\text{loc}}(\Omega),$$

which is weaker than (4.6) in view of first inequality in (4.5).

**THEOREM 4.1.** *If condition (4.7) holds, then there exists  $\varepsilon > 0$  such that  $J_F \mathcal{A}(J_F)^{\beta\varepsilon} \in L^1_{\text{loc}}(\Omega)$ .*

**REMARK 4.2.** Even assuming  $F$  of finite outer distortion, trying to deduce Theorem 4.1 by the mentioned result of [8], if  $n > 2$  we find the following difficulty. By second inequality in (4.5), from assumption (4.7) we can only deduce

$$\Phi(\beta K_O^{1/(n-1)^2}) \in L^1_{\text{loc}}(\Omega)$$

and hence we cannot apply [8, Theorem 4.1] since in general  $t \mapsto \log \Phi(t^{1/(n-1)^2})$  does not satisfy the divergence condition.

Condition (4.7) and the local integrability of the Jacobian imply integrability properties of  $|D^\# F|$ . Indeed, recalling that  $\Psi = (\Phi^*)^{-1}$  is concave and increasing, from distortion inequality (4.2) we deduce

$$(4.8) \quad \begin{aligned} \Psi(|D^\# F|^{n/(n-1)}) &\leq \Psi(K_I^{1/(n-1)} J_F) \leq K_I^{1/(n-1)} \Psi(J_F) \\ &\leq \frac{1}{\beta} [\Phi(\beta K_I^{1/(n-1)}) + J_F] \end{aligned}$$

and hence

$$(4.9) \quad \Psi(|D^\# F|^{n/(n-1)}) \in L^1_{\text{loc}}(\Omega).$$

Note that by (2.13) this means that  $|D^\# F| \in L^p_{\text{loc}}(\Omega)$ , for all  $p < \frac{n}{n-1}$ .

We shall use isoperimetric inequality. For the mappings that we consider here, a version of the inequality can be deduced by elementary arguments from the case of smooth mappings.

**LEMMA 4.3.** *For a given ball  $B = B(x_0, R)$  let the mapping  $F \in W^{1, n-1}(B; \mathbb{R}^n) \cap L^\infty(B; \mathbb{R}^n)$  verify  $J_F \in L^1(B)$  and obey to the rule of integration by parts, that is,*

$$(4.10) \quad \int_B \varphi(x) J_F(x) dx = - \int_B F^1 d\varphi \wedge dF^2 \wedge \cdots \wedge dF^n,$$

for all  $\varphi \in C_0^\infty(B)$ . Then

$$(4.11) \quad \int_{B(x_0, r)} J_F(x) dx \leq C(n) \left( \int_{\partial B(x_0, r)} |D^\# F(x)| d\sigma \right)^{n/(n-1)}$$

for a.e.  $r \in (0, R)$ .

PROOF. Integrating by parts, we easily get

$$(4.12) \quad \int_{B(x_0, r)} J_F(x) dx = \int_{\partial B(x_0, r)} F^1 \left\langle \frac{x}{|x|}, \nabla F^2 \times \cdots \times \nabla F^n \right\rangle d\sigma,$$

for a.e.  $r \in (0, R)$ . On the other hand, if  $F_h \in C^\infty(B; \mathbb{R}^n)$ ,  $h \in \mathbb{N}$ , converge to  $F$  in  $W^{1, n-1}(B; \mathbb{R}^n)$  and a.e. in  $B$ , and verify  $\|F_h\|_\infty \leq \|F\|_\infty$ , then

$$\int_{B(x_0, r)} J_{F_h}(x) dx = \int_{\partial B(x_0, r)} F_h^1 \left\langle \frac{x}{|x|}, \nabla F_h^2 \times \cdots \times \nabla F_h^n \right\rangle d\sigma$$

and

$$\int_{\partial B(x_0, r)} |D^\# F_h| d\sigma$$

converge to the corresponding expressions for  $F$  in  $L^1(0, R)$  and hence (for a subsequence) for a.e.  $r \in (0, R)$ . Therefore, (4.11) follows writing the isoperimetric inequality for  $F_h$  and passing to the limit as  $h \rightarrow \infty$ .  $\square$

**COROLLARY 4.4.** *If  $F \in W_{\text{loc}}^{1, n-1}(\Omega; \mathbb{R}^n)$  verifies  $\Psi(|D^\# F|^{n/(n-1)}) \in L^1_{\text{loc}}(\Omega)$  and  $J \geq 0$  a.e., then (4.11) holds, for all  $x_0 \in \Omega$ .*

