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Partial Differential Equations — *Local integrability for solutions to some quasilinear elliptic systems*, by Francesco Leonetti and Pier Vincenzo Petricca.

Abstract. — In this paper we prove local integrability for weak solutions u of quasilinear elliptic systems whose off-diagonal coefficients are small when $|u|$ is large.

Key words: Local, integrability, solution, quasilinear, elliptic, system.

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1. Introduction

In this paper we deal with weak solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of quasilinear systems

$$-\sum_{i=1}^{n} D_i\Big(\sum_{j=1}^{n}\sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x,u(x))D_j u^\beta(x)\Big) = 0, \quad x \in \Omega,\ \alpha = 1,\dots,N. \tag{1.1}$$

We assume that $a_{ij}^{\alpha\beta}(x,u)$ are measurable with respect to x and continuous with respect to u; they are also bounded and elliptic. De Giorgi's counterexample [1] shows that, in general, weak solutions u of elliptic systems (1.1) are not regular; see also [2], [13] and chapter 3 of the survey [10]. In order to get regularity, we need additional assumptions on the coefficients. If $a_{ij}^{\alpha\beta}(x,u)$ are diagonal

$$a_{ij}^{\gamma\beta}(x,u) = 0 \quad \text{for } \beta \neq \gamma \tag{1.2}$$

then the N equations (1.1) are decoupled and maximum principle applies to every component u^γ of $u = (u^1,\dots,u^N)$:

$$\sup_{\Omega} u^\gamma \leq \sup_{\partial\Omega} u^\gamma. \tag{1.3}$$

In [12] the authors assume that coefficients are diagonal only for large values of u^γ:

$$\theta^\gamma \leq u^\gamma \quad \Rightarrow \quad a_{ij}^{\gamma\beta}(x,u) = 0 \quad \text{for } \beta \neq \gamma; \tag{1.4}$$

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then it results that

$$\sup_{\Omega} u^{\gamma} \leq \max\left[\theta^{\gamma}; \sup_{\partial\Omega} u^{\gamma}\right], \tag{1.5}$$

see also [9] and [7]. Now we no longer assume that off-diagonal coefficients vanish; we only know that they are small when $|u|$ is large:

$$|a_{ij}^{\gamma\beta}(x,u)| \leq \frac{c}{(1+|u|)^{q}} \quad \text{for } \beta \neq \gamma, \tag{1.6}$$

for some constants $c, q \in (0, +\infty)$. We assume ellipticity only for diagonal coefficients $a_{ij}^{\gamma\gamma}(x,u)$ and only for large values of $|u|$:

$$\theta \leq |u| \quad \Rightarrow \quad \nu|\xi|^{2} \leq \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)\xi_{j}\xi_{i} \tag{1.7}$$

for some constants $\theta \in [0, +\infty)$ and $\nu \in (0, +\infty)$. Also diagonal coefficients are assumed to be bounded:

$$|a_{ij}^{\gamma\gamma}(x,u)| \leq \tilde{c} \tag{1.8}$$

for some constant $\tilde{c} \in (0, +\infty)$. In this paper we prove that every weak solution $u : \Omega \subset \mathbb{R}^{n} \to \mathbb{R}^{N}$, with $u \in W^{1,2}(\Omega, \mathbb{R}^{N})$, of quasilinear system (1.1) verifying (1.6)–(1.8) enjoys the following higher integrability

$$u \in L_{loc}^{2^{*}(t+1)}(\Omega, \mathbb{R}^{N}) \tag{1.9}$$

for every t such that

$$0 < t < \frac{\nu}{n\tilde{c}2N} \quad \text{and} \quad t \leq \frac{q}{2} \tag{1.10}$$

where 2^{*} is the Sobolev exponent $\frac{2n}{n-2}$ and $n \geq 3$, see Theorem 2.1 in section 2. Note that reverse Hölder inequality gives us higher integrability of the gradient: $Du \in L_{loc}^{2+\varepsilon}$, see [3], chapter 6 in [4] and [5]; this improves on the integrability of u by means of Sobolev embedding: $u \in L_{loc}^{(2+\varepsilon)^{*}}$. It seems that strong ellipticity of $a_{ij}^{\alpha\beta}$ is required; in the present paper we need ellipticity only for diagonal entries $a_{ij}^{\gamma\gamma}$ and only for large values of $|u|$. Please, note that we do not assume ellipticity for small values of $|u|$: on such a set u is bounded but it might oscillate very much and the gradient Du might loose regularity. Let us come back to the regularity result (1.9) of this paper. Let us assume, in addition, that all diagonal coefficients come from the same matrix

$$a_{ij}^{\gamma\gamma}(x,u) = b_{ij}(x,u) \tag{1.11}$$

for all $\gamma = 1, \dots, N$; then we are able to improve on the integrability (1.9)

$$u \in L^{2^*(q/2+1)}_{loc}(\Omega, \mathbb{R}^N), \tag{1.12}$$

see Theorem 2.2 in section 2. This means that the additional assumption (1.11) allows us to remove the left restriction $t < \frac{\nu}{n\tilde{c}2N}$ in (1.10). In this last theorem, the faster off-diagonal $a^{\gamma\beta}_{ij}(x,u)$ decay when $|u| \to \infty$ the higher u is integrable. Note that in this paper we prove local integrability: $u \in L^s_{loc}$; global integrability has been studied in [8] by means of a different technique, under the assumption that $u_{|\partial\Omega}$ is bounded.

2. Assumptions and results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$. For $N \geq 2$, let $a^{\alpha\beta}_{ij} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be Carathéodory functions, that is, $a^{\alpha\beta}_{ij}(x, y)$ are measurable with respect to x and continuous with respect to y. We assume that diagonal coefficients $a^{\gamma\gamma}_{ij}$ are bounded: there exists $c_1 \in (0, +\infty)$ such that

$$|a^{\gamma\gamma}_{ij}(x, y)| \leq c_1 \tag{2.1}$$

for almost every $x \in \Omega$, for every $y \in \mathbb{R}^N$, for all $i, j \in \{1, \dots, n\}$, for any $\gamma \in \{1, \dots, N\}$. Now we assume ellipticity of diagonal coefficients $a^{\gamma\gamma}_{ij}$ for large values of y: there exist $\theta \in [0, +\infty)$ and $\nu \in (0, +\infty)$ such that

$$\theta \leq |y| \quad \Rightarrow \quad \nu|\xi|^2 \leq \sum_{i,j=1}^{n} a^{\gamma\gamma}_{ij}(x, y)\xi_j\xi_i \tag{2.2}$$

for almost every $x \in \Omega$, for any $\xi \in \mathbb{R}^n$ and for any $\gamma \in \{1, \dots, N\}$. Now we assume that off-diagonal coefficients $a^{\gamma\beta}_{ij}(x, y)$ do not vanish any more, but they are small when y is large: there exist $q \in (0, +\infty)$ and $c_2 \in (0, +\infty)$ such that

$$|a^{\gamma\beta}_{ij}(x, y)| \leq \frac{c_2}{(1 + |y|)^q} \quad \text{for } \beta \neq \gamma. \tag{2.3}$$

Note that both diagonal and off-diagonal coefficients are bounded.

