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Mathematical Physics — *Phragmén-Lindelöf alternative of exponential type for the solutions of a fourth order dispersive equation*, by Ramon Quintanilla and Giuseppe Saccomandi, presented on 11 November 2011 by Tommaso Ruggeri.

Abstract. — In a recent paper (R. Quintanilla, G. Saccomandi. Quarterly Applied Mathematics, **44**, (2006) 547–560.) we investigated the spatial behavior of a linear equation of fourth order which models several mechanical situations where dispersive effects are taken into account. We proved a polynomial decay estimate for the solutions of this equation. In the present note we improve this result and we show a Phragmén-Lindelöf alternative of exponential type.

Key words: Phragmén-Lindelöf alternative, spatial decay estimates, dispersive effects.

MSC 2010 Classification: 35Q72.

1. Introduction and basic equations

It has been pointed out recently [1] that, on studying solitary waves propagating on the free surface of a constant depth fluid, Boussinesq derived initially a well-posed equation which contained a *mixed fourth order* derivative for the dispersion alongside with the purely spatial fourth order derivative. Then, in an attempt to find an analytical solution he went on to replace the time derivatives in the mixed derivative term by purely spatial ones using the ansatz $u_{,t} = -cu_{,x}$ which is true for steady linear waves propagating with phase speed c. If this unnecessary ansatz was not applied by Boussinesq the form of his celebrated equation in one spatial dimension would it be in the linear regime

$$u_{,tt} - c_0^2 u_{,xx} = (\beta_1 u_{,tt} - \beta_2 u_{,xx})_{,xx}, \tag{1.1}$$

where $\beta_1, \beta_2 > 0$.

The equation we usually denote as the Boussinesq equation (BE) is obtained from (1.1) considering $\beta_1 = 0$ and $\beta_2 < 0$. This is an *incorrect* equation called for this reason the *bad* BE. Here incorrect means that the corresponding initial-value problem is ill-posed in the sense of Hadamard. To make the coefficient β_2 positive we need to add a sufficiently strong surface tension to the model.

On the other hand Rosenau [8] considering the long-wave-length limit of a chain of atoms interacting nonlinearly and considering a special expansion in the discrete-continuum approximation was able to deduce (in the linear regime) the equation (1.1) with $\beta_2 = 0$ which is denoted the regularized long-wave (BE) equation and which is correct in the sense of Hadamard. Several papers have shown that this approach is interesting and effective to introduce mesoscopic informations in mathematical models of solid mechanics [2, 4], fluid mechanics [5] and in the context of the heat conduction equation with two temperatures and without energy dissipation [6].

For these reasons in [7], we investigated the spatial behavior of the dispersive fourth-order linear equation

$$u_{,tt} - \varepsilon \triangle u_{,tt} = a \triangle u. \tag{1.2}$$

Here ε and a are two positive constants. We obtained a *polynomial* decay estimate for the solutions whenever we assume the decay of solutions. Now, it is clear that if we assume solutions of (1.2) in the form

$$u(x, y; t) = \exp(-mx) \cos(\omega t - ny),$$

(for the sake of brevity we are considering only two spatial dimensions), it is easily checked that exponential decaying solutions are possible. In [7] the exponential decay was established also for the harmonic vibration problem in a cylindrical domain. It is interesting therefore to improve the result in [7] about the polynomial decay estimate for the whole class of solutions of (1.2). The aim of the present note is exactly to show a Phragmén-Lindelöf alternative of exponential type for the solutions of this equation.

In this article we use the summation and differentiation conventions. Summation over repeated indexes is assumed and differentiation in the direction x_k is denoted by $,k$.

As we want to study the spatial behavior, we will denote by R the cylinder $(0, \infty) \times D$, where D is a two dimensional bounded domain such that the boundary ∂D is smooth enough to apply the divergence theorem. Let $D(z)$ be the cross section of the points in R such that $x_1 = z$, and let $R(z)$ be the points of the cylinder such that $x_1 > z$. By $R(z_0, z)$, we denote the points in R such that the first component is greater than z_0 and lower than z. The equations we study here are determined on the semi-infinite cylinder R. u is constrained to be zero on the lateral sides of the cylinder. Thus, we add to our equation the following conditions

$$u = 0, \quad \mathbf{x} \in [0, \infty) \times \partial D. \tag{1.3}$$

Moreover, we impose boundary conditions on the finite end of the cylinder. Thus, we take as assumptions

$$u(0, x_2, x_3, t) = f(x_2, x_3, t), \quad (x_2, x_3) \in D. \tag{1.4}$$

To have a well determined problem we need to impose initial conditions. Here, we assume null initial conditions. Thus

$$u(\mathbf{x}, 0) = u_{,t}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in R. \tag{1.5}$$

