Topology — *Poincaré series for maximal De Concini–Procesi models of root arrangements*, by Giovanni Gaiffi and Matteo Serventi.

Abstract. — In this paper we focus on maximal complex De Concini–Procesi models associated to root arrangements of types A, B, C, D and we compute inductive formulas for their Poincaré series.

1. Introduction

In [4], [5], De Concini and Procesi constructed *wonderful models* for the complement of a subspace arrangement in a vector space. In general, given a subspace arrangement, there are several De Concini–Procesi models associated to it, depending on distinct sets of initial combinatorial data ("building sets", see Section 2.1).

The interest in these varieties was at first motivated by an approach to Drinfeld construction of special solutions for Khniznik–Zamolodchikov equation (see [8]). Then real and complex De Concini–Procesi models turned out to play a central role in several fields of mathematical research: subspace (and toric) arrangements, toric varieties, moduli spaces of curves, configuration spaces, box splines, index theory and discrete geometry (see for instance [6], [7], [9], [10], [11], [13], [17], [18] and [23]).

Among the building sets associated to a given subspace arrangement there are always a minimal one and a maximal one with respect to inclusion: as a consequence there are always a minimal and a maximal De Concini–Procesi model. Several examples of minimal models (associated to the minimal building set of *irreducible subspaces*) have been studied in detail. More recently, the relevance of real and complex maximal models was pointed out (maximal models appear for instance in [3], [16], [21], [1] and in the context of toric varieties, see [14] for further references).

The case of root arrangements is particularly interesting. In this paper we will compute inductive formulas for the Poincaré series of maximal complex models associated to root arrangements of types A, B, C, D.

Let us consider for instance the arrangements of type A_n. Our purpose is to compute the series

$$\phi_A(q,t) = t + \sum_{n \geq 2} P_{\mathscr{C}_{A_{n-1}}}(q) \frac{t^n}{n!} \in \mathbb{Q}[q][[t]]$$

Here, for every $n \geq 2$,

$$P_{\mathscr{C}_{A_{n-1}}}(q) = \sum \mathrm{rk}(H^{2i}(Y_{\mathscr{C}_{A_{n-1}}}, \mathbb{Z}))q^i$$

is the Poincaré polynomial of the maximal complex De Concini–Procesi model $Y_{\mathscr{C}_{A_{n-1}}}$ and the variable q has degree two (in odd degree, the integer cohomology of De Concini–Procesi models is 0—see [5]).

By carefully counting the elements of a basis for the integer cohomology which was first described by Yuzvinski (see [22] and also [12]), we find an inductively defined series in infinite variables $g_0, g_1, g_2, \ldots g_n, \ldots$ with the following property: when we replace g_0 with t and, for every $i, r \geq 1$, g_i^r with $\frac{q^r - q}{q - 1} t^r$, we obtain the series $\phi_A(q,t)$ (Theorem 3.1).

Some explicit computations (using the Computer Algebra system Axiom) show that our method is effective (see Section 3.2). The same technique can be extended to the case B_n (see Theorem 3.3), which, from the point of view of subspaces and models, is equal to C_n, and also to the D_n case (see Section 3.4).

In Section 4 we show that the series in infinite variables computed in the preceding sections encode more general results. For instance, they allow us to obtain the Poincaré series of the families of De Concini–Procesi models whose building sets are the maximal building sets $\mathscr{C}_{A_n}, \mathscr{C}_{B_n}, \mathscr{C}_{D_n}$ tensored by $\mathbb{C}^h$: it is sufficient to perform different substitutions of the variables $g_0, g_1, \ldots g_n, \ldots$ We observe that in the A_n case the complements of these tensored arrangements are classical generalizations of the pure braid space (see [2] and [20]).

We also point out the connection of our formulas with the rich combinatorics of the corresponding real maximal De Concini–Procesi models: we show how to specialize our series in infinite variables to obtain series whose coefficients are the Euler characteristics of these real models.

2. Basic concepts

2.1. Some combinatorics of subspace arrangements

Let V be a finite dimensional vector space over an infinite field $\mathbb{K}$ and denote by V^* its dual. Let now $\mathscr{G}$ be a finite set of subspaces of V^* and denote by $\mathscr{C}_{\mathscr{G}}$ its closure under the sum.

DEFINITION 2.1. *Given a subspace $U \in \mathscr{C}_{\mathscr{G}}$, a decomposition of U in $\mathscr{C}_{\mathscr{G}}$ is a collection $\{U_1, \ldots, U_k\}$ $(k > 1)$ of non zero subspaces in $\mathscr{C}_{\mathscr{G}}$ such that*

1. $U = U_1 \oplus \cdots \oplus U_k$
2. *for every subspace $A \subset U$, $A \in \mathscr{C}_{\mathscr{G}}$, we have $A \cap U_1, \ldots, A \cap U_k \in \mathscr{C}_{\mathscr{G}}$ and $A = (A \cap U_1) \oplus \cdots \oplus (A \cap U_k)$.*

Definition 2.2. *A subspace $F \in \mathscr{C}_{\mathscr{G}}$ which does not admit a decomposition is called* irreducible *and the set of irreducible subspaces is denoted by $\mathscr{F}_{\mathscr{G}}$.*

One can prove (see [5]) that every subspace $U \in \mathscr{C}_{\mathscr{G}}$ has a unique decomposition into irreducible subspaces.

Definition 2.3. *A collection $\mathscr{G}$ of subspaces of V^* is called* building *if every element $C \in \mathscr{C}_{\mathscr{G}}$ is the direct sum $G_1 \oplus \cdots \oplus G_k$ of the set of maximal elements $G_1, \ldots, G_k$ of $\mathscr{G}$ contained in C.*

Remark 2.1 (see [5]). *The set of irreducible subspaces of a given family of subspaces of V^* is building. A set of subspaces of V^* which is closed under the sum is building.*

Given a family $\mathscr{G}$ of subspaces of V^* there are different sets $\mathscr{B}$ of subspaces of V^* such that $\mathscr{C}_{\mathscr{B}} = \mathscr{C}_{\mathscr{G}}$; if we order by inclusion the collection of such sets, it turns out that the minimal element is $\mathscr{F}_{\mathscr{G}}$ and the maximal one is $\mathscr{C}_{\mathscr{G}}$.

Definition 2.4 (see [5]). *Let $\mathscr{G}$ be a building set of subspaces of V^*. A subset $\mathscr{S} \subset \mathscr{G}$ is called $\mathscr{G}$-nested if and only if for every subset $\{A_1, \ldots, A_k\}$ $(k \geq 2)$ of pairwise non comparable elements of $\mathscr{S}$ the subspace $A = A_1 + \cdots + A_k$ does not belong to $\mathscr{G}$.*

Remark 2.2. *If $\mathscr{C}$ is a building family of subspaces closed under the sum, then the subspaces of a $\mathscr{C}$-nested set are totally ordered* (*with respect to inclusion*).

2.2. Wonderful models

Let now V be a finite dimensional complex vector space and denote by V^* its dual.