PROOF. We truncate the components of  $F$ , setting for  $i = 1, \dots, n$  and  $k > 0$ ,

$$G_k^i = \text{sign}(F^i) \min\{|F^i|, k\},$$

apply Lemma 4.3 to  $G_k = (G_k^1, \dots, G_k^n)$ , and finally let  $k \rightarrow \infty$ . Indeed, we have  $|D^\# G_k| \leq |D^\# F|$ ,  $\det DG_k(x) = \det DF(x)$  if for every component  $|F^i(x)| \leq k$  and  $\det DG_k(x) = 0$  otherwise. To show that each  $G_k$  obeys to the rule of integration by parts we use [6, Theorem 1.3 and Corollary 1.4]. We only remark that we do not need to assume the technical condition

$$(4.13) \quad [t^{-1}\Psi(t)]' \leq 0 \leq [t^{-s}\Psi(t)]', \quad s = \frac{n^2 - 2n + 1}{n^2 - n - 1}.$$

Indeed, first inequality in (4.13) holds because  $\Psi$  is concave. Second inequality in (4.13) is used in [6] to show that

$$(4.14) \quad \Psi(t) \geq Ct^s$$

for large values of  $t$  and for a suitable constant, and to deduce

$$(4.15) \quad \inf_{t \geq 1} t^{1/(n-1)} \int_{|D^\# F| > t} |D^\# F| dx = 0$$

from (4.9). Actually, (4.14) is a particular case of (2.13). On the other hand, (4.15) follows by [9, Lemma 2.10].  $\square$

**PROOF OF THEOREM 4.1.** We shall prove that the Jacobian satisfies a reverse Hölder's inequality and then apply Lemma 1.2.

There is no loss of generality in assuming that  $\Phi(\beta K_I^{1/(n-1)})$  and  $J_F$  are integrable in  $\Omega$ , hence  $\Psi(|D^\# F|^{n/(n-1)}) \in L^1(\Omega)$ .

Now from (4.11), integrating with respect to  $r$ , we get

$$(4.16) \quad \int_B J_F dx \leq C(n) \left( \int_{2B} |D^\# F| dx \right)^{n/(n-1)},$$

for all balls  $B \subset 2B \subset \Omega$ . Hence recalling (4.2) we have the reverse Hölder's inequality

$$(4.17) \quad \left( \int_B J_F dx \right)^{1/m} \leq \int_{2B} (C(n) K_I^{1/(n-1)} J_F)^{1/m} dx,$$

where  $m = \frac{n}{n-1} > 1$ . Therefore, we are in a position to apply Lemma 1.2 with

$$(4.18) \quad g = J_F^{1/m}, \quad f = (C(n) K_I^{1/(n-1)})^{1/m}, \quad h = 0,$$

and  $C(n)^{-1}\beta$  in place of  $\beta$ , concluding the proof.  $\square$

**REMARK 4.5.** We add a few comments to what we said in Remark 1.3, where we noted that optimality of Lemma 1.2 follows from examples in [8, Section 6]. Actually, these examples are given in terms of mappings of finite distortion; however, it is well known that such a mapping satisfies the estimate

$$(4.19) \quad \int_B J_F dx \leq C(n) \left( \int_{2B} |DF|^{n^2/(n+1)} dx \right)^{(n+1)/n},$$

see e.g. [10], which together with distortion inequality (4.4) obviously give reverse inequalities of the type (1.1). Indeed, (4.19) follows clearly also from (4.16).

## 5. A COUNTEREXAMPLE

In this section we show that the Conjecture 1.1 has in general a negative answer. Recall our notation:  $\Phi(t) = \exp(P(t))$ ,  $t \geq 1$ ,

$$E(t) = 1 + \int_1^t \frac{P(\tau)}{\tau^2} d\tau,$$

and  $R(t) = P(t)/E(t)$ .



Actually, it is easy to see that the conjecture fails when  $P$  is very large. Indeed, if  $P(t) = e^t$ , then  $R(t) \sim t^2$ , hence we should have  $|DF| \in L^4_{\text{loc}}$ , while in general we only have

$$|DF|^2 \exp \left[ \varepsilon \frac{\log(e + |DF|)}{\log \log(e^e + |DF|)} \right] \in L^1_{\text{loc}}$$

for some  $\varepsilon > 0$ , see [8].