Theorem 2.1. *Under the previous assumptions* (2.1), (2.2), (2.3) *let* $u = (u^1, \dots, u^N)$ *be a weak solution of the system* (1.1), *that is,* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *and*

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a^{\alpha\beta}_{ij}(x, u(x))D_j u^\beta(x)D_i v^\alpha(x)\, dx = 0 \quad \forall v \in W^{1,2}_0(\Omega, \mathbb{R}^N). \tag{2.4}$$

Then

$$u \in L^{2^*(t+1)}_{loc}(\Omega, \mathbb{R}^N) \tag{2.5}$$

for every t such that

$$0 < t < \frac{\nu}{nc_1 2N} \quad \textit{and} \quad t \le \frac{q}{2}. \tag{2.6}$$

If we also assume that diagonal coefficients are taken from the same array, namely, there exists $b_{ij}(x, y)$ such that

$$\theta \le |y| \quad \Rightarrow \quad a^{\gamma\gamma}_{ij}(x, y) = b_{ij}(x, y) \tag{2.7}$$

for every $\gamma = 1, \dots, N$, then we can improve on the previous theorem as follows.

THEOREM 2.2. *Under the previous assumptions* (2.1), (2.2), (2.3) *and* (2.7), *let* $u = (u^1, \dots, u^N)$ *be a weak solution of the system* (1.1). *Then*

$$u \in L^{2^*(q/2+1)}_{loc}(\Omega, \mathbb{R}^N). \tag{2.8}$$

3. PROOF OF THEOREM 2.1

Let $\phi : [0, +\infty) \to [0, +\infty)$ be increasing and $C^1([0, +\infty))$; moreover we assume that there exists a constant $\tilde{c} \in [1, +\infty)$ such that

$$0 \le \phi(t) \le \tilde{c} \quad \forall t \in [0, +\infty) \tag{3.1}$$

$$0 \le \phi'(t) \le \tilde{c} \quad \forall t \in [0, +\infty) \tag{3.2}$$

$$0 \le \phi'(t)t \le \tilde{c} \quad \forall t \in [0, +\infty). \tag{3.3}$$

Let $B_\rho = B(x_0, \rho)$ and $B_R = B(x_0, R)$ be open balls with the same center x_0 and radii $0 < \rho < R \le 1$, with $\overline{B_R} \subset \Omega$. We assume that $\eta : \mathbb{R}^n \to \mathbb{R}$, $\eta \in C^1_0(B_R)$ with $0 \le \eta \le 1$ in $\mathbb{R}^n$, $\eta = 1$ on B_ρ, $|D\eta| \le 4/(R - \rho)$ in $\mathbb{R}^n$. Note that $0 < R - \rho < R \le 1$ so $4/(R - \rho) > 4$. We fix $\gamma \in \{1, \dots, N\}$. We consider the test function $v = (v^1, \dots, v^N)$ defined as follows

$$v^\alpha = \begin{cases} 0 & \text{if } \alpha \ne \gamma, \\ \phi(|u|)u^\alpha \eta^2 & \text{if } \alpha = \gamma; \end{cases} \tag{3.4}$$

it results that

$$v \in W^{1,2}_0(B_R; \mathbb{R}^N) \subset W^{1,2}_0(\Omega; \mathbb{R}^N) \tag{3.5}$$

and

$$D_i v^\gamma = \left[\phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma + \phi(|u|)D_i u^\gamma\right]\eta^2 + [\phi(|u|)u^\gamma]D_i(\eta^2) \tag{3.6}$$

where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. We insert such a test function v into (2.4):

$$\begin{aligned}
0 &= \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x))D_j u^\beta(x) D_i v^\alpha(x)\, dx \\
&= \int_\Omega \sum_{i,j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\gamma\beta}(x,u)D_j u^\beta \phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma \eta^2 \\
&\quad + \int_\Omega \sum_{i,j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\gamma\beta}(x,u)D_j u^\beta \phi(|u|)D_i u^\gamma \eta^2 \\
&\quad + \int_\Omega \sum_{i,j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\gamma\beta}(x,u)D_j u^\beta \phi(|u|)u^\gamma D_i(\eta^2) \\
&= \int_\Omega \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)D_j u^\gamma \phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma \eta^2 \\
&\quad + \int_\Omega \sum_{i,j=1}^{n} \sum_{\beta\neq\gamma} a_{ij}^{\gamma\beta}(x,u)D_j u^\beta \phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma \eta^2 \\
&\quad + \int_{\{|u|\geq\theta\}} \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)D_j u^\gamma \phi(|u|)D_i u^\gamma \eta^2 \\
&\quad + \int_{\{|u|<\theta\}} \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)D_j u^\gamma \phi(|u|)D_i u^\gamma \eta^2 \\
&\quad + \int_\Omega \sum_{i,j=1}^{n} \sum_{\beta\neq\gamma} a_{ij}^{\gamma\beta}(x,u)D_j u^\beta \phi(|u|)D_i u^\gamma \eta^2 \\
&\quad + \int_\Omega \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)D_j u^\gamma \phi(|u|)u^\gamma D_i(\eta^2) \\
&\quad + \int_\Omega \sum_{i,j=1}^{n} \sum_{\beta\neq\gamma} a_{ij}^{\gamma\beta}(x,u)D_j u^\beta \phi(|u|)u^\gamma D_i(\eta^2).
\end{aligned} \tag{3.7}$$

Then

$$(3.8)\qquad \int_{\{|u|\geq\theta\}}\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma\phi(|u|)D_iu^\gamma\eta^2$$

$$= -\int_\Omega\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_iu^\delta)u^\gamma\eta^2$$

$$-\int_\Omega\sum_{i,j=1}^{n}\sum_{\beta\neq\gamma}a_{ij}^{\gamma\beta}(x,u)D_ju^\beta\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_iu^\delta)u^\gamma\eta^2$$

$$-\int_{\{|u|<\theta\}}\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma\phi(|u|)D_iu^\gamma\eta^2$$

$$-\int_\Omega\sum_{i,j=1}^{n}\sum_{\beta\neq\gamma}a_{ij}^{\gamma\beta}(x,u)D_ju^\beta\phi(|u|)D_iu^\gamma\eta^2$$

$$-\int_\Omega\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma\phi(|u|)u^\gamma D_i(\eta^2)$$

$$-\int_\Omega\sum_{i,j=1}^{n}\sum_{\beta\neq\gamma}a_{ij}^{\gamma\beta}(x,u)D_ju^\beta\phi(|u|)u^\gamma D_i(\eta^2).$$