2. Phragmén-Lindelöf alternative

In this Section we study the spatial asymptotic behavior of the solutions of the problem determined by equation, boundary conditions, and initial conditions proposed previously. In fact, we are going to prove that either the measure on the solutions

$$\int_0^t \int_{R(0,z)} \left(\left[|\nabla u|^2 + a^{-1} u_{,s}^2 + \frac{2\varepsilon}{a} |\nabla u_{,s}|^2 + \frac{\varepsilon^2}{a} (\triangle u_{,s})^2 + \varepsilon (\triangle u)^2 \right] \right) dv\, ds, \quad z \geq 0, \tag{2.1}$$

grows in a exponential way or the function

$$\mathscr{E}(z,t) = \frac{1}{2} \int_0^t \int_{R(z)} \left(\left[|\nabla u|^2 + a^{-1} u_{,s}^2 + \frac{2\varepsilon}{a} |\nabla u_{,s}|^2 + \frac{\varepsilon^2}{a} (\triangle u_{,s})^2 + \varepsilon (\triangle u)^2 \right] \right) dv\, ds, \tag{2.2}$$

decays in a exponential way.

Our analysis starts noting that the following relations

$$(u_{,t} u_{,i})_{,i} = \frac{1}{2} \frac{d}{dt} (|\nabla u|^2 + a^{-1} (u_{,t} - \varepsilon \triangle u_{,t})^2 + \varepsilon (\triangle u)^2), \tag{2.3}$$

$$(u_{,t} u_{,ti})_{,i} = |\nabla u_{,t}|^2 + u_{,t} \triangle u_{,t}, \tag{2.4}$$

are satisfied for every solution of the equation (1.2).

We consider the function

$$\Phi(z,t) = -\int_0^t \int_{D(z)} \left((t-s) u_{,s} u_{,1} + \frac{\varepsilon}{a} u_{,s} u_{,s1} \right) da\, ds. \tag{2.5}$$

Using the divergence theorem and the initial and boundary conditions from (2.5) we obtain

$$\Phi(z,t) = \Phi(z_0,t) - \frac{1}{2} \int_0^t \int_{R(z_0,z)} \left(\left[|\nabla u|^2 + a^{-1} u_{,s}^2 + \frac{2\varepsilon}{a} |\nabla u_{,s}|^2 + \frac{\varepsilon^2}{a} (\triangle u_{,s})^2 + \varepsilon (\triangle u)^2 \right] \right) dv\, ds \quad z \geq z_0 \geq 0. \tag{2.6}$$

If $\Phi(z,t) \to 0$ as $z \to \infty$, then

$$\Phi(z,t) = \frac{1}{2} \int_0^t \int_{R(z)} \left(\left[|\nabla u|^2 + a^{-1} u_{,s}^2 + \frac{2\varepsilon}{a} |\nabla u_{,s}|^2 + \frac{\varepsilon^2}{a} (\triangle u_{,s})^2 + \varepsilon (\triangle u)^2 \right] \right) dv\, ds. \tag{2.7}$$

Moreover, by a simple differentiation,

$$\frac{\partial \Phi}{\partial z}(z,t) = -\frac{1}{2}\int_0^t \int_{D(z)} \left(\left[|\nabla u|^2 + a^{-1}u_{,s}^2 + \frac{2\varepsilon}{a}|\nabla u_{,s}|^2 + \frac{\varepsilon^2}{a}(\triangle u_{,s})^2 + \varepsilon(\triangle u)^2 \right] \right) da\, ds. \tag{2.8}$$

The next step is to estimate the absolute value of $\Phi(z,t)$ in terms of its spatial derivative to obtain an inequality of the type

$$|\Phi(z,t)| \leq -\lambda(t)\frac{\partial \Phi}{\partial z}. \tag{2.9}$$

Using the Holder inequality it is possible to obtain

$$|\Phi| \leq t\Big(\int_0^t \int_D u_{,s}^2\, da\, ds\Big)^{1/2}\Big(\int_0^t \int_D u_{,1}^2\, da\, ds\Big)^{1/2} + \frac{\varepsilon}{a}\Big(\int_0^t \int_D u_{,s}^2\, da\, ds\Big)^{1/2}\Big(\int_0^t \int_D u_{,s1}^2\, da\, ds\Big)^{1/2}. \tag{2.10}$$