The case  $P(t) = e^t$  is the first example considered in the table in the Introduction. We remark that in all the other cases considered there, the conjecture has a positive answer. Surprisingly, the conjecture fails also in the subexponential integrability case.

**PROPOSITION 5.1.** *There exist a function  $P : [0, \infty[ \rightarrow [0, \infty[$  verifying divergence condition (1.3) and a homeomorphism  $F$  of the closed ball  $\bar{B} = \bar{B}(0; 1/e)$  onto itself, having finite distortion  $K = K_F$  satisfying  $\Phi(K) \in L^1(B)$ , but*

$$(5.1) \quad R(|DF|^n) \notin L^1_{\text{loc}}(B), \quad J \log E(J) \notin L^1_{\text{loc}}(B).$$

The function  $P$  can be found of class  $C^1$ , concave, and verifying

$$(5.2) \quad \lim_{t \rightarrow \infty} \frac{P(t)}{tP'(t)} = 1.$$

In particular, for each  $\vartheta \in ]0, 1[$ , the function  $t \mapsto P(t)t^{-\vartheta}$  is increasing in a neighborhood of  $\infty$ .

**PROOF.** We note that in (5.1), the second condition implies the first one, by the theory of higher integrability of the Jacobian. This follows e.g. by [10, Theorem 1] and maximal inequalities.

To construct  $P$ , we shall use the function

$$\varphi(t) = \frac{t}{\log^2(e + t)}, \quad t \geq 0,$$

which is concave, strictly increasing, and verifies

$$(5.3) \quad \int_1^\infty \frac{\varphi(\tau)}{\tau^2} d\tau < \infty,$$

$$(5.4) \quad \lim_{t \rightarrow \infty} \varphi'(t)\varphi(t) = \infty, \quad \lim_{t \rightarrow \infty} \varphi'(t) \log t = 0,$$

$$(5.5) \quad \lim_{t \rightarrow \infty} \frac{\varphi(t)}{t\varphi'(t)} = 1.$$

We start setting  $P(t) = t$ , for all  $t \in [0, 1]$ . Then, we shall define a sequence of intervals  $[a_k, b_k]$ ,  $k = 0, 1, \dots$ , with  $[a_0, b_0] = [0, 1]$ ,  $b_{k-1} < a_k$ ,  $\forall k \in \mathbb{N}$ , and  $a_k \rightarrow \infty$ . Moreover, we shall define  $P$  recursively, setting

$$(5.6) \quad P(t) = P(b_{k-1}) + \frac{P'(b_{k-1})}{\varphi'(b_{k-1})} [\varphi(t) - \varphi(b_{k-1})], \quad t \in [b_{k-1}, a_k]$$

(where  $P(b_{k-1}) = P(b_{k-1}-)$  and  $P'(b_{k-1}) = P'(b_{k-1}-)$  depend on the definition of  $P$  over  $[a_{k-1}, b_{k-1}]$ ), and

$$(5.7) \quad P(t) = P(a_k) + P'(a_k)(t - a_k), \quad t \in [a_k, b_k]$$

(where  $P(a_k) = P(a_k-)$  and  $P'(a_k) = P'(a_k-)$  depend on the definition of  $P$  over  $[b_{k-1}, a_k]$ ). Hence,

$$P : [0, \infty[ \rightarrow [0, \infty[$$

is strictly increasing, concave of class  $C^1$ , for any choice of the intervals  $[a_k, b_k]$ . We shall set  $b_k = \Phi(a_k) = \exp(P(a_k))$ , for each  $k \in \mathbb{N}$ , and choose the points  $a_k$  so that

$$(5.8) \quad \limsup_{t \rightarrow \infty} \frac{\log \log E(\Phi(t))}{E(t)} = \infty.$$

Note that this implies divergence condition (1.3), as  $E$  cannot be bounded.