Now we use ellipticity (2.2):

$$(3.9)\qquad \nu\int_{\{|u|\geq\theta\}}\phi(|u|)|Du^\gamma|^2\eta^2\leq\int_{\{|u|\geq\theta\}}\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma\phi(|u|)D_iu^\gamma\eta^2.$$

Applying twice Cauchy Schwartz inequality we get

$$(3.10)\qquad \left|\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)\xi_jh_i\right|\leq\Big(\sum_{i,j=1}^{n}|a_{ij}^{\gamma\gamma}(x,u)|^2\Big)^{1/2}|\xi|\,|h|;$$

if we keep in mind boundedness (2.1) we get

$$(3.11)\qquad \left|\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)\xi_jh_i\right|\leq nc_1|\xi|\,|h|;$$

now we use this inequality with $\xi_i=h_i=D_iu^\gamma$ and we get

$$(3.12)\qquad \left|\int_{\{|u|<\theta\}}\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma\phi(|u|)D_iu^\gamma\eta^2\right|\leq\int_{\{|u|<\theta\}}nc_1\phi(|u|)|Du^\gamma|^2\eta^2;$$

then we use inequality (3.11) with $\xi_i = D_i u^\gamma$, $h_i = \sum_{\delta=1}^N \frac{u^\delta}{|u|}(D_i u^\delta) u^\gamma$; since $|h| \le |Du|\,|u^\gamma|$, we get

$$\left| \int_\Omega \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x,u) D_j u^\gamma \phi'(|u|) 1_{\{|u|>0\}} \sum_{\delta=1}^N \frac{u^\delta}{|u|}(D_i u^\delta) u^\gamma \eta^2 \right| \le \int_\Omega n c_1 |Du^\gamma| \phi'(|u|) |Du|\,|u^\gamma| \eta^2; \tag{3.13}$$

eventually we use inequality (3.11) with $\xi_i = D_i u^\gamma$, $h_i = u^\gamma D_i(\eta^2)$; since $|h| \le |u^\gamma| 2\eta |D\eta|$, we get

$$\left| \int_\Omega \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x,u) D_j u^\gamma \phi(|u|) u^\gamma D_i(\eta^2) \right| \le \int_\Omega n c_1 |Du^\gamma| \phi(|u|) |u^\gamma| 2\eta |D\eta|. \tag{3.14}$$

Now we keep in mind that off-diagonal coefficients satisfy (2.3) and we get

$$\left| \int_\Omega \sum_{i,j=1}^n \sum_{\beta\ne\gamma} a_{ij}^{\gamma\beta}(x,u) D_j u^\beta \phi'(|u|) 1_{\{|u|>0\}} \sum_{\delta=1}^N \frac{u^\delta}{|u|}(D_i u^\delta) u^\gamma \eta^2 \right| \le \int_\Omega \frac{n^2 N c_2}{(1+|u|)^q} |Du| \phi'(|u|) |Du|\,|u^\gamma| \eta^2, \tag{3.15}$$

$$\left| \int_\Omega \sum_{i,j=1}^n \sum_{\beta\ne\gamma} a_{ij}^{\gamma\beta}(x,u) D_j u^\beta \phi(|u|) D_i u^\gamma \eta^2 \right| \le \int_\Omega \frac{n^2 N c_2}{(1+|u|)^q} |Du|^2 \phi(|u|) \eta^2 \tag{3.16}$$

and

$$\left| \int_\Omega \sum_{i,j=1}^n \sum_{\beta\ne\gamma} a_{ij}^{\gamma\beta}(x,u) D_j u^\beta \phi(|u|) u^\gamma D_i(\eta^2) \right| \le \int_\Omega \frac{n^2 N c_2}{(1+|u|)^q} |Du| \phi(|u|) |u| 2\eta |D\eta|. \tag{3.17}$$

Since ϕ increases, we have

$$\begin{aligned} \left| \int_{\{|u|<\theta\}} \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x,u) D_j u^\gamma \phi(|u|) D_i u^\gamma \eta^2 \right| &\le \int_{\{|u|<\theta\}} n c_1 |Du^\gamma|^2 \phi(|u|) \eta^2 \\ &\le \int_{\{|u|<\theta\}} n c_1 |Du^\gamma|^2 \phi(\theta) \eta^2 \\ &\le n c_1 \phi(\theta) \int_\Omega |Du^\gamma|^2 \end{aligned} \tag{3.18}$$

and

$$\nu \int_{\{|u|<\theta\}} |Du^\gamma|^2 \phi(|u|)\eta^2 \le \nu \int_{\{|u|<\theta\}} |Du^\gamma|^2 \phi(\theta)\eta^2 \le \nu\phi(\theta) \int_\Omega |Du^\gamma|^2. \tag{3.19}$$

Let us collect all these inequalities:

$$\begin{aligned} \nu \int_\Omega |Du^\gamma|^2 \phi(|u|)\eta^2 & \\ \le (\nu + nc_1)\phi(\theta) \int_\Omega |Du^\gamma|^2 &+ n^2 N c_2 \int_\Omega \frac{\phi(|u|) + \phi'(|u|)|u|}{(1+|u|)^q} |Du|^2 \\ &+ n^2 N c_2 \int_\Omega \frac{\phi(|u|)}{(1+|u|)^q} 2\eta |Du|\,|u|\,|D\eta| \\ &+ nc_1 \int_\Omega 2\eta |Du^\gamma|\,|u|\,|D\eta| \phi(|u|) + nc_1 \int_\Omega |Du|^2 \phi'(|u|)|u|\eta^2. \end{aligned} \tag{3.20}$$

We recall the inequality $2AB \le \varepsilon A^2 + B^2/\varepsilon$, provided $\varepsilon > 0$; then

$$\begin{aligned} nc_1 \int_\Omega 2\eta |Du^\gamma|\,|u|\,|D\eta|\phi(|u|) & \\ \le \varepsilon \int_\Omega \eta^2 |Du^\gamma|^2 \phi(|u|) &+ \frac{n^2 c_1^2}{\varepsilon} \int_\Omega |u|^2 |D\eta|^2 \phi(|u|) \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} n^2 N c_2 \int_\Omega \frac{\phi(|u|)}{(1+|u|)^q} 2\eta |Du|\,|u|\,|D\eta| & \\ \le n^2 N c_2 \int_\Omega \frac{\phi(|u|)}{(1+|u|)^q} \eta^2 |Du|^2 &+ n^2 N c_2 \int_\Omega \frac{\phi(|u|)}{(1+|u|)^q} |u|^2 |D\eta|^2. \end{aligned} \tag{3.22}$$