After the use of the arithmetic-geometric mean inequality we obtain the existence of two positive constants A, B such that

$$|\Phi| \leq -(At+B)\frac{\partial \Phi}{\partial z}: \tag{2.11}$$

which is the estimate (2.9). This inequality is well-known and often used in the study of spatial decay estimates for partial differential equation. This is because it implies that

$$\Phi \leq -(At+B)\frac{\partial \Phi}{\partial z} \quad \text{and} \quad -\Phi \leq -(At+B)\frac{\partial \Phi}{\partial z}, \tag{2.12}$$

and from these statements we obtain an alternative of Phragmén-Lindelöf type which states (see [3]) that the solutions either grow exponentially for z sufficiently large with the measure given at (2.1) or the function (2.2) decays exponentially in the form

$$\mathscr{E}(z,t) \leq \mathscr{E}(0,t)\exp(-(At+B)^{-1}z): \tag{2.13}$$

for all $z \geq 0$.

In summary the estimates just derived allow to state the following result:

Theorem 2.1. *Let u be a solution of the initial-boundary value problem determined by* (1.2), (1.3)–(1.5). *Then, either the function* (2.1) *becomes exponentially* *unbounded when z goes to infinity or the measure* (2.2) *satisfies the spatial decay estimate* (2.13).

It is important to emphasize that the Phragmén-Lindelöf result here obtained is quite different from the one usually obtained in the framework of the classical hyperbolic problems.

3. The amplitude term

To have a more detailed knowledge and understanding of the decay estimates here proposed, we give an estimate for the term $\mathscr{E}(0,t)$ which is the amplitude term.

Let us assume that $u(\mathbf{x},t)$ is a solution of the initial-boundary value problem satisfying the decay estimate. Let $\xi(\mathbf{x},t)$ be a regular function which satisfies the boundary conditions and such that tends to zero uniformly (in x_2, x_3 and t) when z goes to infinity. We know that

$$\Phi(0,t) = -\int_0^t \int_{D(0)} \Big((t-s)\xi_{,s}u_{,1} + \frac{\varepsilon}{a}\xi_{,s}u_{,1s} \Big)\, da\, ds. \tag{3.1}$$

It follows that

$$\Phi(0,t) = \int_0^t \int_R \Big((t-s)\xi_{,is}u_{,i} + \frac{\varepsilon}{a}\xi_{,is}u_{,is} + (t-s)\xi_{,s}\triangle u + \frac{\varepsilon}{a}\xi_{,s}\triangle u_{,s} \Big)\, dv\, ds. \tag{3.2}$$

Using the Holder inequality, we see

$$\begin{aligned}\Phi(0,t)^2 \leq{}& \int_0^t \int_R (t-s)\xi_{,is}\xi_{,is}\, dv\, ds \int_0^t \int_R (t-s)u_{,i}u_{,i}\, dv\, ds\\ &+ \int_0^t \int_R \frac{\varepsilon}{a}\xi_{,is}\xi_{,is}\, dv\, ds \int_0^t \int_R \frac{\varepsilon}{a}u_{,is}u_{,is}\, dv\, ds\\ &+ \int_0^t \int_R (t-s)\xi_{,s}\xi_{,s}\, dv\, ds \int_0^t \int_R (t-s)(\triangle u)^2\, dv\, ds\\ &+ \int_0^t \int_R \frac{\varepsilon}{a}\xi_{,s}\xi_{,s}\, dv\, ds \int_0^t \int_R \frac{\varepsilon}{a}(\triangle u_{,s})^2\, dv\, ds.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\Phi(0,t) \leq{}& \left(\int_0^t \int_R \Big[\Big((t-s) + \frac{\varepsilon}{a} \Big) |\nabla \xi_{,s}|^2 + \Big((t-s) + \frac{\varepsilon}{a} \Big) |\xi_{,s}|^2 \Big]\, dv\, ds \right)^{1/2}\\ &\times \left(\int_0^t \int_R \Big[(t-s)|\nabla u|^2 + \frac{\varepsilon}{a}|\nabla u_{,s}|^2 \right.\\ &\left. + (t-s)(\triangle u)^2 + \frac{\varepsilon}{a}(\triangle u_{,s})^2 \Big]\, dv\, ds \right)^{1/2}.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3) it follows

$$\Phi(0,t) \le K(t)\Phi(0,t)^{1/2}\Big(\int_0^t \int_R \Big[\Big((t-s)+\frac{\varepsilon}{a}\Big)|\nabla\xi_{,s}|^2 + \Big((t-s)+\frac{\varepsilon}{a}\Big)|\xi_{,s}|^2\Big]\, dv\, ds\Big)^{1/2}, \tag{3.4}$$

where $K(t)$ is a calculable function of time[1].

