Rend. Lincei Mat. Appl. 23 (2012), 97–103
DOI 10.4171/RLM/619

Partial Differential Equations — *A semilinear problem with a $W_0^{1,1}$ solution*, by

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ABSTRACT. — We study a degenerate elliptic equation, proving the existence of a $W_0^{1,1}$ distributional solution.

KEY WORDS: Elliptic equations, $W^{1,1}$ solutions, Degenerate equations.

MSC 1991 CLASSIFICATION: 35J61, 35J70, 35J75.

In the study of elliptic problems, it is quite standard to find solutions belonging either to $BV(\Omega)$ or to $W^{1,s}(\Omega)$, with $s > 1$. In this paper we prove the existence of a $W_0^{1,1}$ distributional solution for the following boundary value problem:

$$(1) \qquad \begin{cases} -\operatorname{div}\Big(\dfrac{a(x)\nabla u}{(1+b(x)|u|)^2}\Big) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $a(x)$, $b(x)$ are measurable functions such that

$$(2) \qquad 0 < \alpha \le a(x) \le \beta, \quad 0 \le b(x) \le B,$$

with $\alpha, \beta \in \mathbb{R}^+$, $B \in \mathbb{R}$ and

$$(3) \qquad f(x) \text{ belongs to } L^2(\Omega).$$

We are going to prove that problem (1) has a distributional solution u belonging to the non-reflexive Sobolev space $W_0^{1,1}(\Omega)$.

Problems like (1) have been extensively studied in the past. In [4], existence and regularity results were obtained for

$$(4) \qquad \begin{cases} -\operatorname{div}\Big(\dfrac{a(x)\nabla u}{(1+|u|)^\theta}\Big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \theta \le 1$ and f belongs to $L^m(\Omega)$ for some $m \ge 1$. A whole range of existence results was proved, yielding solutions belonging to some Sobolev space $W_0^{1,q}(\Omega)$, with $q = q(2,m) \le 2$ or entropy solutions. In the case where $\theta > 1$ a non-existence result for constant sources has been proved in [1].

As pointed out in [2], existence of solutions can be recovered for any value of $\theta > 0$, by adding a lower order term of order zero. If we consider the problem

$$(5) \qquad \begin{cases} -\mathrm{div}\Big(\dfrac{a(x)\nabla u}{(1+|u|)^2}\Big) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with f in $L^m(\Omega)$, then the following results can be proved (see [2] and [5]):

i) if $2 < m < 4$, then there exists a distributional solution in $W_0^{1,2m/m+2}(\Omega) \cap L^m(\Omega)$;
ii) if $1 \le m \le 2$, then there exists an entropy solution in $L^m(\Omega)$ whose gradient belongs to the Marcinkiewicz space $M^{m/2}(\Omega)$.

In this paper we deal with the borderline case $m = 2$, improving the above results as follows.

Theorem 1. *Assume* (2) *and* (3). *Then there exists a distributional solution* $u \in W_0^{1,1}(\Omega) \cap L^2(\Omega)$ *to problem* (1), *in the sense that*

$$\int_\Omega \frac{a(x)\nabla u \cdot \nabla\varphi}{(1+b(x)|u|)^2} + \int_\Omega u\varphi = \int_\Omega f\varphi,$$

for all $\varphi \in W_0^{1,\infty}(\Omega)$.

Remark 2. If the operator is nonlinear with respect to the gradient, existence of distributional solutions are studied in [3].

Proof of Theorem 1.

Step 1. We begin by approximating our boundary value problem (1) and we consider a sequence $\{f_n\}$ of $L^\infty(\Omega)$ functions such that f_n strongly converges to f in $L^2(\Omega)$, and $|f_n| \le |f|$ for every n in $\mathbb{N}$. The same technique of [2] assures the existence of a solution u_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of

$$(6) \qquad \begin{cases} -\mathrm{div}\Big(\dfrac{a(x)\nabla u_n}{(1+b(x)|u_n|)^2}\Big) + u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Indeed, let $M_n = \|f_n\|_{L^\infty(\Omega)} + 1$, and consider the problem

$$(7) \qquad \begin{cases} -\mathrm{div}\Big(\dfrac{a(x)\nabla w}{(1+b(x)|T_{M_n}(w)|)^2}\Big) + w = f_n & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

where $T_k(s) = \max(-k, \min(s,k))$ for $k \ge 0$ and s in $\mathbb{R}$. The existence of a weak solution w in $H_0^1(\Omega)$ of (7) follows from Schauder's theorem. Choosing

$(|w| - \|f_n\|_{L^\infty(\Omega)})_+ \operatorname{sgn}(w)$ as a test function we obtain, dropping the nonnegative first term,

$$\int_\Omega |w|(|w| - \|f_n\|_{L^\infty(\Omega)})_+ \leq \int_\Omega \|f_n\|_{L^\infty(\Omega)}(|w| - \|f_n\|_{L^\infty(\Omega)})_+.$$

Thus,

$$\int_\Omega (|w| - \|f_n\|_{L^\infty(\Omega)})(|w| - \|f_n\|_{L^\infty(\Omega)})_+ \leq 0,$$

so that $|w| \leq \|f_n\|_{L^\infty(\Omega)} < M_n$. Therefore, $T_{M_n}(w) = w$, and w is a bounded weak solution of (6).