Let us consider a finite subspace arrangement $\mathscr{G}$ in V^* and, for every $A \in \mathscr{G}$, let us denote by $A^\perp$ its annihilator in V.

For every $A \in \mathscr{G}$ we have a rational map

$$\pi_A : V \to V/A^\perp \to \mathbb{P}(V/A^\perp)$$

which is regular on $V - A^\perp$.

We then consider the embedding

$$\phi_{\mathscr{G}} : \mathscr{A}_{\mathscr{G}} \to V \times \prod_{A \in \mathscr{G}} \mathbb{P}(V/A^\perp)$$

given by inclusion on the first component and the maps π_A on the other components. The De Concini–Procesi model $Y_{\mathscr{G}}$ associated to $\mathscr{G}$ is the closure of $\phi_{\mathscr{G}}(\mathscr{A}_{\mathscr{G}})$ in $V \times \prod_{A \in \mathscr{G}} \mathbb{P}(V/A^\perp)$.

These *wonderful models* are particularly interesting when the arrangement $\mathscr{G}$ is building: they turn out to be smooth varieties and the complement of $\mathscr{A}_{\mathscr{G}}$ in $Y_{\mathscr{G}}$ is a divisor with normal crossings, described in terms of $\mathscr{G}$-nested sets. Moreover,

their integer cohomology rings are torsion free (see [5]). In [22] Yuzvinski explicitly described $\mathbb{Z}$-bases of these rings (see also [12]): we briefly recall these results concerning cohomology.

Let $\mathscr{G}$ be a building set of subspaces of V^*. Let $\mathscr{H} \subset \mathscr{G}$, let $B \in \mathscr{G}$ such that $A \subsetneq B$ for each $A \in \mathscr{H}$ and define

$$d_{\mathscr{H}, B} := \dim B - \dim\Big(\sum_{A \in \mathscr{H}} A\Big).$$

With these notations, in the polynomial ring $\mathbb{Z}[c_A]_{A \in \mathscr{G}}$, we put

$$P_{\mathscr{H}, B} := \prod_{A \in \mathscr{H}} c_A \Big(\sum_{C \supset B} c_C\Big)^{d_{\mathscr{H}, B}}$$

and we denote by I the ideal generated by these polynomials as $\mathscr{H}$ and B vary.

THEOREM 2.1 (see [5]). *There is a surjective ring homomorphism*

$$\phi : \mathbb{Z}[c_A]_{A \in \mathscr{G}} \to H^*(Y_{\mathscr{G}}, \mathbb{Z})$$

whose kernel is I and such that $\phi(c_A) \in H^2(Y_{\mathscr{G}}, \mathbb{Z})$.

DEFINITION 2.5. *Let $\mathscr{G}$ be a building set of subspaces of V^*. A function*

$$f : \mathscr{G} \to \mathbb{N}$$

is $\mathscr{G}$-admissible (or simply admissible) if $f = 0$ or, if $f \neq 0$, $\mathrm{supp}(f)$ is $\mathscr{G}$-nested and for all $A \in \mathrm{supp}(f)$ one has

$$f(A) < d_{\mathrm{supp}(f)_A, A}$$

where $\mathrm{supp}(f)_A := \{C \in \mathrm{supp}(f) : C \subsetneq A\}$.

DEFINITION 2.6. *A monomial $m_f = \prod_{A \in \mathscr{G}} c_A^{f(A)} \in \mathbb{Z}[c_A]_{A \in \mathscr{G}}$ is admissible if f is admissible.*

THEOREM 2.2 (see [22] and also [12]). *The set $\mathscr{B}_{\mathscr{G}}$ of all admissible monomials corresponds to a $\mathbb{Z}$-basis of $H^*(Y_{\mathscr{G}}, \mathbb{Z})$.*

3. MAXIMAL MODELS OF REFLECTION ARRANGEMENTS OF CLASSICAL TYPE

We now focus on the cohomology rings of maximal De Concini–Procesi models for root arrangements of type A_n, B_n (C_n) and D_n.

3.1. *Type A_{n-1}*

Let W be a complex vector space of dimension n and consider the arrangement given by hyperplanes $H_{ij} := \{z_i - z_j = 0\}$ where z_i $(i = 1, \ldots, n)$ are the coordi-

nates. The intersection of these hyperplanes is the line $N = \{z_1 = \cdots = z_n\}$: we consider the quotient $V = W/N$ and the arrangement provided by the images of the hyperplanes H_{ij} via the quotient map $W \xrightarrow{\pi} V$.

We can choose linear functionals f_{ij} in V^* such that the zeroes of f_{ij} form the hyperplane $\pi(H_{ij})$ and the set $\{f_{ij}\}$ is a root system of type A_{n-1}.

In V^* we consider the subspace arrangement $\mathscr{A}_{A_{n-1}}$ given by the lines $\langle f_{ij} \rangle$ and denote for brevity by $\mathscr{C}_{A_{n-1}}$ its closure under the sum and by $\mathscr{F}_{A_{n-1}}$ the set of irreducible subspaces in $\mathscr{C}_{A_{n-1}}$.

Our purpose is to compute the series

(1) $$\phi_A(q,t) = t + \sum_{n \geq 2} P_{\mathscr{C}_{A_{n-1}}}(q) \frac{t^n}{n!} \in \mathbb{Q}[q][[t]]$$

where, for every $n \geq 2$,

$$P_{\mathscr{C}_{A_{n-1}}}(q) = \sum \mathrm{rk}(H^{2i}(Y_{\mathscr{C}_{A_{n-1}}}, \mathbb{Z}))q^i$$

is the Poincaré polynomial of $Y_{\mathscr{C}_{A_{n-1}}}$ (the variable q has degree 2).

In [22], Yuzvinsky noticed there is a bijective correspondence (actually an isomorphism of partially ordered sets) between the elements of $\mathscr{F}_{A_{n-1}}$ and the subsets of $\{1, \ldots, n\}$ of cardinality at least two: this correspondence identifies the subset $\{i_1, \ldots, i_k\}$ with $\langle f_{i_1 i_2}, \ldots, f_{i_{k-1} i_k} \rangle$. Since every subspace in $\mathscr{C}_{A_{n-1}}$ has a unique decomposition into irreducible subspaces, we can identify elements of $\mathscr{C}_{A_{n-1}}$ with families of disjoint subsets of cardinality at least two of $\{1, \ldots, n\}$. Furthermore, given two such collections $X = \{X_1, \ldots, X_k\}$ and $Y = \{Y_1, \ldots, Y_r\}$ we say that Y is included in X (and write $Y \subset X$) if for every $i \in \{1, \ldots, r\}$ there exists $j \in \{1, \ldots, k\}$ such that $Y_i \subset X_j$; our identification thus becomes an isomorphism of partially ordered sets (we order $\mathscr{C}_{A_{n-1}}$ by inclusion).