This results in

$$\begin{aligned} (\nu - \varepsilon) \int_\Omega |Du^\gamma|^2 \phi(|u|)\eta^2 & \\ \le (\nu + nc_1)\phi(\theta) \int_\Omega |Du^\gamma|^2 &+ n^2 N c_2 \int_\Omega \frac{2\phi(|u|) + \phi'(|u|)|u|}{(1+|u|)^q} |Du|^2 \\ &+ n^2 N c_2 \int_\Omega \frac{\phi(|u|)}{(1+|u|)^q} |u|^2 |D\eta|^2 + \frac{n^2 c_1^2}{\varepsilon} \int_\Omega |u|^2 |D\eta|^2 \phi(|u|) \\ &+ nc_1 \int_\Omega |Du|^2 \phi'(|u|)|u|\eta^2. \end{aligned} \tag{3.23}$$

Let us sum over γ from 1 to N:

$$(3.24)\qquad (\nu-\varepsilon)\int_\Omega |Du|^2\phi(|u|)\eta^2$$
$$\le (\nu+nc_1)\phi(\theta)\int_\Omega |Du|^2 + n^2N^2c_2\int_\Omega \frac{2\phi(|u|)+\phi'(|u|)|u|}{(1+|u|)^q}|Du|^2$$
$$+ n^2N^2c_2\int_\Omega \frac{\phi(|u|)}{(1+|u|)^q}|u|^2|D\eta|^2 + \frac{n^2Nc_1^2}{\varepsilon}\int_\Omega |u|^2|D\eta|^2\phi(|u|)$$
$$+ nNc_1\int_\Omega |Du|^2\phi'(|u|)|u|\eta^2$$

so that

$$(3.25)\qquad \int_\Omega |Du|^2[(\nu-\varepsilon)\phi(|u|) - nNc_1\phi'(|u|)|u|]\eta^2$$
$$\le \int_\Omega \Big((\nu+nc_1)\phi(\theta) + n^2N^2c_2\frac{2\phi(|u|)+\phi'(|u|)|u|}{(1+|u|)^q}\Big)|Du|^2$$
$$+ \int_\Omega \Big(\frac{n^2N^2c_2}{(1+|u|)^q} + \frac{n^2Nc_1^2}{\varepsilon}\Big)\phi(|u|)|u|^2|D\eta|^2.$$

Let us consider $p \in \left(0,\frac{1}{2}\right)$ and let us assume that

$$(3.26)\qquad |u|^{2(p+1)} \in L^1(B_R).$$

For $t \in [0,+\infty)$ we set

$$(3.27)\qquad \psi(t) = (p+1)^2 t^{2p};$$

we would like to take $\phi = \psi$ in (3.25) but we cannot do that, since ϕ must satisfy (3.1), (3.2) and (3.3). So we approximate ψ in this way: for every $k \in \mathbb{N}$ we consider $\psi_k : [0,+\infty) \to \mathbb{R}$ as follows

$$(3.28)\qquad \psi_k(t) = \begin{cases} \psi\left(\frac{1}{k}\right) + \psi'\left(\frac{1}{k}\right)\left(t-\frac{1}{k}\right) & \text{if } t \in \left[0,\frac{1}{k}\right) \\ \psi(t) & \text{if } t \in \left[\frac{1}{k},k\right] \\ \psi(k) + \frac{\psi'(k)}{2}(1-(k+1-t)^2) & \text{if } t \in (k,k+1) \\ \psi(k) + \frac{\psi'(k)}{2} & \text{if } t \in [k+1,+\infty) \end{cases}$$

then $\psi_k : [0,+\infty) \to [0,+\infty)$ is increasing, $C^1([0,+\infty))$ and

$$0 \le \psi_k'(t)t \le 2p\psi_k(t) \quad \forall t \in [0, +\infty), \tag{3.29}$$

$$0 \le \psi_k(t) \le \psi(1) + \psi(t) \quad \forall t \in [0, +\infty); \tag{3.30}$$

in addition, there exists $c_k \in [0, +\infty)$ such that

$$0 \le \psi_k(t) \le c_k \quad \forall t \in [0, +\infty), \tag{3.31}$$

$$0 \le \psi_k'(t) \le c_k \quad \forall t \in [0, +\infty), \tag{3.32}$$

$$0 \le \psi_k'(t)t \le c_k \quad \forall t \in [0, +\infty). \tag{3.33}$$

Moreover

$$\lim_{k \to +\infty} \psi_k(t) = \psi(t) \quad \forall t \in [0, +\infty). \tag{3.34}$$

Now we can take $\phi = \psi_k$ in (3.25) and we get

$$\begin{aligned} \int_\Omega |Du|^2[(\nu - \varepsilon)\psi_k(|u|) - nNc_1\psi_k'(|u|)|u|]\eta^2 \\ \le \int_\Omega \Big((\nu + nc_1)\psi_k(\theta) + n^2N^2c_2\frac{2\psi_k(|u|) + \psi_k'(|u|)|u|}{(1+|u|)^q}\Big)|Du|^2 \\ + \int_\Omega \Big(\frac{n^2N^2c_2}{(1+|u|)^q} + \frac{n^2Nc_1^2}{\varepsilon}\Big)\psi_k(|u|)|u|^2|D\eta|^2. \end{aligned} \tag{3.35}$$

In order to lower the left hand side we use (3.29) and we get

$$\begin{aligned} (\nu - \varepsilon)\psi_k(|u|) - nNc_1\psi_k'(|u|)|u| &\ge (\nu - \varepsilon)\psi_k(|u|) - nNc_12p\psi_k(|u|) \\ &= (\nu - \varepsilon - nNc_12p)\psi_k(|u|). \end{aligned} \tag{3.36}$$