Step 2. We prove some *a priori* estimates on the sequence $\{u_n\}$. Let $k \geq 0$, $i > 0$, and let $\psi_{i,k}(s)$ be the function defined by

$$\psi_{i,k}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq k, \\ i(s-k) & \text{if } k < s \leq k + \frac{1}{i}, \\ 1 & \text{if } s > k + \frac{1}{i}, \\ \psi_{i,k}(s) = -\psi_{i,k}(-s) & \text{if } s < 0. \end{cases}$$

Note that

$$\lim_{i \to +\infty} \psi_{i,k}(s) = \begin{cases} 1 & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ -1 & \text{if } s < -k. \end{cases}$$

We choose $|u_n|\psi_{i,k}(u_n)$ as a test function in (6), and we obtain

$$\int_\Omega \frac{a(x)|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} |\psi_{i,k}(u_n)| + \int_\Omega \frac{a(x)|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} \psi'_{i,k}(u_n)|u_n| + \int_\Omega u_n|u_n|\psi_{i,k}(u_n)$$
$$= \int_\Omega f_n|u_n|\psi_{i,k}(u_n).$$

Since $\psi'_{i,k}(s) \geq 0$, we can drop the second term; using (2), and the assumption $|f_n| \leq |f|$, we have

$$\alpha \int_\Omega \frac{|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} |\psi_{i,k}(u_n)| + \int_\Omega u_n|u_n|\psi_{i,k}(u_n) \leq \int_\Omega |f|\,|u_n|\,|\psi_{i,k}(u_n)|.$$

Letting i tend to infinity, we thus obtain, by Fatou's lemma (on the left hand side) and by Lebesgue's theorem (on the right hand side, recall that u_n belongs to $L^\infty(\Omega)$),

$$\alpha \int_{\{|u_n| \geq k\}} \frac{|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} + \int_{\{|u_n| \geq k\}} |u_n|^2 \leq \int_{\{|u_n| \geq k\}} |f|\,|u_n|. \tag{8}$$

Dropping the nonnegative first term in (8) and using Hölder's inequality on the right hand side, we obtain

$$\int_{\{|u_n|\geq k\}} |u_n|^2 \leq \left[\int_{\{|u_n|\geq k\}} |f|^2\right]^{1/2} \left[\int_{\{|u_n|\geq k\}} |u_n|^2\right]^{1/2}.$$

Simplifying equal terms we thus have

$$\int_{\{|u_n|\geq k\}} |u_n|^2 \leq \int_{\{|u_n|\geq k\}} |f|^2. \tag{9}$$

For $k = 0$, (9) gives

$$\int_\Omega |u_n|^2 \leq \int_\Omega |f|^2, \tag{10}$$

so that $\{u_n\}$ is bounded in $L^2(\Omega)$. This fact implies in particular that

$$\lim_{k\to+\infty} \operatorname{meas}(\{|u_n| \geq k\}) = 0, \quad \text{uniformly with respect to } n. \tag{11}$$

From (8), written for $k = 0$, dropping the nonnegative second term and using that $b(x) \leq B$, we have

$$\alpha \int_\Omega \frac{|\nabla u_n|^2}{(1 + B|u_n|)^2} \leq \int_\Omega |f|\,|u_n|.$$

Hölder's inequality on the right hand side then gives

$$\alpha \int_\Omega \frac{|\nabla u_n|^2}{(1 + B|u_n|)^2} \leq \left[\int_\Omega |f|^2\right]^{1/2} \left[\int_\Omega |u_n|^2\right]^{1/2},$$

so that, by (10), we infer that

$$\alpha \int_\Omega \frac{|\nabla u_n|^2}{(1 + B|u_n|)^2} \leq \int_\Omega |f|^2. \tag{12}$$

Step 3. We prove that, up to subsequences, the sequence $\{u_n\}$ strongly converges in $L^2(\Omega)$ to some function u.

From (12) we deduce that $v_n = \log(1 + B|u_n|)\operatorname{sgn}(u_n)$ is bounded in $H_0^1(\Omega)$. Therefore, up to subsequences, it converges to some function v weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω. If we define $u = \frac{e^{|v|}-1}{B}\operatorname{sgn}(v)$, then u_n converges almost everywhere to u in Ω. Let now E be a measurable subset of Ω; then

$$\begin{aligned}\int_E |u_n|^2 &\leq \int_{E\cap\{|u_n|\geq k\}} |u_n|^2 + \int_{E\cap\{|u_n|<k\}} |u_n|^2 \\ &\leq \int_{\{|u_n|\geq k\}} |f|^2 + k^2 \operatorname{meas}(E),\end{aligned}$$

where we have used (9) in the last passage. Thanks to (11), we may choose k large enough so that the first integral is small, uniformly with respect to n; once k is chosen, we may choose the measure of E small enough such that the second term is small. Thus, the sequence $\{u_n^2\}$ is equiintegrable and so, by Vitali's theorem, u_n strongly converges to u in $L^2(\Omega)$.