This means that

$$\Phi(0,t) \le \Big(t+\frac{\varepsilon}{a}\Big)K^2(t)\Big(\int_0^t \int_R [|\nabla\xi_{,s}|^2 + |\xi_{,s}|^2]\, dv\, ds\Big). \tag{3.5}$$

Let us take the function

$$\xi(x_i,t) = f(x_2,x_3,t)\exp(-vx_1), \tag{3.6}$$

where v is an arbitrary positive constant. By simple computations we may derive that

$$\begin{aligned} &\xi_{,t} = f_{,t}(x_2,x_3,t)\exp(-vx_1),\\ &\xi_{,1t} = -vf_{,t}(x_2,x_3,t)\exp(-vx_1), \quad \xi_{,\beta t} = f_{,\beta t}(x_2,x_3,t)\exp(-vx_1), \end{aligned} \tag{3.7}$$

where $\beta = 2,3$ and

$$\xi_{,t}^2 = f_{,t}^2\exp(-2vx_1), \quad |\nabla\xi_{,t}|^2 = (v^2 f_{,t}f_{,t} + f_{,\beta t}f_{,\beta t})\exp(-2vx_1). \tag{3.8}$$

Because the function ξ defined in (3.6) is a regular function which as required satisfies the boundary conditions and converges to zero uniformly when $z \to \infty$ in all the remaining independent variables we introduce the following notation

$$\begin{aligned} B_1 &= \int_0^t \int_R |\nabla\xi_{,s}|^2\, dv\, ds = \frac{1}{2v}\int_0^t \int_D (v^2 f_{,s}f_{,s} + f_{,\beta s}f_{,\beta s})\, da\, ds\\ B_2 &= \int_0^t \int_R \xi_{,s}^2\, dv\, ds = \frac{1}{2v}\int_0^t \int_D f_{,s}^2\, da\, ds, \end{aligned} \tag{3.9}$$

and we evaluate as

$$\Phi(0,t) \le \Big(t+\frac{\varepsilon}{a}\Big)K^2(t)(B_1+B_2). \tag{3.10}$$

This relation is an upper bound for the amplitude term in the estimate (2.13). We note that the estimate depends on an arbitrary positive constant v which can be optimized to have the right hand side of (3.10) as small as possible.

[1] The explicit expression of $K(t)$ is easy to be computed but cumbersome. We note that an upper-bound for such quantity is given by $\big(2t+1+\frac{2}{\varepsilon}+2t\varepsilon\big)^{1/2}$.

4. Further comments

We point out that our rate of decay depends on time. It is possible to save this fact by selecting an alternative measure. If we define the function

$$\Phi_\omega(z,t) = -\int_0^t \int_{D(z)} \exp(-2\omega s)\left(u_{,s}u_{,1} + \frac{2\omega\varepsilon}{a} u_{,s}u_{,s1}\right) da\, ds, \tag{4.1}$$

where $\omega > 0$, we obtain

$$\begin{aligned} \Phi_\omega(z,t) = {} & \Phi_\omega(z_0,t) \\ & - \frac{1}{2}\exp(-2\omega t)\int_{R(z_0,z)} ([|\nabla u|^2 + a^{-1}(u_{,s}^2 - \varepsilon\triangle u_{,s})^2 + \varepsilon(\triangle u)^2])\, dv \\ & - \omega\int_0^t \int_{R(z_0,z)} \exp(-2\omega s)\left(\left[|\nabla u|^2 + a^{-1}u_{,s}^2 + \frac{2\varepsilon}{a}|\nabla u_{,s}|^2 \right.\right. \\ & \left.\left. + \frac{\varepsilon^2}{a}(\triangle u_{,s})^2 + \varepsilon(\triangle u)^2\right]\right) dv\, ds, \end{aligned} \tag{4.2}$$

for $z \geq z_0 \geq 0$. Similar manipulations to the ones proposed previously bring us to an estimate of the type

$$|\Phi_\omega(z,t)| \leq -\lambda(\omega)\frac{\partial \Phi_\omega}{\partial z}. \tag{4.3}$$

As $\lambda(\omega)$ does not depend on time, we could also obtain a P-L alternative where the rate of decay depends on ω, but it is independent of the time. However, if we want to obtain a decay estimate for the function $\mathscr{E}$ we will have the dependence on the time in the amplitude term.