We require that

$$0 < \nu - nNc_12p \tag{3.37}$$

that is

$$p < \frac{\nu}{nc_12N}. \tag{3.38}$$

Note that (2.2) and (3.11) imply

$$\nu|\xi|^2 \le \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x,u)\xi_j\xi_i \le nc_1|\xi|^2 \tag{3.39}$$

thus, for $\xi \ne 0$, we get

$$\nu \le nc_1 \tag{3.40}$$

so that

$$0 < \frac{\nu}{nc_1 2N} \le \frac{1}{2N} \le \frac{1}{4}. \tag{3.41}$$

Under (3.38) we have (3.37) thus we can choose

$$\varepsilon = \frac{\nu - nNc_1 2p}{2} \tag{3.42}$$

and it turns out that $\varepsilon > 0$ with

$$\nu - \varepsilon - nNc_1 2p = \frac{\nu - nNc_1 2p}{2} > 0. \tag{3.43}$$

In order to estimate the right hand side of (3.35) we recall (3.29), (3.30) and (3.27):

$$\begin{aligned} \psi_k(\theta) &\le \psi(1) + \psi(\theta) = (p+1)^2 + (p+1)^2\theta^{2p} = (p+1)^2(1+\theta^{2p}) \\ &\le (p+1)^2 2(1+\theta)^{2p} \le 2^2 2(1+\theta) = 8(1+\theta) \end{aligned} \tag{3.44}$$

where we also used $0 < p < \frac{1}{2}$; moreover

$$\begin{aligned} 2\psi_k(|u|) + \psi_k'(|u|)|u| &\le 2\psi_k(|u|) + 2p\psi_k(|u|) = 2(p+1)\psi_k(|u|) \\ &\le 2(p+1)(\psi(1) + \psi(|u|)) \\ &= 2(p+1)((p+1)^2 + (p+1)^2|u|^{2p}) \\ &= 2(p+1)^3(1+|u|^{2p}) \le 2 \times 2^3 \times 2(1+|u|)^{2p} \\ &= 32(1+|u|)^{2p}; \end{aligned} \tag{3.45}$$

on the other hand

$$\frac{n^2N^2c_2}{(1+|u|)^q} \le n^2N^2c_2 \tag{3.46}$$

and

$$\frac{n^2Nc_1^2}{\varepsilon} = \frac{2n^2Nc_1^2}{\nu - nNc_1 2p}; \tag{3.47}$$

moreover

$$\begin{aligned} \psi_k(|u|)|u|^2 &\le (\psi(1) + \psi(|u|))|u|^2 = ((p+1)^2 + (p+1)^2|u|^{2p})|u|^2 \\ &= (p+1)^2(1+|u|^{2p})|u|^2 \le 2^2 \times 2(1+|u|)^{2p}(1+|u|)^2 \\ &= 8(1+|u|)^{2(p+1)}. \end{aligned} \tag{3.48}$$

We insert these estimates into (3.35):

$$\frac{\nu - nNc_1 2p}{2}\int_\Omega |Du|^2\psi_k(|u|)\eta^2 \tag{3.49}$$
$$\le \int_\Omega \Big((\nu + nc_1)8(1+\theta) + 32n^2N^2c_2\frac{(1+|u|)^{2p}}{(1+|u|)^q}\Big)|Du|^2$$
$$+ \int_\Omega \Big(n^2N^2c_2 + \frac{2n^2Nc_1^2}{\nu - nNc_1 2p}\Big)8(1+|u|)^{2(p+1)}|D\eta|^2.$$

Now we further require that

$$2p \le q \tag{3.50}$$

so that (3.49) becomes

$$\frac{\nu - nNc_1 2p}{2}\int_\Omega |Du|^2\psi_k(|u|)\eta^2 \tag{3.51}$$
$$\le ((\nu + nc_1)8(1+\theta) + 32n^2N^2c_2)\|Du\|^2_{L^2(\Omega)}$$
$$+ \frac{2^7}{(R-\rho)^2}\Big(n^2N^2c_2 + \frac{2n^2Nc_1^2}{\nu - nNc_1 2p}\Big)\int_{B_R}(1+|u|)^{2(p+1)}.$$

Positivity of ψ_k and pointwise convergence (3.34) allow us to use Fatou lemma:

$$\frac{\nu - nNc_1 2p}{2}\int_\Omega |Du|^2(p+1)^2|u|^{2p}\eta^2 \tag{3.52}$$
$$\le ((\nu + nc_1)8(1+\theta) + 32n^2N^2c_2)\|Du\|^2_{L^2(\Omega)}$$
$$+ \frac{2^7}{(R-\rho)^2}\Big(n^2N^2c_2 + \frac{2n^2Nc_1^2}{\nu - nNc_1 2p}\Big)\int_{B_R}(1+|u|)^{2(p+1)}.$$

Let us set

$$w = |u|^{p+1}\eta; \tag{3.53}$$

then

$$w \in W_0^{1,2}(B_R) \tag{3.54}$$

and

$$|Dw|^2 \le 2(p+1)^2|u|^{2p}|Du|^2\eta^2 + 2n|u|^{2(p+1)}\Big(\frac{4}{R-\rho}\Big)^2. \tag{3.55}$$

The previous inequality and (3.52) give

$$\begin{aligned}
(3.56)\quad \int_{B_R} |Dw|^2 &\le 2\int_{B_R} (p+1)^2|u|^{2p}|Du|^2\eta^2 + 2n\Big(\frac{4}{R-\rho}\Big)^2\int_{B_R}|u|^{2(p+1)} \\
&\le \frac{4}{v-nNc_1 2p}((v+nc_1)8(1+\theta)+32n^2N^2c_2)\|Du\|^2_{L^2(\Omega)} \\
&\quad + \frac{4}{v-nNc_1 2p}\frac{2^7}{(R-\rho)^2} \\
&\quad \times\Big(n^2N^2c_2 + \frac{2n^2Nc_1^2}{v-nNc_1 2p}\Big)\int_{B_R}(1+|u|)^{2(p+1)} \\
&\quad + 2n\Big(\frac{4}{R-\rho}\Big)^2\int_{B_R}|u|^{2(p+1)} \\
&\le \frac{32\Big((v+nc_1)(1+\theta)+4n^2N^2c_2\Big)}{v-nNc_1 2p}\|Du\|^2_{L^2(\Omega)} \\
&\quad + \frac{2^9}{(R-\rho)^2}\Big(\frac{n^2N^2c_2}{v-nNc_1 2p} + \frac{2n^2Nc_1^2}{(v-nNc_1 2p)^2}\Big)\int_{B_R}(1+|u|)^{2(p+1)} \\
&\quad + n\frac{2^5}{(R-\rho)^2}\int_{B_R}(1+|u|)^{2(p+1)}.
\end{aligned}$$