A $\mathscr{C}_{A_{n-1}}$-nested set is a subset of $\mathscr{C}_{A_{n-1}}$ strictly ordered by inclusion. Now we will see how to associate a graph (actually a forest of levelled oriented rooted trees) to a $\mathscr{C}_{A_{n-1}}$-nested set. Let us first recall by an example the $\mathscr{F}_{A_{n-1}}$ case (see [22]). Let us take, as $\mathscr{F}_{A_8}$-nested set, the collection $\mathscr{S} := \{(1,2,3,4,5), (6,7,8,9), (1,4,5), (6,7)\}$; we associate to $\mathscr{S}$ the following graph:
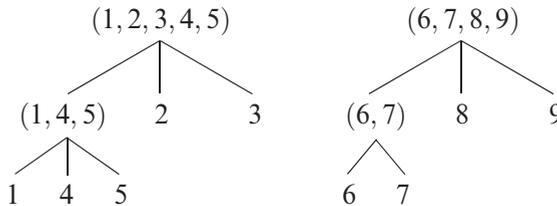
where the edges are directed from the top to the bottom.

Let us now consider the $\mathscr{C}_{A_{15}}$-nested set

$$\mathscr{S} := \{(1,2,3,4,5)(8,10,12,13,14,16), (1,2,4,5)(8,10,12), (1,2)(10,12)\}.$$

We associate to it the *levelled* forest $\Gamma(\mathscr{S})$:
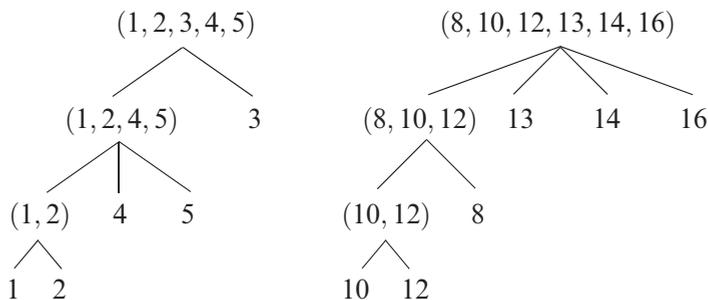
where, again, the orientation is from the top to the bottom, and the elements of the nested set can be read "level by level" from the vertices which are not leaves (we call level 1 the level which contains the roots, and level $k+1$ the one which contains the vertices which are k steps away from a root).

Let now $\mathscr{S}$ be a $\mathscr{C}_{A_{n-1}}$-nested set and let us denote by B (resp. A) the element of $\mathscr{S}$ determined by the vertices (not leaves) at level k (resp. $k+1$). Then A is the maximal element of $\mathscr{S}$ strictly contained in B.

Hence if B is given by the family $\{B_1, \ldots, B_k\}$, A by $\{A_1, \ldots, A_r\}$ and, for every $i \in \{1, \ldots, k\}$, we set $I_{B_i} := \{j \in \{1, \ldots, r\} : A_j \subset B_i\}$, we have

$$\dim B_i - \sum_{j \in I_{B_i}} \dim A_j = |\text{out}(v_{B_i})| - 1$$

where v_{B_i} is the vertex of $\Gamma(\mathscr{S})$ associated to B_i and $\text{out}(v_{B_i})$ is the set of outgoing edges from v_{B_i}. Then we obtain:

$$(2) \qquad d_{\{A\}, B} = \sum_{v \in \text{Lev}(k)} (|\text{out}(v)| - 1)$$

where $\text{Lev}(k)$ is the set of vertices (not leaves) of $\Gamma(\mathscr{S})$ belonging to level k.

DEFINITION 3.1. *A levelled forest Γ is* admissible *if, for any level k, one has*

$$\sum_{v \in \text{Lev}(k)} (|\text{out}(v)| - 1) = \left(\sum_{v \in \text{Lev}(k)} |\text{out}(v)| \right) - |\text{Lev}(k)| \geq 2.$$

DEFINITION 3.2. *Let Γ be a levelled admissible forest on $n \geq 2$ leaves. We denote by* cont(Γ) *the contribution given to the series* (1) *by all the monomials m_f of the basis such that* supp f *is a nested set whose graph is (up to a relabelling of the leaves) isomorphic to Γ.*

PROPOSITION 3.1. *Let Γ be a levelled admissible forest on $n \geq 2$ leaves. Then we have*

$$\text{cont}(\Gamma) = \frac{n!}{|\text{Aut}(\Gamma)|} C_\Gamma(q) \frac{t^n}{n!}$$

where

$$C_\Gamma(q) = \prod_{k \text{ level}} \frac{q^{\sum_{v \in \text{Lev}(k)}(|\text{out}(v)|-1)} - q}{q - 1}$$

and $\text{Aut}(\Gamma)$ *is the group of automorphisms of* Γ.

PROOF. We notice that there are $\frac{n!}{|\text{Aut}(\Gamma)|}$ different $\mathscr{C}_{A_{n-1}}$-nested sets whose associated graph is isomorphic to Γ.

The thesis follows by observing that, if $\Gamma = \Gamma(\mathscr{S})$, where $\mathscr{S}$ is a $\mathscr{C}_{A_{n-1}}$-nested set, the contribution to the Poincaré polynomial $P_{\mathscr{C}_{A_{n-1}}}(q)$ of the monomials m_f such that $\text{supp} f = \mathscr{S}$ is $C_\Gamma(q)$. □

Our idea to compute the series (1) is to consider all levelled forests (not necessarily admissible) on at least two leaves and to associate, to each of them, a monomial that encodes the data we are interested in: number of levels and, for each level k, the number $\sum_{v \in \text{Lev}(k)}(|\text{out}(v)| - 1)$. We will put together these monomials in a series, which will be calculated inductively, and from which one can obtain the series (1).

DEFINITION 3.3. *A trivial tail of a (levelled) oriented forest* $\mathscr{T}$ *is given by a subtree* $\mathscr{T}'$ *which has a single leaf and stems from a vertex* v *of* $\mathscr{T}$ *with* $|\text{out}(v)| = 1$.

DEFINITION 3.4. *Two (levelled) oriented forests* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *are* equivalent *if they differ only by trivial tails.*

DEFINITION 3.5. *Given an equivalence class of (levelled) rooted oriented forests modulo trivial tails we call* minimal representative *the forest in this class with no trivial tails.*

Let us now define the following series

$$\tilde{p}_A(g_0, g_1, g_2, g_3, \ldots) := g_0 + \sum_\Gamma \frac{g_0^{\text{tr}(\Gamma)}}{|\text{Aut}(\Gamma)|} \prod_v g_{l(v)}^{|\text{out}(v)|-1}$$

where Γ runs among minimal representatives of levelled oriented forests on at least two leaves, $\text{tr}(\Gamma)$ is the number of trees of Γ (we are considering also the degenerate tree given by a single leaf), v varies among the vertices (not leaves) of Γ and $l(v)$ is the level of v.

DEFINITION 3.6. *A monomial of* $\tilde{p}_A$ *is* bad *if we can find i and j, with* $1 \le i < j$, *such that* g_j *appears in the monomial but* g_i *doesn't.*

We notice that bad monomials correspond to forests with a level k such that $\sum_{v \in \text{Lev}(k)}(|\text{out}(v)| - 1) = 0$.

