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Functional Analysis — *A spectral Schwarz lemma, II*, by Edoardo Vesentini.

Abstract. — The spectral Schwarz lemma holding for any operator-valued holomorphic map of the open unit disc $\Delta \subset \mathbb{C}$ into the algebra of all continuous linear operators acting on a complex Banach space is extended to other scalar- and vector-valued gauges of continuous linear operators, with particular attention to the numerical range and the numerical radius of any continuous linear mapping acting on a complex Banach space.

Key words: Banach algebra, subharmonic function, spectrum, mumerical range.

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The main results of [12], devoted to establishing a spectral version of the classical Schwarz inequality holding for a holomorphic operator-valued function of a complex variable, are based on the fact that, given a complex Banach space $\mathscr{E}$[1], and the unital Banach algebra $\mathscr{L}(\mathscr{E})$ of all linear operators $X \in \mathscr{L}(\mathscr{E})$, the spectral radius $X \mapsto \rho(X)$ is (a logarithmic-plurisubharmonic function and therefore) a plurisubharmonic function of X.

This short *addendum* to [12] will develop a similar analysis, for some other gauges associated to X—with special attention to the numerical range $W(X)$ and the numerical radius $v(X)$ when $\mathscr{E}$ is a complex Hilbert space and to their extensions—due to G. Lumer [6], and to B. B. Bonsall and J. Duncan [1]—to the case in which $\mathscr{E}$ is any complex Banach space.

1

Let $B = B_{\mathscr{E}}$ be the open unit ball of a complex Banach space $\mathscr{E}$, and let $\phi : B \to [0,1]$ be a plurisubharmonic function on B such that $\phi(0) = 0$. For any $u \in \partial B$ the function $\varphi_u : \Delta \to [0,1]$ defined on $z \in \Delta$ by $\varphi_u(z) = \phi(zu)$ is subharmonic on Δ, and $\varphi_u(0) = 0$.

According to Lemma 1 in [12], if f is any scalar-valued holomorphic function on Δ such that $f(0) = 0$ and $f(\Delta) \subset \Delta$, then $\varphi_u(f(z)) \le |z|$ for all $z \in \Delta$ and $\varphi_u(f'(0)) \le 1$; if either $\varphi_u(f'(0)) = 1$ or there is $z \in \Delta \backslash \{0\}$ for which $\varphi_u(f(z)) = |z|$, then this latter equality holds for all $z \in \Delta$.

Since $u \in \partial B$ is arbitrary, the following lemma holds (in which $\| \ \|_{\mathscr{E}}$ is the norm in $\mathscr{E}$):

[1] Throughout the following, all normed spaces will be assumed to be complete.

Lemma 1. *If $F : B \to B$ is a holomorphic map such that $F(0) = 0$, then*

$$\phi(F(x)) \le \|x\|_{\mathscr{E}} \quad \forall x \in B$$

and

$$\phi((dF(0))v) \le 1 \quad \forall v \in \partial B.$$

If either $\phi(dF(0)v) = 1$ for some $v \in \partial B$ or there is $x \in B \setminus \{0\}$ for which $\phi(F(x)) = \|x\|_{\mathscr{E}}$, then $\phi(F(zx)) = |z|\, \|x\|_{\mathscr{E}}$ for all $z \in \Delta$.

The first candidate to the role of ϕ is the norm in $\mathscr{E}$, for which Lemma 1 holds with $\phi = \| \ \|_{\mathscr{E}}$.

But also the Carathéodory pseudodistance (see, e.g. [2], pp. 84–85) plays a role, as the following example will show.

Let E be a domain in $\mathscr{E}$. It was shown in [10] (pp. 212–217; see also Theorem 1.4, pp. 480–482, and Theorem 1.10, p. 488, in [11]) that the Carathéodory pseudodistance $c_E(x, y)$ between any two points x, y in E is a (logarithmic) plurisubharmonic function of x and y.

Let the domain E contain the open unit ball B of $\mathscr{E}$, and let $G : E \to E$ be a holomorphic map such that $G(0) = 0$ and $G(B) \subset B$.

Letting $\phi(x) = c_E(0, x)$ for any $x \in E$, then

$$\phi(G(x)) = c_E(0, G(x)) = c_E(G(0), G(x)) \le c_E(0, x) = \phi(x) \quad \forall x \in E.$$

Since, if $x \in B$,

$$\phi(x) = c_E(0, x) \le c_B(0, x) = \|x\|_{\mathscr{E}},$$

then

$$\begin{aligned}\phi(G(x)) &= c_E(0, G(x)) = c_E(G(0), G(x)) \le c_E(0, x)\\ &= \phi(x) \le \|x\|_{\mathscr{E}} \quad \forall x \in B.\end{aligned}$$