We recall (3.37) and (3.40):

$$(3.57)\qquad 0 < v - Nc_1 2p < v \le nc_1$$

then

$$(3.58)\qquad 1 \le \frac{nc_1}{v-Nc_1 2p}.$$

Thus (3.56) becomes

$$\begin{aligned}
(3.59)\quad \int_{B_R}|Dw|^2 &\le \frac{32((v+nc_1)(1+\theta)+4n^2N^2c_2)}{v-nNc_1 2p}\|Du\|^2_{L^2(\Omega)} \\
&\quad + \frac{2^9}{(R-\rho)^2}\Big(\frac{n^2N^2c_2}{v-nNc_1 2p} + \frac{(2N+n)n^2c_1^2}{(v-nNc_1 2p)^2}\Big)\int_{B_R}(1+|u|)^{2(p+1)}.
\end{aligned}$$

Since $2 < n$, we can use Sobolev embedding

$$(3.60)\qquad \int_{B_R}|w|^{2^*} \le \Big(\frac{2(n-1)}{n-2}\int_{B_R}|Dw|^2\Big)^{2^*/2}$$

and we get

$$\int_{B_\rho} |u|^{(p+1)2^*} \le \int_{B_R} |\,|u|^{p+1}\eta|^{2^*} = \int_{B_R} |w|^{2^*} \le \Big(\frac{2(n-1)}{n-2}\int_{B_R} |Dw|^2\Big)^{2^*/2}$$
$$\le \Big(\frac{32((v+nc_1)(1+\theta)+4n^2N^2c_2)}{v-nNc_1 2p}\|Du\|^2_{L^2(\Omega)}$$
$$+\frac{2^9}{(R-\rho)^2}\Big(\frac{n^2N^2c_2}{v-nNc_1 2p}+\frac{(2N+n)n^2c_1^2}{(v-nNc_1 2p)^2}\Big)$$
$$\times \int_{B_R}(1+|u|)^{2(p+1)}\Big)^{2^*/2} \times \Big(\frac{2(n-1)}{n-2}\Big)^{2^*/2}. \tag{3.61}$$

Let us summarize as follows: if for some $p \in \big(0, \frac{1}{2}\big)$ with

$$p < \frac{v}{nc_1 2N} \quad \text{and} \quad p \le \frac{q}{2} \tag{3.62}$$

and for some $0 < \rho < R \le 1$ with $\overline{B_R} \subset \Omega$ we have

$$|u|^{2(p+1)} \in L^1(B_R) \tag{3.63}$$

then it results that

$$|u|^{2^*(p+1)} \in L^1(B_\rho). \tag{3.64}$$

Since $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ and $\overline{B_R} \subset \Omega$, Sobolev embedding gives us

$$|u|^{2(n/(n-2))} \in L^1(B_R) \tag{3.65}$$

thus (3.63) is fulfilled, provided

$$p + 1 \le \frac{n}{n-2} \tag{3.66}$$

that is

$$p \le \frac{n}{n-2} - 1 = \frac{2}{n-2}. \tag{3.67}$$

Let us fix t such that

$$0 < t < \frac{v}{nc_1 2N} \le \frac{1}{4} \quad \text{and} \quad t \le \frac{q}{2}; \tag{3.68}$$

moreover, we fix $0 < \sigma \le 1$ with $\overline{B_\sigma} \subset \Omega$; we claim that

$$|u|^{2^*(t+1)} \in L^1(B_{\sigma/2}). \tag{3.69}$$

Indeed, we start using (3.63) with (3.65): this improves the integrability accordingly to (3.64); then we start again using (3.63) with a larger p and we improve the integrability accordingly to (3.64) and so on. Let us evaluate how much we gain at every step:

$$2^*(p+1) - 2(p+1) = (2^* - 2)(p+1) > (2^* - 2) > 0; \tag{3.70}$$

thus, in a finite number of steps we reach the exponent $2^*(t+1)$ and claim (3.69) is proved: this is a finite Moser's iteration, [11]. This ends the proof of Theorem 2.1. □

4. Proof of Theorem 2.2

We start as in the proof of Theorem 2.1 and we arrive at formula (3.12). Instead of writing (3.13) we remark that our new assumption (2.7) guarantees that, on the set $\{|u| \ge \theta\}$,

$$\begin{aligned}
&\sum_{\gamma=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma\eta^2 \\
&= \sum_{\gamma=1}^{N}\sum_{i,j=1}^{n} b_{ij}(x,u)(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma\eta^2 \\
&= \sum_{i,j=1}^{n} b_{ij}(x,u)\sum_{\gamma=1}^{N}(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma\eta^2 \\
&= \phi'(|u|)1_{\{|u|>0\}}\frac{1}{|u|}\eta^2\sum_{i,j=1}^{n} b_{ij}(x,u)\sum_{\gamma=1}^{N}u^\gamma(D_j u^\gamma)\sum_{\delta=1}^{N}u^\delta(D_i u^\delta) \\
&= \phi'(|u|)1_{\{|u|>0\}}\frac{1}{|u|}\eta^2\sum_{i,j=1}^{n} b_{ij}(x,u)\langle u; D_j u\rangle\langle u; D_i u\rangle = (*);
\end{aligned} \tag{4.1}$$

then (2.7) and (2.2) with $\xi_i = \langle u; D_i u\rangle$ give, on the set $\{|u| \ge \theta\}$,

$$(*) \ge 0; \tag{4.2}$$

on the set $\{|u| < \theta\}$ we use inequality (3.11) with $\xi_i = D_i u^\gamma$ and $h_i = \sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma$; since $|h| \le |Du|\,|u^\gamma|$, we get

$$(4.3)\qquad \left|\sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma\eta^2\right|$$
$$\leq nc_1|Du^\gamma|\,|Du|\,|u^\gamma|\phi'(|u|)1_{\{|u|>0\}} \leq nc_1\phi'(|u|)|u|\,|Du|^2.$$

We use inequalities (3.14), (3.15), (3.16), (3.17), (3.18), (3.19) as in the proof of theorem 2.1 and we arrive at the analog of (3.20):

$$(4.4)\qquad v\int_\Omega |Du^\gamma|^2\phi(|u|)\eta^2$$
$$\leq (v+nc_1)\phi(\theta)\int_\Omega |Du^\gamma|^2$$
$$+n^2Nc_2\int_\Omega \frac{\phi(|u|)+\phi'(|u|)|u|}{(1+|u|)^q}|Du|^2$$
$$+n^2Nc_2\int_\Omega \frac{\phi(|u|)}{(1+|u|)^q}2\eta|Du|\,|u|\,|D\eta|$$
$$+nc_1\int_\Omega 2\eta|Du^\gamma|\,|u|\,|D\eta|\phi(|u|)+nc_1\int_{\{|u|<\theta\}}\phi'(|u|)|u|\,|Du|^2$$
$$-\int_{\{|u|\geq\theta\}}\sum_{i,j=1}^{n} b_{ij}(x,u)(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma\eta^2.$$