DEFINITION 3.7. *A monomial m of $\tilde{p}_A$ has* valency *k if $k = \max\{j \geq 1 : g_j$ appears in $m\}$.*

PROPOSITION 3.2. *Given a levelled oriented forest Γ on $n \geq 2$ leaves, let m_Γ be the monomial of $\tilde{p}_A$ associated to Γ. Then the degree of m_Γ is n.*

PROOF. One can restrict to trees and then proceed by induction on the valency of m_Γ. $\qquad\qquad\square$

THEOREM 3.1. *Removing bad monomials from $\tilde{p}_A$ and replacing g_0 with t and, for every $i, r \geq 1$, g_i^r with $\frac{q^r - q}{q - 1} t^r$ we obtain the Poincaré series (1).*

PROOF. Let's start by observing that, if we remove bad monomials, we haven't removed all monomials corresponding to non admissible forests; in fact we still have the ones corresponding to forests in which there is (at least) a level k such that

$$\sum_{v \in \text{Lev}(k)} (|\text{out}(v)| - 1) = 1.$$

Anyhow the contribution of such monomials is killed by our substitution; indeed they have a variable whose exponent is 1 and $\frac{q^r - q}{q - 1} = 0$ if $r = 1$.

Let now Γ be an admissible forest on $n \geq 2$ leaves; let m_Γ be the monomial of Γ in $\tilde{p}_A$ and $k \geq 1$ be its valency. For all $1 \leq j \leq k$ the exponent of the variable g_j in m_Γ is

$$\sum_{v \in \text{Lev}(j)} (|\text{out}(v)| - 1).$$

Our claim then follows from Proposition 3.1 and Proposition 3.2. $\qquad\square$

The problem is now reduced to the computation of $\tilde{p}_A$. To this end we define

$$\tilde{\tilde{p}}_A := 1 + \sum_{\mathcal{T}} \frac{1}{|\text{Aut}(\mathcal{T})|} \prod_v g_{l(v)}^{|\text{out}(v)| - 1}$$

where $\mathcal{T}$ runs among minimal representatives of levelled oriented trees on $n \geq 2$ leaves and v among the vertices (not leaves) of $\mathcal{T}$. One immediately checks that:

$$(3) \qquad\qquad\qquad \tilde{p}_A = e^{g_0 \tilde{\tilde{p}}_A} - 1,$$

therefore, all we need is a formula for $\tilde{\tilde{p}}_A$.

THEOREM 3.2. *The following recursive formula holds:*

$$(4) \qquad\qquad\qquad \tilde{\tilde{p}}_A = \frac{e^{g_1 \tilde{\tilde{p}}_A[1]} - 1}{g_1}$$

where $\tilde{\tilde{p}}_A[1]$ is $\tilde{\tilde{p}}_A(g_1, g_2, g_3, \ldots)$ evaluated in $(g_2, g_3, g_4, \ldots)$.

PROOF. The formula is recursive, as one can easily check by induction on the valency.

Let $\mathscr{T}$ be a tree on $n \geq 2$ leaves (recall that we are taking into account only minimal representatives modulo trivial tails). Let $i_1^{m_1} \ldots i_r^{m_r}$ be a partition of n of length k made by positive integers $i_1, \ldots, i_r$ such that, for each $j \in \{1, \ldots, r\}$, i_j occurs m_j times (and $k = \sum_{j=1}^r m_j$). Suppose that exactly k edges stem from the root; furthermore, suppose that, if we cut off the root of $\mathscr{T}$ and these edges, we get a forest of k trees, $\{\mathscr{T}_1, \ldots, \mathscr{T}_k\}$, such that, for each $j \in \{1, \ldots, r\}$, m_j of them are isomorphic and have i_j leaves (here we are considering also the degenerate tree given by a single leaf).

If, for every $i \in \{1, \ldots, k\}$, we call $m_{\mathscr{T}_i}$ the monomial of $\mathscr{T}_i$ in $\tilde{\tilde{p}}_A$ and $m_{\mathscr{T}}$ the one of $\mathscr{T}$ we have:

$$m_{\mathscr{T}} = g_1^{k-1} \frac{1}{m_1! m_2! \ldots m_r!} \prod_{i=1}^k m_{\mathscr{T}_i}[1].$$

We conclude by observing that $\prod_{i=1}^k m_{\mathscr{T}_i}[1]$ appears exactly $\frac{k!}{m_1! m_2! \ldots m_r!}$ times in $(\tilde{\tilde{p}}_A[1])^k$. $\qquad\square$

3.2. Some examples

Theorem 3.2 allows us to compute $\tilde{\tilde{p}}_A$; once we have $\tilde{\tilde{p}}_A$ we can compute $\tilde{p}_A$ and, using Theorem 3.1, the series ϕ_A; here we exhibit some examples of these computations made with the help of the Computer Algebra system Axiom.

As a first example, we show the monomials of $\tilde{\tilde{p}}_A$ of valency less than or equal to 3 and degree less than or equal to 3:

$$\frac{1}{24} g_3^3 + \left(\frac{7}{24} g_2 + \frac{7}{24} g_1 + \frac{1}{6} \right) g_3^2 + \left(\frac{1}{4} g_2^2 + \left(\frac{3}{4} g_1 + \frac{1}{2} \right) g_2 + \frac{1}{4} g_1^2 + \frac{1}{2} g_1 + \frac{1}{2} \right) g_3$$

$$+ \frac{1}{24} g_2^3 + \left(\frac{7}{24} g_1 + \frac{1}{6} \right) g_2^2 + \left(\frac{1}{4} g_1^2 + \frac{1}{2} g_1 + \frac{1}{2} \right) g_2 + \frac{1}{24} g_1^3 + \frac{1}{6} g_1^2 + \frac{1}{2} g_1 + 1.$$

If we look at terms of degree 3 and valency 3 we find:

$$\frac{g_3^3}{4!} + \frac{g_2 g_3^2}{8} + \frac{g_2 g_3^2}{6} + \frac{g_2^2 g_3}{4} + \frac{g_1 g_3^2}{8} + \frac{g_1 g_3^2}{6} + \frac{g_1 g_2 g_3}{2} + \frac{g_1 g_2 g_3}{4} + \frac{g_1^2 g_3}{4}.$$