Assuming that we already defined  $a_{k-1}$  and  $b_{k-1} = \Phi(a_{k-1})$ , now we choose  $a_k$ . In view of (5.3), we easily see that  $E(a_k)$  can be bounded independently of  $a_k$ :

$$\begin{aligned} E(a_k) &= E(b_{k-1}) + \int_{b_{k-1}}^{a_k} \frac{P(\tau)}{\tau^2} d\tau \\ &\leq E(b_{k-1}) + \frac{P(b_{k-1})}{b_{k-1}} + \frac{P'(b_{k-1})}{\varphi'(b_{k-1})} \int_{b_{k-1}}^{\infty} \frac{\varphi(\tau)}{\tau^2} d\tau. \end{aligned}$$

On the other hand, as by concavity

$$(5.9) \quad \frac{P(t)}{tP'(t)} \geq 1,$$

we have

$$\begin{aligned} E(\Phi(a_k)) &\geq \int_{a_k}^{b_k} \frac{P(\tau)}{\tau^2} d\tau \geq \frac{P'(b_{k-1})}{\varphi'(b_{k-1})} \varphi'(a_k) [P(a_k) - \log a_k] \\ &\geq \frac{P'(b_{k-1})}{\varphi'(b_{k-1})} \varphi'(a_k) \left\{ \frac{P'(b_{k-1})}{\varphi'(b_{k-1})} [\varphi(a_k) - \varphi(b_{k-1})] - \log a_k \right\} \end{aligned}$$

By (5.4) we see that  $E(\Phi(a_k))$  can be made arbitrarily large. Therefore, we can choose  $a_k$  sufficiently large so that

$$(5.10) \quad \log \log E(\Phi(a_k)) \geq k[E(a_k) + 1].$$

In order to get also (5.2), for  $t \in [b_{k-1}, a_k]$ , we compute

$$(5.11) \quad \frac{P(t)}{tP'(t)} \frac{t\varphi'(t)}{\varphi(t)} - 1 = \frac{\varphi(b_{k-1})}{\varphi(t)} \left[ \frac{P(b_{k-1})}{b_{k-1}P'(b_{k-1})} \frac{b_{k-1}\varphi'(b_{k-1})}{\varphi(b_{k-1})} - 1 \right]$$

In particular, making  $\varphi(b_{k-1})/\varphi(a_k)$  small enough, we can also impose the condition

$$(5.12) \quad \left| \frac{P(a_k)}{a_k P'(a_k)} \frac{a_k \varphi'(a_k)}{\varphi(a_k)} - 1 \right| \leq \frac{1}{k}.$$

By (5.5) this implies

$$(5.13) \quad \lim_k \frac{P(a_k)}{a_k P'(a_k)} = 1.$$

On the other hand, on  $[a_k, b_k]$  the function  $P$  is affine with positive coefficients, thus

$$t \mapsto \frac{P(t)}{tP'(t)}$$

is decreasing. Hence by (5.9) we have also

$$(5.14) \quad \lim_k \frac{P(b_k)}{b_k P'(b_k)} = 1.$$

Finally, by (5.11) we conclude with (5.2).

Once we have defined the function  $P$ , we consider a radial stretching  $F : \bar{B} \rightarrow \bar{B}$ ,

$$(5.15) \quad F(x) = \rho(|x|) \frac{x}{|x|},$$

where

$$(5.16) \quad \rho(r) = \exp \left[ -1 - \int_e^{1/r} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \right], \quad 0 < r \leq 1/e.$$

Note that divergence condition implies

$$\lim_{r \rightarrow 0} \rho(r) = 0.$$

It is clear that  $F$  is a homeomorphism of  $\bar{B}$  onto itself. Moreover,  $\rho \in C^1(0, 1/e)$  and we can easily find (setting  $r = |x|$ ):

$$DF(x) = \frac{\rho(r)}{r} \mathbf{I} + \left[ \rho'(r) - \frac{\rho(r)}{r} \right] \frac{x \otimes x}{r^2}, \quad J(x) = \rho'(r) \left[ \frac{\rho(r)}{r} \right]^{n-1},$$

$$\rho'(r) = \frac{\rho(r)}{r} \frac{1}{\Phi^{-1}(1/r)} \leq \frac{\rho(r)}{r}.$$