As in the proof of the previous theorem, we get estimates (3.21) and (3.22): this results in the analog of (3.23)

$$(4.5)\qquad (v-\varepsilon)\int_\Omega |Du^\gamma|^2\phi(|u|)\eta^2$$
$$\leq (v+nc_1)\phi(\theta)\int_\Omega |Du^\gamma|^2$$
$$+n^2Nc_2\int_\Omega \frac{2\phi(|u|)+\phi'(|u|)|u|}{(1+|u|)^q}|Du|^2$$
$$+n^2Nc_2\int_\Omega \frac{\phi(|u|)}{(1+|u|)^q}|u|^2|D\eta|^2$$
$$+\frac{n^2c_1^2}{\varepsilon}\int_\Omega |u|^2|D\eta|^2\phi(|u|)+nc_1\int_{\{|u|<\theta\}}\phi'(|u|)|u|\,|Du|^2$$
$$-\int_{\{|u|\geq\theta\}}\sum_{i,j=1}^{n} b_{ij}(x,u)(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma\eta^2.$$

Let us sum over γ from 1 to N:

$$
\begin{aligned}
(4.6)\quad (\nu-\varepsilon)\int_\Omega |Du|^2\phi(|u|)\eta^2
&\le (\nu+nc_1)\phi(\theta)\int_\Omega |Du|^2\\
&\quad + n^2N^2c_2\int_\Omega \frac{2\phi(|u|)+\phi'(|u|)|u|}{(1+|u|)^q}|Du|^2\\
&\quad + n^2N^2c_2\int_\Omega \frac{\phi(|u|)}{(1+|u|)^q}|u|^2|D\eta|^2\\
&\quad + \frac{n^2Nc_1^2}{\varepsilon}\int_\Omega |u|^2|D\eta|^2\phi(|u|) + nNc_1\int_{\{|u|<\theta\}}\phi'(|u|)|u|\,|Du|^2\\
&\quad - \int_{\{|u|\ge\theta\}}\sum_{\gamma=1}^N\sum_{i,j=1}^n b_{ij}(x,u)(D_ju^\gamma)\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^N\frac{u^\delta}{|u|}(D_iu^\delta)u^\gamma\eta^2.
\end{aligned}
$$

We use (4.2) and we get

$$
(4.7)\quad -\int_{\{|u|\ge\theta\}}\sum_{\gamma=1}^N\sum_{i,j=1}^n b_{ij}(x,u)(D_ju^\gamma)\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^N\frac{u^\delta}{|u|}(D_iu^\delta)u^\gamma\eta^2\le 0;
$$

then

$$
\begin{aligned}
(4.8)\quad (\nu-\varepsilon)\int_\Omega |Du|^2\phi(|u|)\eta^2
&\le \int_\Omega\Big((\nu+nc_1)\phi(\theta)+n^2N^2c_2\frac{2\phi(|u|)+\phi'(|u|)|u|}{(1+|u|)^q}\Big)|Du|^2\\
&\quad + \int_\Omega\Big(\frac{n^2Nc_1^2}{\varepsilon}+\frac{n^2N^2c_2}{(1+|u|)^q}\Big)\phi(|u|)|u|^2|D\eta|^2\\
&\quad + nNc_1\int_{\{|u|<\theta\}}\phi'(|u|)|u|\,|Du|^2.
\end{aligned}
$$

We want to remark that additional assumption (2.7) and ellipticity (2.2) guarantee inequality (4.7) by means of (4.1) and (4.2) as in the sign condition (H3) of [6]. Let us consider $p\in(0,+\infty)$ and let us assume that

$$
(4.9)\qquad |u|^{2(p+1)}\in L^1(B_R).
$$

For $t \in [0, +\infty)$ we set

$$\psi(t) = (p+1)^2 t^{2p}; \tag{4.10}$$

we would like to take $\phi = \psi$ in (4.8) but we cannot do that, since ϕ must satisfy (3.1), (3.2) and (3.3). So we approximate ψ in this way; for every $k \in \mathbb{N}$ we consider $\psi_k : [0, +\infty) \to \mathbb{R}$ as follows: when $p < \frac{1}{2}$ we take

$$\psi_k(t) = \begin{cases} \psi\left(\frac{1}{k}\right) + \psi'\left(\frac{1}{k}\right)\left(t - \frac{1}{k}\right) & \text{if } t \in \left[0, \frac{1}{k}\right) \\ \psi(t) & \text{if } t \in \left[\frac{1}{k}, k\right] \\ \psi(k) + \frac{\psi'(k)}{2}(1 - (k+1-t)^2) & \text{if } t \in (k, k+1) \\ \psi(k) + \frac{\psi'(k)}{2} & \text{if } t \in [k+1, +\infty) \end{cases} \tag{4.11}$$

when $p \geq \frac{1}{2}$ we take

$$\psi_k(t) = \begin{cases} \psi(t) & \text{if } t \in [0, k] \\ \psi(k) + \frac{\psi'(k)}{2}(1 - (k+1-t)^2) & \text{if } t \in (k, k+1) \\ \psi(k) + \frac{\psi'(k)}{2} & \text{if } t \in [k+1, +\infty) \end{cases} \tag{4.12}$$

in both cases $\psi_k : [0, +\infty) \to [0, +\infty)$ is increasing, $C^1([0, +\infty))$ and satisfies (3.29), (3.30), (3.31), (3.32), (3.33) and (3.34). Now we can take $\phi = \psi_k$ in (4.8) and we get

$$\begin{aligned} (v - \varepsilon) \int_\Omega |Du|^2 \psi_k(|u|)\eta^2 & \\ \leq \int_\Omega \Big((v + nc_1)\psi_k(\theta) + n^2 N^2 c_2 \frac{2\psi_k(|u|) + \psi_k'(|u|)|u|}{(1+|u|)^q} \Big) |Du|^2 & \\ + \int_\Omega \Big(\frac{n^2 N c_1^2}{\varepsilon} + \frac{n^2 N^2 c_2}{(1+|u|)^q} \Big) \psi_k(|u|)|u|^2 |D\eta|^2 & \\ + nNc_1 \int_{\{|u|<\theta\}} \psi_k'(|u|)|u|\,|Du|^2. & \end{aligned} \tag{4.13}$$