These nine monomials correspond to the levelled trees with 4 leaves and 3 levels (modulo equivalence):
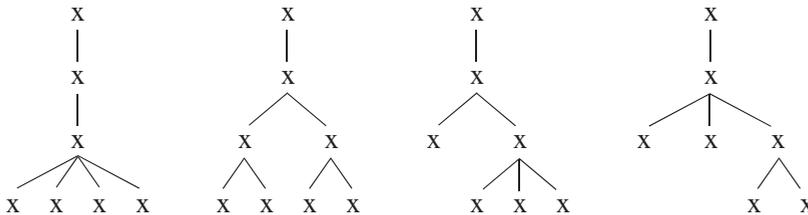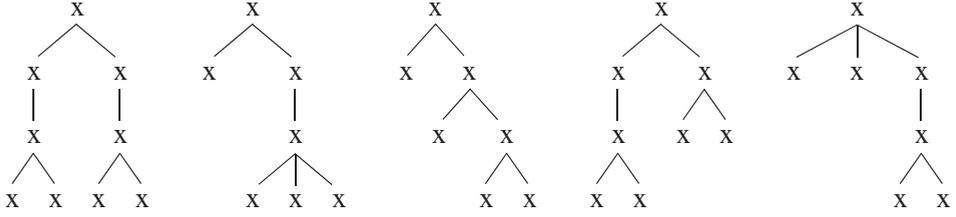
Then we show $\tilde{p}$ up to degree 5, without bad monomials:

degrees 1, 2 and 3

$$g_0 + \frac{1}{2}g_0g_1 + \frac{1}{2}g_0^2 + \frac{1}{2}g_0g_1g_2 + \frac{1}{6}g_0g_1^2 + \frac{1}{2}g_0^2g_1 + \frac{1}{6}g_0^3;$$

degree 4

$$\frac{3}{4}g_0g_1g_2g_3 + \frac{7}{24}g_0g_1g_2^2 + \left(\frac{1}{4}g_0g_1^2 + \frac{3}{4}g_0^2g_1\right)g_2 + \frac{1}{24}g_0g_1^3$$

$$+ \frac{7}{24}g_0^2g_1^2 + \frac{1}{4}g_0^3g_1 + \frac{1}{24}g_0^4;$$

degree 5

$$\frac{3}{2}g_0g_1g_2g_3g_4 + \frac{5}{8}g_0g_1g_2g_3^2 + \left(\frac{7}{12}g_0g_1g_2^2 + \left(\frac{1}{2}g_0g_1^2 + \frac{3}{2}g_0^2g_1\right)g_2\right)g_3$$

$$+ \frac{1}{8}g_0g_1g_2^3 + \left(\frac{5}{24}g_0g_1^2 + \frac{5}{8}g_0^2g_1\right)g_2^2 + \left(\frac{1}{12}g_0g_1^3 + \frac{7}{12}g_0^2g_1^2 + \frac{1}{2}g_0^3g_1\right)g_2$$

$$+ \frac{1}{120}g_0g_1^4 + \frac{1}{8}g_0^2g_1^3 + \frac{5}{24}g_0^3g_1^2 + \frac{1}{12}g_0^4g_1 + \frac{1}{120}g_0^5.$$

At last here it is the series ϕ_A up to degree 7 (with respect to t):

$$\phi_A(q,t) = t + \frac{1}{2}t^2 + \left(\frac{1}{6}q + \frac{1}{6}\right)t^3 + \left(\frac{1}{24}q^2 + \frac{1}{3}q + \frac{1}{24}\right)t^4$$

$$+ \left(\frac{1}{120}q^3 + \frac{41}{120}q^2 + \frac{41}{120}q + \frac{1}{120}\right)t^5$$

$$+ \left(\frac{1}{720}q^4 + \frac{187}{720}q^3 + \frac{61}{60}q^2 + \frac{187}{720}q + \frac{1}{720}\right)t^6$$

$$+ \left(\frac{1}{5040}q^5 + \frac{19}{112}q^4 + \frac{2389}{1260}q^3 + \frac{2389}{1260}q^2 + \frac{19}{112}q + \frac{1}{5040}\right)t^7 + \cdots$$

3.3. Type B_n

Let V be a complex vector space of dimension n. We consider in V^* the line arrangement corresponding to a root system of type B_n and denote by $\mathscr{C}_{B_n}$ its closure under the sum and by $\mathscr{F}_{B_n}$ the set of irreducible subspaces of $\mathscr{C}_{B_n}$. Our aim is to compute the series:

$$(5) \qquad \phi_B(q,t) := \frac{t}{2} + \sum_{n \geq 2} P_{\mathscr{C}_{B_n}}(q) \frac{t^n}{2^n n!} \in \mathbb{Q}[q][[t]]$$

where, for each $n \geq 2$, $P_{\mathscr{C}_{B_n}}(q)$ is the Poincaré polynomial of $Y_{\mathscr{C}_{B_n}}$.

In [22], Yuzvinsky divided the elements of $\mathscr{F}_{B_n}$ into two classes: strong elements and weak elements; if we denote by $x_1, \ldots, x_n \in V^*$ the coordinate functions, strong elements are the subspaces of V^* like $\langle x_{i_1}, \ldots, x_{i_k} \rangle$ $(k \geq 1)$, whose annihilator in V is the subspace $H_{i_1, \ldots, i_k} := \{x_{i_1} = \cdots = x_{i_k} = 0\}$. They can be put in bijective correspondence with subsets of $\{1, \ldots, n\}$ of cardinality greater than or equal to 1 (such subsets will be called strong). A weak element is a subspace whose annihilator is of type $L_{i_1, \ldots, i_k, j_1, \ldots, j_s} := \{x_{i_1} = \cdots = x_{i_k} = -x_{j_1} = \cdots = -x_{j_s}\}$ $(r + s \geq 2)$; therefore weak elements can be put in a bijective correspondence with subsets of $\{1, \ldots, n\}$ of cardinality greater than or equal to 2 (such subsets will be called weak) equipped with a partition (possibly trivial) into 2 parts.

Moreover, if we order $\mathscr{F}_{B_n}$ by inclusion of subspaces, we can read this order as follows:

- a subset that includes a strong subset of $\{1, \ldots, n\}$ is strong;
- a weak subset A is smaller than a strong subset B if and only if $A \subset B$;
- a weak subset $A = A_1 \cup A_2$ is smaller than a weak subset $B = B_1 \cup B_2$ if and only if either $A_i \subset B_i$ $(i = 1, 2)$ or $A_1 \subset B_2$ and $A_2 \subset B_1$.

Coming to the maximal building set, we observe that there is a bijective correspondence between elements of $\mathscr{C}_{B_n}$ and families of disjoint subsets of $\{1, \ldots, n\}$ in which at most one is strong and in each of the weak ones a partition into two parts is fixed. Given two such families $X = \{X_1, \ldots, X_k\}$ and $Y := \{Y_1, \ldots, Y_h\}$ we say that X is greater than Y (and write $X \supset Y$) if for every $i \in \{1, \ldots, h\}$ there exists $j \in \{1, \ldots, k\}$ such that $Y_i \subset X_j$ as elements of $\mathscr{F}_{B_n}$.

It is again possible to associate levelled forests to $\mathscr{C}_{B_n}$-nested sets (a $\mathscr{C}_{B_n}$-nested is a subset of $\mathscr{C}_{B_n}$ strictly ordered by inclusion); the rules are the same as in the case A_{n-1} but we have to divide the vertices of our graphs into two classes: weak vertices and strong vertices. We notice that we lose the information concerning partitions of weak sets. From now we call "strong tree" a tree with at least one strong vertex and "weak tree" a tree with no strong vertices; a forest is "strong" if it contains a strong tree, otherwise is weak.