Step 4. We prove that, up to subsequences, the sequence $\{u_n\}$ weakly converges to u in $W_0^{1,1}(\Omega)$.

Let again E be a measurable subset of Ω, and let i be in $\{1,\dots,N\}$. Then

$$\begin{aligned}\int_E |\partial_i u_n| \leq \int_E |\nabla u_n| &= \int_E \frac{|\nabla u_n|}{1+B|u_n|}(1+B|u_n|) \\ &\leq \left[\int_E \frac{|\nabla u_n|^2}{(1+B|u_n|)^2}\right]^{1/2} \left[\int_E (1+B|u_n|)^2\right]^{1/2} \\ &\leq \left[\frac{1}{\alpha}\int_\Omega |f|^2\right]^{1/2} \left[2\operatorname{meas}(E) + 2B^2\int_E |u_n|^2\right]^{1/2},\end{aligned}$$

where we have used (12) in the last passage. Since the sequence $\{u_n\}$ is compact in $L^2(\Omega)$, we have that the sequence $\{\partial_i u_n\}$ is equiintegrable. Thus, by Dunford-Pettis theorem, and up to subsequences, there exists Y_i in $L^1(\Omega)$ such that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$. Since $\partial_i u_n$ is the distributional derivative of u_n, we have, for every n in $\mathbb{N}$,

$$\int_\Omega \partial_i u_n \varphi = -\int_\Omega u_n \partial_i \varphi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

We now pass to the limit in the above identities, using that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$, and that u_n strongly converges to u in $L^2(\Omega)$; we obtain

$$\int_\Omega Y_i \varphi = -\int_\Omega u \partial_i \varphi, \quad \forall \varphi \in C_0^\infty(\Omega),$$

which implies that $Y_i = \partial_i u$, and this result is true for every i. Since Y_i belongs to $L^1(\Omega)$ for every i, u belongs to $W_0^{1,1}(\Omega)$, as desired.

Note now that, since $s \mapsto \log(1+Bs)$ is Lipschitz continuous on $\mathbb{R}^+$, and u belongs to $W_0^{1,1}(\Omega)$, by the chain rule we have

$$\nabla[\log(1+B|u|)\operatorname{sgn}(u)] = \frac{\nabla u}{1+B|u|}, \quad \text{almost everywhere in } \Omega.$$

Hence, from the weak convergence of v_n to v in $H_0^1(\Omega)$ we deduce that

$$\lim_{n\to+\infty} \frac{\nabla u_n}{1+B|u_n|} = \frac{\nabla u}{1+B|u|}, \quad \text{weakly in } (L^2(\Omega))^N. \tag{13}$$

Step 5. We now pass to the limit in the approximate problems (6).

Both the lower order term and the right hand side give no problems, due to the strong convergence of u_n to u, and of f_n to f, in $L^2(\Omega)$.

For the operator term we can write, if φ belongs to $W_0^{1,\infty}(\Omega)$,

$$\int_\Omega \frac{a(x)\nabla u_n \cdot \nabla\varphi}{(1+b(x)|u_n|)^2} = \int_\Omega a(x)\frac{\nabla u_n}{1+B|u_n|}\cdot\nabla\varphi\frac{1+B|u_n|}{(1+b(x)|u_n|)^2}. \tag{14}$$

In the last integral, the first term is fixed in $L^\infty(\Omega)$, the second is weakly convergent in $(L^2(\Omega))^N$ by (13), the third is fixed in $(L^\infty(\Omega))^N$, and the fourth is strongly convergent in $L^2(\Omega)$, since is bounded from above by $1+B|u_n|$, which is compact in $L^2(\Omega)$. Therefore, we can pass to the limit to have that

$$\lim_{n\to+\infty} \int_\Omega \frac{a(x)\nabla u_n\cdot\nabla\varphi}{(1+b(x)|u_n|)^2} = \int_\Omega \frac{a(x)\nabla u\cdot\nabla\varphi}{(1+b(x)|u|)^2},$$

as desired.

REMARK 3. Note that if $b(x)\geq b>0$ in Ω, then we can choose test functions φ in $H_0^1(\Omega)$. Indeed,

$$0\leq \frac{1+B|u_n|}{(1+b(x)|u_n|)^2} \leq \frac{1+B|u_n|}{(1+b|u_n|)^2} \leq C(B,b),$$

for some nonnegative constant $C(B,b)$, so that we can rewrite (14) as

$$\int_\Omega \frac{a(x)\nabla u_n\cdot\nabla\varphi}{(1+b(x)|u_n|)^2} = \int_\Omega a(x)\frac{\nabla u_n}{1+B|u_n|}\cdot\frac{\nabla\varphi(1+B|u_n|)}{(1+b(x)|u_n|)^2},$$

with the first term fixed in $L^\infty(\Omega)$, the second weakly convergent in $(L^2(\Omega))^N$, and the third strongly convergent in the same space by Lebesgue's theorem.

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Received 28 April 2011,
and in revised form 19 September 2011.

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