We select

$$\varepsilon = \frac{v}{2}. \tag{4.14}$$

In order to estimate the right hand side of (4.13) we recall (3.29), (3.30) and (4.10):

$$\psi_k(t) \le \psi(1) + \psi(t) = (p+1)^2 + (p+1)^2 t^{2p} \\ = (p+1)^2(1+t^{2p}) \le (p+1)^2 2(1+t)^{2p} \tag{4.15}$$

and

$$\psi_k'(t)t \le 2p\psi_k(t) \le 2p(p+1)^2 2(1+t)^{2p} \tag{4.16}$$

then (4.13) becomes

$$\begin{aligned}
\frac{v}{2}\int_\Omega |Du|^2\psi_k(|u|)\eta^2 \\
&\le \int_\Omega \Big[(v+nc_1)(p+1)^2 2(1+\theta)^{2p} \\
&\quad + n^2N^2c_2\frac{2(p+1)^2 2(1+|u|)^{2p} + 2p(p+1)^2 2(1+|u|)^{2p}}{(1+|u|)^q}\Big]|Du|^2 \\
&\quad + \int_\Omega \Big(\frac{2n^2Nc_1^2}{v} + \frac{n^2N^2c_2}{(1+|u|)^q}\Big)(p+1)^2 2(1+|u|)^{2p}|u|^2|D\eta|^2 \\
&\quad + nNc_1\int_{\{|u|<\theta\}} 2p(p+1)^2 2(1+|u|)^{2p}|Du|^2 \\
&\le \int_\Omega (p+1)^2 2\Big[(v+nc_1)(1+\theta)^{2p} \\
&\qquad + n^2N^2c_2 2(1+p)\frac{(1+|u|)^{2p}}{(1+|u|)^q}\Big]|Du|^2 \\
&\quad + \int_\Omega \Big(\frac{2n^2Nc_1^2}{v} + \frac{n^2N^2c_2}{1}\Big)(p+1)^2 2(1+|u|)^{2p+2}|D\eta|^2 \\
&\quad + nNc_1\int_\Omega 2p(p+1)^2 2(1+\theta)^{2p}|Du|^2.
\end{aligned} \tag{4.17}$$

We further require that

$$p \le \frac{q}{2} \tag{4.18}$$

and (4.17) becomes

$$\frac{\nu}{2}\int_{\Omega}|Du|^2\psi_k(|u|)\eta^2 \tag{4.19}$$
$$\leq \int_{\Omega}(p+1)^2 2[(\nu+(1+2pN)nc_1)(1+\theta)^{2p}+n^2N^2c_2 2(1+p)]|Du|^2$$
$$+\int_{\Omega}\Big(\frac{2n^2Nc_1^2}{\nu}+n^2N^2c_2\Big)(p+1)^2 2(1+|u|)^{2p+2}\frac{4^2}{(R-\rho)^2}$$
$$\leq \int_{\Omega}(q+1)^2 2[(\nu+(1+qN)nc_1)(1+\theta)^{2q}+n^2N^2c_2 2(1+q)]|Du|^2$$
$$+\int_{\Omega}\Big(\frac{2n^2Nc_1^2}{\nu}+n^2N^2c_2\Big)2^5\frac{(q+1)^2}{(R-\rho)^2}(1+|u|)^{2(p+1)}$$
$$= c_3 + c_4\int_{\Omega}(1+|u|)^{2(p+1)}$$

where

$$c_3 = \int_{\Omega}(q+1)^2 2[(\nu+(1+qN)nc_1)(1+\theta)^{2q}+n^2N^2c_2 2(1+q)]|Du|^2 \tag{4.20}$$

and

$$c_4 = \Big(\frac{2n^2Nc_1^2}{\nu}+n^2N^2c_2\Big)2^5\frac{(q+1)^2}{(R-\rho)^2}. \tag{4.21}$$

Then

$$\frac{\nu}{2}\int_{\Omega}|Du|^2\psi_k(|u|)\eta^2 \leq c_3 + c_4\int_{\Omega}(1+|u|)^{2(p+1)}; \tag{4.22}$$

positivity of ψ_k and pointwise convergence (3.34) allow us to use Fatou lemma:

$$\frac{\nu}{2}\int_{\Omega}|Du|^2(p+1)^2|u|^{2p}\eta^2 \leq c_3 + c_4\int_{\Omega}(1+|u|)^{2(p+1)}. \tag{4.23}$$

Let us set

$$w = |u|^{p+1}\eta; \tag{4.24}$$

then

$$w \in W_0^{1,2}(B_R) \tag{4.25}$$

and

$$|Dw|^2 \le 2(p+1)^2|u|^{2p}|Du|^2\eta^2 + 2n|u|^{2(p+1)}\left(\frac{4}{R-\rho}\right)^2. \tag{4.26}$$

The previous inequality and (4.23) give

$$\begin{aligned}\int_{B_R} |Dw|^2 &\le 2\int_{B_R} (p+1)^2|u|^{2p}|Du|^2\eta^2 + 2n\left(\frac{4}{R-\rho}\right)^2\int_{B_R} |u|^{2(p+1)} \\ &\le \frac{4}{\nu}\left(c_3 + c_4\int_\Omega (1+|u|)^{2(p+1)}\right) + 2n\left(\frac{4}{R-\rho}\right)^2\int_\Omega (1+|u|)^{2(p+1)} \\ &= c_5 + c_6\int_\Omega (1+|u|)^{2(p+1)}\end{aligned} \tag{4.27}$$

where

$$c_5 = \frac{4}{\nu}c_3 \tag{4.28}$$

and

$$c_6 = \frac{4}{\nu}c_4 + 2n\left(\frac{4}{R-\rho}\right)^2. \tag{4.29}$$

Since $2 < n$ we can use Sobolev embedding (3.60) and we get

$$\begin{aligned}\int_{B_\rho} |u|^{(p+1)2^*} &\le \int_{B_R} ||u|^{p+1}\eta|^{2^*} = \int_{B_R} |w|^{2^*} \le \left(\frac{2(n-1)}{n-2}\int_{B_R} |Dw|^2\right)^{2^*/2} \\ &\le \left(\frac{2(n-1)}{n-2}\left(c_5 + c_6\int_\Omega (1+|u|)^{2(p+1)}\right)\right)^{2^*/2}.\end{aligned} \tag{4.30}$$

Let us summarize as follows: if for some $p \in (0, +\infty)$ with

$$p \le \frac{q}{2} \tag{4.31}$$

and for some $0 < \rho < R \le 1$ with $\overline{B_R} \subset \Omega$ we have

$$|u|^{2(p+1)} \in L^1(B_R) \tag{4.32}$$

then it results that

$$|u|^{2^*(p+1)} \in L^1(B_\rho). \tag{4.33}$$

Now we argue as in the proof of previous Theorem 2.1 and we get the desired integrability. This ends the proof of Theorem 2.2. □

Added in proofs: after the paper was completed and accepted, we were told about [*] where similar results are contained.

References

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