Let now $\mathscr{S}$ be a $\mathscr{C}_{B_n}$-nested set and $\Gamma(\mathscr{S})$ be its associated forest; let us denote by B (resp. A) the element of $\mathscr{S}$ determined by the vertices (not leaves) at level k (resp. $k+1$); then A is the maximal element of $\mathscr{S}$ strictly contained in B. If B is given by a family of weak subsets, $d_{\{A\}, B}$ can be computed, in terms of

outgoing edges, exactly as in the A_n case. Otherwise, B is associated to a family $\{B_1, \ldots, B_k\}$ ($k \geq 1$) of subsets of $\{1, \ldots, n\}$, where B_1 is strong. Then we have

$$(6) \qquad d_{\{A\}, B} = |\text{wout}(v_{B_1})| + \sum_{i=2}^{k} (|\text{out}(v_{B_i})| - 1)$$

where v_{B_i} is the vertex of $\Gamma(\mathscr{S})$ which corresponds to B_i and $\text{wout}(v_{B_1})$ is the set of outgoing edges from v_{B_1} to a weak vertex (we are considering the leaves as weak vertices).

The following lemma and corollary explain how to take in account the information on the partitions associated to weak sets, which is not contained in the graphs.

LEMMA 3.1 (see [22]). *Let $\mathscr{S}$ be a $\mathscr{F}_{B_n}$-nested set and $\Gamma(\mathscr{S})$ be its associated forest. If we denote by $\pi(\Gamma(\mathscr{S}))$ the number of different $\mathscr{F}_{B_n}$-nested sets $\mathscr{U}$ such that $\Gamma(\mathscr{U}) = \Gamma(\mathscr{S})$, then*

$$\log_2 \pi(\Gamma(\mathscr{S})) = \sum_{v_B} \dim B$$

where v_B ranges over all the maximal weak vertices (not leaves), i.e. the weak vertices (not leaves) which are not preceded, according to the orientation, by other weak vertices.

COROLLARY 3.1. *Let $\Gamma = \Gamma(\mathscr{S})$ be a levelled forest associated to a $\mathscr{C}_{B_n}$-nested set $\mathscr{S}$. Let $\{v_{X_1}, \ldots, v_{X_j}\}$ be the maximal weak vertices of $\Gamma(\mathscr{S})$. Then Γ corresponds to $2^{\sum_{i=1}^{j}(|X_i|-1)}$ different $\mathscr{C}_{B_n}$-nested sets.*

As in the A_n case, to compute the Poincaré series (5) we define a series in infinite variables $g_0, g_1, g_2, \ldots$. We need to extend the definition of trivial tail to strong trees.

DEFINITION 3.8. *A trivial tail of a levelled oriented strong tree $\mathscr{T}$ is given by a weak subtree $\mathscr{T}'$ which has a single leaf and stems from a vertex v of $\mathscr{T}$ with $|\text{out}(v)| = 1$.*

Then we define a series which will take into account the contribution of strong forests to the Poincaré series:

$$\tilde{Q}_B(g_0, g_1, g_2, \ldots) := \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \left(\frac{g_0}{2}\right)^{\text{tr}(\Gamma)-1} \prod_{v \in \Gamma_s} \left(\frac{g_{l(v)}}{2}\right)^{|\text{wout}(v)|} \prod_{v \in \Gamma_w} g_{l(v)}^{|\text{out}(v)|-1}$$

where Γ runs among minimal representatives (modulo trivial tails) of levelled oriented strong forests on $n \geq 1$ leaves, Γ_s is the set of strong vertices of Γ, Γ_w is the set of weak vertices (not leaves); we notice that an automorphism sends strong vertices to strong vertices and weak vertices to weak vertices.

Then we put:

$$(7) \qquad Q_B(g_0, g_1, g_2, \dots) := \tilde{p}_A\left(\frac{g_0}{2}, g_1, g_2, \dots\right) + \tilde{Q}_B(g_0, g_1, g_2, \dots).$$

REMARK 3.1. *From equation* (3) *it follows* $\tilde{p}_A\left(\frac{g_0}{2}, g_1, \dots\right) = e^{(g_0/2)\tilde{\tilde{p}}_A(g_1, g_2, \dots)} - 1.$

We notice that, if Γ is a (strong or weak) levelled oriented forest on $n \geq 2$ leaves, then the corresponding monomial m_Γ in $Q_B(g_0, g_1, g_2, \dots)$ has degree n.

THEOREM 3.3. *If we remove bad monomials from Q_B and replace g_0 with t and, for every $i, r \geq 1$, g_i^r with $\frac{q^r - q}{q - 1} t^r$, we obtain the series* (5).

PROOF. It is a computation very similar to the one of Theorem 3.1: $\tilde{p}_A\left(\frac{g_0}{2}, g_1, g_2, \dots\right)$ counts the contribution of weak forests, $\tilde{Q}_B$ of strong forests. $\qquad \square$

We now need to compute $\tilde{Q}_B$; to this end we define

$$\tilde{\tilde{Q}}_B(g_1, g_2, \dots) := \sum_{\mathscr{T}} \frac{1}{|\mathrm{Aut}(\mathscr{T})|} \prod_{v \in \mathscr{T}_s} \left(\frac{g_{l(v)}}{2}\right)^{|\mathrm{wout}(v)|} \prod_{v \in \mathscr{T}_w} g_{l(v)}^{|\mathrm{out}(v)| - 1}$$

where $\mathscr{T}$ runs among minimal representatives of classes of strong trees on $n \geq 1$ leaves, $\mathscr{T}_s$ is the set of strong vertices of $\mathscr{T}$ and $\mathscr{T}_w$ is the set of the weak ones (not leaves). Then, since each strong forest has exactly one strong tree we have:

$$\tilde{Q}_B = \left(\tilde{p}_A\left(\frac{g_0}{2}, g_1, g_2, \dots\right) + 1\right)\tilde{\tilde{Q}}_B.$$

THEOREM 3.4. *The following inductive formula holds:*

$$(8) \qquad \tilde{\tilde{Q}}_B = \tilde{p}_A\left(\frac{g_0}{2}, g_1, g_2, \dots\right)[1] + \tilde{\tilde{Q}}_B[1]\left(1 + \tilde{p}_A\left(\frac{g_0}{2}, g_1, g_2, \dots\right)[1]\right).$$

PROOF. We will prove the equivalent formula

$$(9) \quad \tilde{\tilde{Q}}_B = \left(\sum_{j \geq 0} \frac{g_1^j\left(\frac{1}{2}\tilde{\tilde{p}}_A(g_1, g_2, \dots)[1]\right)^j}{j!}\right)\tilde{\tilde{Q}}_B[1] + \sum_{j \geq 1} \frac{g_1^j\left(\frac{1}{2}\tilde{\tilde{p}}_A(g_1, g_2, \dots)[1]\right)^j}{j!}.$$