As was mentioned at the beginning, the case in which $\mathscr{E}$ is an associative unital Banach algebra $\mathscr{A}$ with no non-zero topologically nilpotent element, and the plurisubharmonic function ϕ is the spectral radius ρ in $\mathscr{A}$ has been considered in [12], showing that,

$$\rho(F(x)) \le \|x\|_{\mathscr{A}} \quad \forall x \in B,$$

and that, if either $\rho(F(x_0)) = \|x_0\|_{\mathscr{A}}$ or $\rho(dF(0)(x_0)) = \|x_0\|_{\mathscr{A}}$ for some $x_0 \in B \setminus \{0\}$, then

$$\rho(F(zx_0)) = |z|\, \|x_0\|_{\mathscr{A}} \quad \forall z \in \Delta.$$

If the Banach space $\mathscr{E}$ is a complex Hilbert space $\mathscr{H}$ with inner product $(\ | \ )_{\mathscr{H}}$, norm $\| \ \|_{\mathscr{H}}$, and if $X \in \mathscr{L}(\mathscr{H})$, the numerical range $W(X) \subset \mathbb{C}$ and the numerical radius $w(X) \in \mathbb{R}_+$ of X are defined by (see, e.g.: [8], p. 130; [3], pp. 108–113. 160–161, 317–321):

$$W(X) = \{(X\xi \mid \xi)_{\mathscr{H}} : \xi \in \partial B_{\mathscr{H}}\}$$

and

$$w(X) = \sup\{|\zeta| : \zeta \in W(X)\}.$$

Let now f be a holomorphic map of the open unit disc Δ of $\mathbb{C}$ into $\mathscr{L}(\mathscr{H})$. It was shown in [9] (Theorem 4.4 and Corollary 4.5; see also [12], Theorem 2) that the function $\Delta \ni z \mapsto w(f(z))$ is (logarithmically subharmonic, hence) subharmonic on Δ: which implies that, the numerical radius $w(X)$ of any $X \in \mathscr{L}(\mathscr{H})$ is a plurisubharmonic function of X. As a consequence of this fact, the following proposition holds:

Proposition 1. *If F is a holomorphic map of $B_{\mathscr{L}(\mathscr{H})}$ into itself and if $F(0) = 0$, then*

$$w(F(X)) \le \|X\|_{\mathscr{L}(\mathscr{H})} \quad \forall X \in B_{\mathscr{L}(\mathscr{H})}$$

and

$$w((dF(0))(X)) \le 1 \quad \forall X \in B_{\mathscr{L}(\mathscr{H})}.$$

If $w((dF(0))(X)) = 1$ or if there exists $X \in B_{\mathscr{L}(\mathscr{H})} \setminus \{0\}$ for which $w(F(X)) = \|X\|_{\mathscr{L}(\mathscr{H})}$, then $w(F(zX)) = |z| \, \|X\|_{\mathscr{L}(\mathscr{H})}$ for all $z \in \Delta$.

2

Thanks to the work of G. Lumer [6] and B. B. Bonsall and J. Duncan [1], the notions of numerical range and numerical radius have found extensions from Hilbert spaces to Banach spaces.

We will briefly review now these extensions in order to show how Proposition 1 can be re-formulated when the Hilbert space $\mathscr{H}$ is replaced by the Banach space $\mathscr{E}$.

Let $\mathscr{A}$ be an associative unital Banach algebra, let $\mathscr{A}'$ be the topological dual of $\mathscr{A}$, $B = B(\mathscr{A})$ and let $\partial B = \partial B(\mathscr{A})$ the open unit ball and the unit sphere in $\mathscr{A}$.

By the Hahn–Banach theorem, for any $x \in \partial B(\mathscr{A})$, the set

$$D(\mathscr{A}, x) := \{\vartheta \in \mathscr{A}' : \langle x, \vartheta \rangle = 1 = \|\vartheta\|_{\mathscr{A}}\} \subset \mathscr{A}'$$

is not empty.

Letting, for $a \in \mathscr{A}$ and $x \in \partial B(\mathscr{A})$,

$$\begin{aligned} V(\mathscr{A}, a, x) &= \{\langle ax, \vartheta \rangle : \vartheta \in D(\mathscr{A}, x)\} \\ &= \{\langle ax, \vartheta \rangle : x \in \partial B, \vartheta \in \mathscr{A}', \langle x, \vartheta \rangle = 1 = \|\vartheta\|_{\mathscr{A}}\}, \end{aligned}$$

the set

$$V(\mathscr{A}, a) = \bigcup\{V(\mathscr{A}, a, x) : x \in \partial B\}$$
$$= \bigcup\{\langle ax, \vartheta\rangle : x \in \partial B, \vartheta \in \mathscr{A}', \langle x, \vartheta\rangle = 1 = \|\vartheta\|_{\mathscr{A}'}\},$$

and the real number

$$v(a) = \sup\{|\zeta| : \zeta \in V(\mathscr{A}, a)\}$$
$$= \sup\{|\langle ax, \vartheta\rangle| : x \in \partial B, \vartheta \in \mathscr{A}', \langle x, \vartheta\rangle = 1 = \|\vartheta\|_{\mathscr{A}'}\} \geq 0$$

are called in [1] the *numerical range* and the *numerical radius* of a.