We can also recall that in the reference [6], a theory of thermoelasticity without energy dissipation with *two temperatures* has been proposed. For isotropic and homogeneous materials the system of equations is

$$\rho\ddot{u}_i = \mu u_{i,jj} + \phi u_{j,ji} - \beta(\dot{\theta} - \varepsilon\triangle\dot{\theta}), \quad \beta\dot{u}_{i,i} + c(\ddot{\theta} - \varepsilon\triangle\ddot{\theta}) = a\triangle\theta, \tag{4.4}$$

where u_i is the displacement, θ is the thermal displacement, $\rho > 0$, $\mu > 0$, $\phi > 0$, $a > 0$ and β are constitutive constants. We assume null initial conditions

$$u_i(\mathbf{x},0) = \dot{u}_i(\mathbf{x},0) = \theta(\mathbf{x},0) = \dot{\theta}(\mathbf{x},0) = 0, \quad \mathbf{x} \in R, \tag{4.5}$$

and boundary conditions

$$u_i(\mathbf{x},t) = \theta(\mathbf{x},t) = 0, \quad \mathbf{x} \in [0,\infty) \times \partial D, \quad t \geq 0. \tag{4.6}$$

Moreover, we impose boundary conditions on the finite end of the cylinder. Thus, we take as assumptions

$$u_i(0,x_2,x_3,t) = f_i(x_2,x_3,t), \quad \theta(0,x_2,x_3,t) = g(x_2,x_3,t), \quad (x_2,x_3) \in D. \tag{4.7}$$

We define the function

$$(4.8)\qquad \Phi_\omega(z,t) = -\int_0^t \int_{D(z)} \exp(-2\omega s)(\mu u_{i,1}u_i + \phi u_{j,j}u_1 - \beta(\theta - \varepsilon\triangle\theta)u_1 + a\theta_{,s}\theta_{,1} + 2\omega c\varepsilon\theta_{,s}\theta_{,s1})\, da\, ds.$$

It satisfies the relation

$$(4.9)\qquad \Phi_\omega(z,t) = \Phi_\omega(z_0,t) - \exp(-2\omega t)\int_{R(z_0,z)} M_1 dv - \omega\int_0^t \int_{R(z_0,z)} \exp(-2\omega s) M_2\, dv\, ds,$$

where

$$2M_1 = \rho\dot{u}_i\dot{u}_i + \mu u_{i,j}u_{i,j} + \phi u_{i,i}u_{j,j} + a|\nabla\theta|^2 + c(\theta_{,s}^2 - \varepsilon\triangle\theta_{,s})^2 + \varepsilon a(\triangle\theta)^2,$$

and

$$M_2 = \rho\dot{u}_i\dot{u}_i + \mu u_{i,j}u_{i,j} + \phi u_{i,i}u_{j,j} + a|\nabla\theta|^2 + c\theta_{,s}^2 + 2\varepsilon c|\nabla\theta_{,s}|^2 + \varepsilon^2 c(\triangle\theta_{,s})^2 + \varepsilon a(\triangle\theta)^2.$$

It is possible to obtain an inequality of the type (4.3) and then an alternative of P-L type for the solutions of the problem of the thermoelasticity without energy dissipation with two temperatures. To be precise, we will obtain the exponential decay of the function

$$(4.10)\qquad \mathscr{E}_0(z,t) = \int_{R(z)} M_1\, dv + \omega\int_0^t \int_{R(z)} M_2\, dv\, ds.$$

5. Concluding remarks

The present note is a complement to the paper [7]. Here by using the relation (2.4) we are able to control the term $u_{,t}$. We note that we had the term

$$(u_{,t} - \varepsilon\triangle u_{,t})^2$$

in (2.3) and then we could not control the time derivative of the solution. In [7] only a polynomial decay result has been provided. It is worth noting that we also need to control $\nabla u_{,t}$, but we also have this in the RHS of (2.4).

The exponential decay of the solutions of the equation (1.2) is an important result because dispersive models are connected with the existence of localized solution (e.g. solitary pulse waves). The linear model here considered and the results obtained are a first step toward a better understanding of localization phenomena in many frameworks as, for example, in the study of disturbances on elastic substrates.

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