Let $\mathscr{T}$ be a strong tree on $n \geq 1$ leaves; suppose that $\mathscr{T}$ has only one strong vertex (therefore its root is strong). Let $i_1^{m_1} \dots i_r^{m_r}$ be a partition of n of length $k \geq 1$ made by positive integers $i_1, \dots, i_r$ such that, for each $j \in \{1, \dots, r\}$, i_j occurs m_j times. Let us suppose that k edges stem from the (strong) root of $\mathscr{T}$ and that, if we cut off the root of $\mathscr{T}$ and these edges, we get a forest of k weak trees, $\{\mathscr{T}_1, \dots, \mathscr{T}_k\}$, such that, for each $j \in \{1, \dots, r\}$, m_j of them are isomorphic

and have i_j leaves. If we call $m_{\mathscr{T}}$ the monomial of $\mathscr{T}$ in $\tilde{\tilde{Q}}_B$ and, for every $i \in \{1, \ldots, k\}$, m_{T_i} the one of $\mathscr{T}_i$ in $\tilde{\tilde{p}}_A(g_1, g_2, \ldots)$, we have

$$m_{\mathscr{T}} = \frac{1}{m_1! m_2! \ldots m_r!} \frac{g_1^k}{2^k} \prod_{i=1}^{k} m_{\mathscr{T}_i}[1].$$

To obtain the second addendum on the right side of formula (9) we observe that $\prod_{i=1}^{k} m_{\mathscr{T}_i}[1]$ appears exactly $\frac{k!}{m_1! m_2! \ldots m_r!}$ times in $(\tilde{\tilde{p}}_A(g_1, g_2, \ldots)[1])^k$.

Suppose now that $\mathscr{T}$ has more than one strong vertex and that the (strong) root of $\mathscr{T}$ is connected to $j + 1 \geq 1$ vertices such that j of them are weak and one (which we will denote by v) is strong.

If $j = 0$ then exactly one edge stems from the root of $\mathscr{T}$ and (by assumption) it reaches v. If we call $\mathscr{T}'$ the tree which stems from v we have that $m_{\mathscr{T}} = m_{\mathscr{T}'}[1]$. Let $j > 0$ and suppose that the j subtrees whose roots are the j weak vertices are divided into h subsets containing respectively $m_1, m_2, \ldots, m_h$ isomorphic trees. Let us denote by $\{\mathscr{T}_1, \ldots, \mathscr{T}_j\}$ these trees and let $\mathscr{T}_{j+1}$ be the strong tree whose root is v. We have

$$m_{\mathscr{T}} = \frac{1}{m_1! m_2! \ldots m_h!} \frac{g_1^j}{2^j} m_{\mathscr{T}_{j+1}}[1] \prod_{q=1}^{j} m_{\mathscr{T}_q}[1]$$

where, as usual, $m_{\mathscr{T}}$ is the monomial of $\mathscr{T}$ in $\tilde{\tilde{Q}}_B$, $m_{\mathscr{T}_{j+1}}$ is the one of $\mathscr{T}_{j+1}$ and, for each $q \in \{1, \ldots, j\}$, $m_{\mathscr{T}_q}$ is the monomial of $\mathscr{T}_q$ in $\tilde{\tilde{p}}_A$.

We end by observing that $\prod_{q=1}^{j} m_{\mathscr{T}_q}[1]$ appears exactly $\frac{j!}{m_1! m_2! \ldots m_h!}$ times in $\tilde{\tilde{p}}_A^j$. $\qquad\square$

3.4. Type D_n

Let V be a complex vector space of dimension n. We consider in V^* the line arrangement corresponding to a root system of type D_n and denote by $\mathscr{C}_{D_n}$ its closure under the sum.

The series we are interested in is the following:

$$(10) \qquad \phi_D(q, t) := t + q\frac{t^2}{4} + \sum_{n \geq 3} P_{\mathscr{C}_{D_n}}(q) \frac{t^n}{n! 2^{n-1}} \in \mathbb{Q}[q][[t]]$$

where, as usual, for each $n \geq 3$, $P_{\mathscr{C}_{D_n}}(q)$ is the Poincaré polynomial of $Y_{\mathscr{C}_{D_n}}$. The combinatorics is essentially the same as in the case B_n: the only difference is that strong sets must have cardinality at least two. Therefore for the computation of the Poincaré series we just need to modify a little what we have done in that case.

We set

$$\tilde{Q}_D(g_0, g_1, g_2, \ldots) := 2 \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \left(\frac{g_0}{2}\right)^{\mathrm{tr}(\Gamma)-1} \prod_{v \in \Gamma_s} \left(\frac{g_{l(v)}}{2}\right)^{|\mathrm{wout}(v)|} \prod_{v \in \Gamma_w} g_{l(v)}^{|\mathrm{out}(v)|-1}$$

where Γ is a strong levelled oriented forest whose strong vertices correspond to subsets of cardinality at least two and

$$\tilde{\tilde{Q}}_D(g_1, g_2, \ldots) := 2 \sum_{\mathcal{T}} \frac{1}{|\mathrm{Aut}(\mathcal{T})|} \prod_{v \in \mathcal{T}_s} \left(\frac{g_{l(v)}}{2} \right)^{|\mathrm{wout}(v)|} \prod_{v \in \mathcal{T}_w} g_{l(v)}^{|\mathrm{out}(v)|-1}$$

where the strong vertices of the strong levelled oriented tree $\mathcal{T}$ correspond to subsets of cardinality at least two. Now, if we define $Q_D := 2\tilde{p}_A\left(\frac{g_0}{2}, g_1, g_2, \ldots\right) + \tilde{Q}_D$ we can compute the Poincaré series as in Theorem 3.3; moreover we have that

$$\tilde{Q}_D = \left(\tilde{p}_A\left(\frac{g_0}{2}, g_1, g_2, \ldots \right) + 1 \right) \tilde{\tilde{Q}}_D$$

and $\tilde{\tilde{Q}}_D$ satisfies a recurrence relation similar to (9):

$$\tilde{\tilde{Q}}_D = \left(\sum_{j \geq 0} \frac{g_1^j \left(\frac{1}{2} \tilde{\tilde{p}}_A(g_1, g_2, \ldots)[1] \right)^j}{j!} \right) \tilde{\tilde{Q}}_D[1] + 2 \left(\sum_{j \geq 1} \frac{g_1^j \left(\frac{1}{2} \tilde{\tilde{p}}_A(g_1, g_2, \ldots)[1] \right)^j}{j!} \right) - g_1.$$

4. Final remarks

4.1. Induced subspace arrangements

The tensor product allows us to obtain new building subspace arrangements $\mathcal{G}_h$ starting from a given building arrangement $\mathcal{G}$ in V^*.

Definition 4.1. *We will call 'induced by $\mathcal{G}$' the subspace arrangement $\mathcal{G}_h$ in $V^* \otimes \mathbb{C}^h$ ($h \geq 1$) given by the subspaces $A \otimes \mathbb{C}^h$, as A varies in $\mathcal{G}$.*

For instance, if $\mathcal{G}$ is a building set associated to a root system of type A, the complements of the arrangements $\mathcal{G}_h$ are classical generalizations of the pure braid space (see [2] and [20]).