Since, for any $x \in \partial B$,

$$\sup\{|\langle ax, \vartheta\rangle| : \vartheta \in \mathscr{A}', \langle x, \vartheta\rangle = \|\vartheta\|_{\mathscr{A}'} = 1\} \leq \|a\|_{\mathscr{A}},$$

then $0 \leq v(a) \leq \|a\|_{\mathscr{A}}$.

For any choice of $a, b \in \mathscr{A}$, $\alpha, \beta \in \mathbb{C}$,

$$V(\mathscr{A}, \alpha a + \beta b) = \alpha V(\mathscr{A}, a) + \beta V(\mathscr{A}, b),$$

and

$$V(\mathscr{A}, \alpha 1_{\mathscr{A}} + \beta b) = \alpha + \beta V(\mathscr{A}, b),$$
$$V(\mathscr{A}, a, 1_{\mathscr{A}}) = \{\langle a, \lambda\rangle : \lambda \in D(\mathscr{A}, 1_{\mathscr{A}})\}.$$

Since

$$D(\mathscr{A}, 1_{\mathscr{A}}) = \{\vartheta \in \mathscr{A}' : \langle 1_{\mathscr{A}}, \vartheta\rangle = 1 = \|\vartheta\|_{\mathscr{A}}\}$$
$$= \{\vartheta \in \mathscr{A}' : \langle 1_{\mathscr{A}}, \vartheta\rangle = 1 \geq \|\vartheta\|_{\mathscr{A}}\},$$

$D(\mathscr{A}, 1_{\mathscr{A}})$ is a convex subset of $\mathscr{A}'$.

For any $a \in \mathscr{A}$, $V(\mathscr{A}, a, 1_{\mathscr{A}})$ is the image of $D(\mathscr{A}, 1_{\mathscr{A}})$ by the weak*-continuous map $\vartheta \mapsto \langle a, \vartheta\rangle$, and is therefore a compact subset of $\mathbb{C}$. That proves, [1]:

Theorem 1. *For each $a \in \mathscr{A}$, $V(\mathscr{A}, a)$ is a compact, convex set in $\mathbb{C}$.*

Let $f : \Delta \to \mathscr{A}$ be a holomorphic map such that $f(0) = 0$, $\|f(z)\|_{\mathscr{A}} \leq 1$ for all $z \in \Delta$, and therefore

$$v(f(z)) \leq 1 \quad \forall z \in \Delta.$$

Lemma 2. *The function $v \circ f : \Delta \ni z \mapsto v(f(z))$ is logarithmically subharmonic.*

Proof. The proof amounts to showing that for any $\alpha \in \mathbb{C}$, the function

$$\Delta \ni z \mapsto |e^{\alpha z}| v(f(z)) = \sup\{|\zeta| : \zeta \in V(\mathscr{A}, e^{\alpha z} f(z))\}$$

is subharmonic on Δ ([7]).

Let $z_0 \in \Delta$, let c and c' be positive constants for which

$$|e^{\alpha z_0}|v(f(z_0)) < c' < c$$

and let ε be such that

$$0 < \varepsilon < c' - |e^{\alpha z_0}|v(f(z_0)).$$

There exists $\delta > 0$ for which, whenever $z \in \Delta$ satisfies the inequality $|z - z_0| < \delta$, then

$$|e^{\alpha z}v(f(z)) - e^{\alpha z_0}v(f(z_0))| < \varepsilon,$$

and therefore, if $|z - z_0| < \delta$,

$$\begin{aligned} |e^{\alpha z}v(f(z))| &\leq \big|\,|e^{\alpha z}v(f(z))| - |e^{\alpha z_0}v(f(z_0))|\,\big| + |e^{\alpha z_0}v(f(z_0))| \\ &\leq |e^{\alpha z}v(f(z)) - e^{\alpha z_0}v(f(z_0))| + |e^{\alpha z_0}v(f(z_0))| \\ &< \varepsilon + |e^{\alpha z_0}v(f(z_0))| < c' < c. \end{aligned}$$

For $\alpha = 0$, that shows, in particular, that the map $z \mapsto v(f(z))$ is upper semi-continuous. □

Choosing $\mu = v$ in Lemma 1 of [12] yields

Theorem 2. *If $f(0) = 0$ and $v(f(z)) \leq 1$ for all $z \in \Delta$, then*

$$v(f(z)) \leq |z| \quad \forall z \in \Delta, \quad \textit{and} \quad v(f'(0)) \leq 1.$$

If $v(f'(0)) = 1$ or if there is $z \in \Delta \backslash \{0\}$ such that $v(f(z)) = |z|$, this latter equality holds for all $z \in \Delta$, and the (non-empty) intersection $V(\mathscr{A}, f(z))/\{z\}) \cap \partial\Delta$ is independent of $z \in \Delta \backslash \{0\}$.