This implies

$$|DF| = \frac{\rho(r)}{r},$$

hence  $F \in W^{1,p}(B)$ , for all  $p < n$ . The distortion is

$$K = K_F = \frac{\rho(r)}{r\rho'(r)} = \Phi^{-1}(1/r),$$

thus clearly  $\Phi(K) = 1/r \in L^1(B)$ . On the other hand, writing

$$J = \left[ \frac{\rho(r)}{r} \right]^n \frac{1}{\Phi^{-1}(1/r)}$$

and observing that

$$\lim_{r \rightarrow 0} \frac{\rho(r)^n}{r\Phi^{-1}(1/r)} = \infty,$$

we see that

$$(5.17) \quad J \geq r^{1-n} \geq \frac{1}{r}$$

in a neighborhood of the origin. To show that  $J \log E(J) \notin L^1_{\text{loc}}(B)$ , therefore, it suffices to show that

$$(5.18) \quad \int_0^{r_0} \rho'(r)\rho(r)^{n-1} \log E(1/r) dr = \infty$$

for  $r_0 > 0$  small enough. To this end, for large  $k \in \mathbb{N}$ , we take  $c_k \in ]a_k, b_k[$  so that

$$(5.19) \quad 1 = \int_{a_k}^{c_k} \frac{P(\tau)}{\tau^2} d\tau.$$

Thus  $E(c_k) = E(a_k) + 1$  and by (5.10)

$$(5.20) \quad \log \log E(\Phi(t)) \geq kE(t), \quad \forall t \in [a_k, c_k].$$

Hence

$$(5.21) \quad \log E(1/r) \geq \exp[kE(\Phi^{-1}(1/r))], \quad \frac{1}{\Phi(c_k)} \leq r \leq \frac{1}{\Phi(a_k)}.$$

Now we observe that

$$E(\Phi^{-1}(1/r)) = 1 + \int_1^{\Phi^{-1}(1/r)} \frac{P(\tau)}{\tau^2} d\tau \geq 1 + \int_1^{\Phi^{-1}(1/r)} \frac{P'(\tau)}{\tau} d\tau.$$

Making the change of variable  $\sigma = \Phi(\tau)$  in the last integral, we have

$$E(\Phi^{-1}(1/r)) \geq 1 + \int_e^{1/r} \frac{d\sigma}{\sigma\Phi^{-1}(\sigma)}.$$

Hence, recalling the definition (5.16) of  $\rho$ , by (5.21) we get

$$(5.22) \quad \log E(1/r) \geq \rho(r)^{-k}, \quad \frac{1}{\Phi(c_k)} \leq r \leq \frac{1}{\Phi(a_k)}.$$

Moreover,

$$(5.23) \quad \int_{1/\Phi(c_k)}^{1/\Phi(a_k)} \rho'(r)\rho(r)^{n-1-k} dr = \frac{1}{k-n} \left[ \rho\left(\frac{1}{\Phi(c_k)}\right)^{n-k} - \rho\left(\frac{1}{\Phi(a_k)}\right)^{n-k} \right]$$

We also note that

$$\rho\left(\frac{1}{\Phi(a_k)}\right)^{n-k} \geq e^{k-n}$$

and

$$\begin{aligned} \left[ \frac{\rho(1/\Phi(c_k))}{\rho(1/\Phi(a_k))} \right]^{n-k} &= \exp \left[ (k-n) \int_{\Phi(a_k)}^{\Phi(c_k)} \frac{d\sigma}{\sigma\Phi^{-1}(\sigma)} \right] \\ &= \exp \left[ (k-n) \int_{a_k}^{c_k} \frac{P'(\tau)}{\tau} d\tau \right] \\ &\geq \exp \left[ \frac{k-n}{2} \int_{a_k}^{c_k} \frac{P(\tau)}{\tau^2} d\tau \right] = \exp \left[ \frac{k-n}{2} \right]. \end{aligned}$$

Above, we used the inequality  $2\tau P'(\tau) \geq P(\tau)$  for large  $\tau$ , and (5.19). In conclusion, by (5.23) we have for large  $k$

$$\int_{1/\Phi(c_k)}^{1/\Phi(a_k)} \rho'(r)\rho(r)^{n-1-k} dr \geq \frac{1}{k-n}$$

and by (5.22) we end up with (5.18). □

**REMARK 5.2.** Since our mapping  $F$  in the proof of Proposition 5.1 is a radial stretching, to show that in (5.1) the second condition implies the first one, we do not really need the general theory of higher integrability of the Jacobian. Indeed, recalling that  $\rho(1/e) = 1/e$  and  $E(1) = 1$ , it suffices to integrate by parts to find

$$\int_0^{1/e} \rho' \rho^{n-1} \log E((\rho/r)^n) dr \leq \int_0^{1/e} R((\rho/r)^n) r^{n-1} dr.$$

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