It is immediate to check that, for any given building arrangement $\mathcal{G}$ in V^*, $\mathcal{G}_h$ is still building, and therefore one can consider its De Concini–Procesi model. Let us focus on the case when the starting arrangements are the maximal building sets of type A, $B(=C)$, D. Our series $\tilde{p}_A$, Q_B, Q_D allow us to obtain quickly the Poincaré series of the families of models associated to the induced building sets: we only have to perform different substitutions of the variables $g_0, g_1, \ldots$

For instance, let us fix $h \geq 1$ and consider the A case: after removing bad monomials from $\tilde{p}_A$, if we replace g_0 with t as before and, for every $i, r \geq 1$, g_i^r with $\frac{q^{rh}-q}{q-1} t^r$, we obtain the Poincaré series for the models $Y_{(\mathcal{G}_{A_n})_h}$ (the same substitutions work also in the other cases).

4.2. The Euler characteristic of maximal compact models

The De Concini–Procesi construction can be repeated also for real (projective) subspace arrangements, producing real (compact) models.

Let us consider a real building set of subspaces in an euclidean vector space and denote by $Y_{\mathscr{G}}(\mathbb{C})$ and $Y_{\mathscr{G}}(\mathbb{R})$ the complex model and the real compact model associated to it; it is well known (see [15] and [19]) that $H^{2i}(Y_{\mathscr{G}}(\mathbb{C}), \mathbb{Z}_2) \cong H^{i}(Y_{\mathscr{G}}(\mathbb{R}), \mathbb{Z}_2)$; given that $\dim H^{2i}(Y_{\mathscr{G}}(\mathbb{C}), \mathbb{Z}_2) = \dim H^{2i}(Y_{\mathscr{G}}(\mathbb{C}), \mathbb{Q})$, we have

$$\sum_i (-1)^i \dim H^{2i}(Y_{\mathscr{G}}(\mathbb{C}), \mathbb{Q}) = \sum_i (-1)^i \dim H^i(Y_{\mathscr{G}}(\mathbb{R}), \mathbb{Z}_2).$$

This, since the homology is finitely generated, is equal to the Euler characteristic $\chi(Y_{\mathscr{G}}(\mathbb{R}))$. Therefore our Poincaré series can be specialized to provide series for the Euler characteristic of the corresponding maximal real compact De Concini–Procesi models: it is sufficient to put $q = -1$.

Moreover, we observe that one can obtain series for the Euler characteristic, in a simpler way, directly from the series $\tilde{p}_A$, Q_B and $\tilde{Q}_D$, by removing bad monomials and replacing g_0 with t, and g_i^r with $-\frac{(1+(-1)^r)}{2} t^r$.

We point out that there are other ways to compute the Euler characteristic of these models. For instance, in the A_n case the maximal real compact model can be obtained by gluing $(n+1)!$ permutohedra of dimension $n-1$. Therefore another formula for the Euler characteristic can be obtained by counting the faces of the $(n+1)!$ permutohedra and taking into account their identifications (every face of codimension i is identified with $2^{i+1} - 1$ other i-codimensional faces): this formula involves eulerian numbers (the components of the h-vector of the permutohedron are the eulerian numbers). At the same way, in the B_n and D_n cases, the maximal real compact model can be obtained by gluing polytopes which are generalized permutohedra (see [17], [18], [23]). Therefore the coefficients of the specializations of the series $\tilde{p}_A$, Q_B and Q_D correspond to formulas where the numbers of faces of such polytopes come into play.

References

[1] F. Callegaro - G. Gaiffi, *An explicit description of Coxeter homology complexes*, ISRN Geometry, Volume 2011 (2011).

[2] F. Cohen - T. Lada - J. May, *The homology of iterated loop spaces*, Springer, 1976.

[3] M. Davis - T. Januszkiewicz - R. Scott, *Fundamental group of blow-ups*, Adv. Math 117 (2003), 115–179.

[4] C. De Concini - C. Procesi, *Hyperplane arrangements and holonomy equations*, Selecta Mathematica 1 (1995), 495–535.

[5] C. De Concini - C. Procesi, *Wonderful models of subspace arrangements*, Selecta Mathematica 1 (1995), 459–494.

[6] C. De Concini - C. Procesi, *On the geometry of toric arrangements*, Transform. Groups 10 (2005), 387–422.

[7] C. De Concini - C. Procesi, *Topics in Hyperplane Arrangements, Polytopes and Box-Splines*, Springer, Universitext, 2010.

[8] V. Drinfeld, *On quasi triangular quasi-hopf algebras and a group closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. 2 (1991), 829–860.

[9] P. ETINGOF - A. HENRIQUES - J. KAMNITZER - E. RAINS, *The cohomology ring of the real locus of the moduli space of stable curves of genus* 0 *with marked points*, Annals of Math. 171 (2010), 731–777.

[10] E. FEICHTNER, *De Concini–Procesi arrangement models—a discrete geometer's point of view*, Combinatorial and Computational Geometry, J. E. Goodman, J. Pach, E. Welzl, eds; MSRI Publications 52, Cambridge University Press (2005), 333–360.

[11] E. FEICHTNER - D. KOZLOV, *A desingularization of real differentiable actions of finite groups*, Int. Math. Res. Not. (2005), 881–898.

[12] G. GAIFFI, *Blow ups and cohomology bases for De Concini–Procesi models of subspace arrangements*, Selecta Mathematica 3 (1997), 315–333.

[13] G. GAIFFI, *Compactification of configuration spaces*, Seminari Algebra e Geometria, Scuola Normale Superiore (1999).

[14] A. HENDERSON, *Rational cohomology of the real coxeter toric variety of type A*, arXiv:1011.3860 (2010).

[15] V. A. KRASNOV, *Real algebraically maximal varieties*, Math Notes 73 (2003), 806–812.

[16] P. LAMBRECHTS - V. TURCHIN - I. VOLIC, *Associahedron, cyclohedron, and permutohedron as compatifications of configuration spaces*, Bull. Belg. Math. Soc. Simon Stevin 17 (2010), 303–332.

[17] A. POSTNIKOV, *Permutohedra, associahedra, and beyond*, Int Math Res Notices (2009), 1026–1106.

[18] A. POSTNIKOV - V. REINER - L. WILLIAMS, *Faces of generalized permutohedra*, Documenta Mathematica 13 (2008), 207–273.

[19] E. RAINS, *The homology of real subspace arrangements*, J Topology 3 (4) (2010), 786–818.

[20] S. SUNDARAM - V. WELKER, *Group actions on arrangements of linear subspaces and applications to configuration spaces*, Trans. Amer. Math. Soc 349 (1997), 1389–1420.

[21] A. SZENES - M. VERGNE, *Toric reduction and a conjecture of Batyrev and Materov*, Invent. Math. 158 (2004), 453–495.

[22] S. YUZVINSKY, *Cohomology bases for De Concini–Procesi models of hyperplane arrangements and sums over trees*, Invent. Math. 127 (1997), 319–335.

[23] A. ZELEVINSKI, *Nested complexes and their polyhedral realizations*, Pure and Applied Mathematics Quarterly 2 (2006), 655–671.

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