Corollary 1. *The numerical radius*

$$\mathscr{A} \ni a \mapsto v(a)$$

is a logarithmically plurisubharmonic function of a.

Letting $B = B_{\mathscr{A}}$ be now the open unit ball of $\mathscr{A}$, Lemma 1 yields the following "Schwarz lemma" for v:

Lemma 3. *If F maps holomorphically B into $\bar{B}$ and $F(0) = 0$, then*

$$v(F(x)) \leq \|x\|_{\mathscr{A}} \quad \forall x \in B_{\mathscr{A}}$$

and

$$v((dF(0))(w) \leq 1 \quad \forall w \in \partial B_{\mathscr{A}}.$$

If either $v((dF(0))w) = 1$ *for some* $w \in \partial B_{\mathscr{A}}$ *or if there is* $w \in B_{\mathscr{A}} \backslash \{0\}$ *for which* $v(F(w)) = \|w\|_{\mathscr{A}}$, *then* $v(F(zw)) = |z|\,\|w\|_{\mathscr{A}}$ *for all* $z \in \Delta$.

Let now X be a continuous linear operator acting on a complex Banach space $\mathscr{E}$. As was noted in [1], a more natural numerical range of X—denoted by $V(X)$ and called *spatial numerical range* of X—can be defined directly in terms of $\mathscr{E}$ and X without involving the algebra $\mathscr{L}(\mathscr{E})$:

$$V(X) = \{\langle Xx; \vartheta\rangle : x \in \mathscr{E}, \vartheta \in \mathscr{E}', \langle x, \vartheta\rangle = 1 = \|x\|_{\mathscr{E}} = \|\vartheta\|_{\mathscr{E}'}\}.$$

Since ([1], Theorem 3, pp. 83–84) its closed convex hull is

$$\overline{co}(V(X)) = V(\mathscr{L}(\mathscr{E}), X),$$

then

$$\sup\{|\zeta| : \zeta \in V(X)\} = v(X).$$

A definition of the numerical range closer to the original one due to Toeplitz and Hausdorff in the case of Hilbert spaces, can be given in terms of the *semi-inner-products* introduced in [6] by G. Lumer, according to whom a semi-inner-product space is a mapping $x, y \mapsto [x, y]$ of $\mathscr{E} \times \mathscr{E}$ into $\mathbb{C}$ such that the mapping $\mathscr{E} \ni x \mapsto [x, y]$ is linear for all $y \in \mathscr{E}$;

$$[x, x] > 0 \quad \forall x \in \mathscr{E}\backslash\{0\}$$
$$|[x, y]|^2 \leq [x, x][y, y] \quad \forall x, y \in \mathscr{E}.$$

By Theorem 2 of [6] (p. 31), the function $\mathscr{E} \ni x \mapsto [x, x]^{1/2}$ is a norm on $\mathscr{E}$ and there is a semi-inner-product $[\,,]$ (in general, infinitely many semi-inner-products) in $\mathscr{E}$ such that

$$\|x\|_{\mathscr{E}}^2 = [x, x] \quad \forall x \in \mathscr{E}. \tag{1}$$

Thus, by the Hahn–Banach theorem, every normed linear space can be made into a semi-inner-product space.

The relationship between semi-inner-products and inner products is clarified by Theorem 3 in [6], according to which a semi-inner-product is an inner product if, and only if, the norm defined by (1) satisfies the parallelogram law; a Hilbert space can be made into a semi-inner-product-space in a unique way.

Given a semi-inner-product $[\,,]$ satisfying (1), the numerical range $W_{[,]}(X)$ of $X \in \mathscr{L}(\mathscr{E})$ is defined by

$$W_{[,]}(X) = \{[Xx, x] : x \in \mathscr{E}, [x, x] = 1\}. \tag{2}$$

It turns out ([1], Theorem 8, p. 86) that

$$W_{[,]}(X) \subset V(X)$$

and, in fact, $V(X)$ is the union of the family $\mathscr{W}(X)$ of the numerical ranges $W_{[\,,\,]}(X)$ represented by (2) for all semi-inner-products $[\,,\,]$ satisfying (1).

Furthermore, if the semi-inner product $[\,,\,]$ satisfies (1), then

$$v(X) = \sup\{|\zeta| : \zeta \in W(X)\} = \sup\{|\zeta| : \zeta \in W_{[\,,\,]}(X)\}.$$

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