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Algebraic Geometry — *Toward a geometric construction of fake projective planes*, by Jonghae Keum, presented on 11 November 2011 by Fabrizio Catanese.

Abstract. — We give a criterion for a projective surface to become a quotient of a fake projective plane. We also give a detailed information on the elliptic fibration of a $(2,3)$-elliptic surface that is the minimal resolution of a quotient of a fake projective plane.

Key words: Fake projective plane, $\mathbb{Q}$-homology projective plane, surface of general type, properly elliptic surface.

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It is known that a compact complex manifold of dimension 2 with the same Betti numbers as the complex projective plane $\mathbb{P}^2$ is projective (see e.g. [BHPV]). Such a manifold is called *a fake projective plane* if it is not isomorphic to $\mathbb{P}^2$.

Let X be a fake projective plane. By definition $b_1(X) = 0$, $b_2(X) = 1$, hence $q(X) = p_g(X) = 0$, $c_2(X) = 3$ and by Noether formula $c_1(X)^2 = 9$. In particular its canonical class K_X or its anti-canonical class $-K_X$ is ample. The latter case cannot occur sice X is not isomorphic to $\mathbb{P}^2$. So a fake projective plane is exactly a smooth surface X of general type with $p_g(X) = 0$ and $c_1(X)^2 = 3c_2(X) = 9$. By [Au] and [Y], its universal cover is the unit 2-ball $\mathbf{B} \subset \mathbb{C}^2$ and hence its fundamental group $\pi_1(X)$ is infinite. More precisely, $\pi_1(X)$ is exactly a discrete torsion-free cocompact subgroup Π of $PU(2,1)$ having minimal Betti numbers and finite abelianization. By Mostow's rigidity theorem [Mos], such a ball quotient is strongly rigid, i.e., Π determines a fake projective plane up to holomorphic or anti-holomorphic isomorphism. By [KK], no fake projective plane can be anti-holomorphic to itself. Thus the moduli space of fake projective planes consists of a finite number of points, and the number is the double of the number of distinct fundamental groups Π. By Hirzebruch's proportionality principle [Hir], Π has covolume 1 in $PU(2,1)$. Furthermore, Klingler [Kl] proved that the discrete torsion-free cocompact subgroups of $PU(2,1)$ having minimal Betti numbers are arithmetic (see also [Ye]).

With these informations, Prasad and Yeung [PY] carried out a classification of fundamental groups of fake projective planes. They describe the algebraic group $\overline{G}(k)$ containing a discrete torsion-free cocompact arithmetic subgroup Π having minimal Betti numbers and finite abelianization as follows. There is a pair

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(k,l) of number fields, k is totally real, l a totally complex quadratic extension of k. There is a central simple algebra D of degree 3 with center l and an involution ι of the second kind on D such that $k = l^\iota$. The algebraic group $\overline{G}$ is defined over k as follows:

$$\overline{G}(k) \cong \{z \in D \mid \iota(z)z = 1\}/\{t \in l \mid \iota(t)t = 1\}.$$

There is one Archimedean place v_0 of k so that $\overline{G}(k_{v_0}) \cong PU(2,1)$ and $\overline{G}(k_v)$ is compact for all other Archimedean places v. The data (k,l,D,v_0) determines $\overline{G}$ up to k-isomorphism. Using Prasad's volume formula [P], they were able to eliminate most 4-tuples (k,l,D,v_0), making a short list of possibilities where such Π's might occur, which yields a short list of maximal arithmetic subgroups $\overline{\Gamma}$ which might contain such a Π. If such a Π is contained, up to conjugacy, in a unique $\overline{\Gamma}$, then the group Π or the fake projective plane $\mathbf{B}/\Pi$ is said to belong to the *class* corresponding to the conjugacy class of $\overline{\Gamma}$. If Π is contained in two non-conjugate maximal arithmetic subgroups, then Π or $\mathbf{B}/\Pi$ is said to form a *class* of its own. They exhibited 28 non-empty classes ([PY], Addendum). It turns out that the index of such a Π in a $\overline{\Gamma}$ is 1, 3, 9, or 21, and all such Π's contained in the same $\overline{\Gamma}$ class have the same index.

Then Cartwright and Steger [CS] have carried out a computer-based but very complicated group-theoretic computation, showing that there are exactly 28 non-empty classes, where 25 of them correspond to conjugacy classes of maximal arithmetic subgroups and each of the remaining 3 to a Π contained in two non-conjugate maximal arithmetic subgroups. This yields a complete list of fundamental groups of fake projective planes: the moduli space consists of exactly 100 points, corresponding to 50 pairs of complex conjugate fake projective planes.

It is easy to see that the automorphism group $Aut(X)$ of a fake projective plane X can be given by

$$Aut(X) \cong N(\pi_1(X))/\pi_1(X),$$

where $N(\pi_1(X))$ is the normalizer of $\pi_1(X)$ in $PU(2,1)$, hence is contained in a suitable $\overline{\Gamma}$.

Theorem 0.1 [PY], [CS], [CS2]. *For a fake projective plane X,*

$$Aut(X) = \{1\}, C_3, C_3^2, \text{ or } 7:3,$$

where C_n denotes the cyclic group of order n, and 7 : 3 *the unique non-abelian group of order* 21. *More precisely, $Aut(X) = \{1\}$ or C_3, when the index of $\pi_1(X)$ in a maximal arithmetic subgroup is* 3, *$Aut(X) = \{1\}$, C_3 or C_3^2, when the index is* 9, *$Aut(X) = \{1\}$, C_3 or* 7 : 3, *when the index is* 21.

According to ([CS], [CS2]), 68 of the 100 fake projective planes admit a non-trivial group of automorphisms.

Let (X,G) be a pair of a fake projective plane X and a non-trivial group G of automorphisms. In [K08], all possible structures of the quotient surface X/G and its minimal resolution were classified.

THEOREM 0.2 [K08].

(1) *If* $G = C_3$, *then* X/G *is a* $\mathbb{Q}$*-homology projective plane with* 3 *singular points of type* $\frac{1}{3}(1,2)$ *and its minimal resolution is a minimal surface of general type with* $p_g = 0$ *and* $K^2 = 3$.
(2) *If* $G = C_3^2$, *then* X/G *is a* $\mathbb{Q}$*-homology projective plane with* 4 *singular points of type* $\frac{1}{3}(1,2)$ *and its minimal resolution is a minimal surface of general type with* $p_g = 0$ *and* $K^2 = 1$.
(3) *If* $G = C_7$, *then* X/G *is a* $\mathbb{Q}$*-homology projective plane with* 3 *singular points of type* $\frac{1}{7}(1,5)$ *and its minimal resolution is a* $(2,3)$-, $(2,4)$-, *or* $(3,3)$*-elliptic surface.*
(4) *If* $G = 7:3$, *then* X/G *is a* $\mathbb{Q}$*-homology projective plane with* 4 *singular points,* 3 *of type* $\frac{1}{3}(1,2)$ *and one of type* $\frac{1}{7}(1,5)$, *and its minimal resolution is a* $(2,3)$-, $(2,4)$-, *or* $(3,3)$*-elliptic surface.*

Here, a $\mathbb{Q}$-homology projective plane is a normal projective surface with the same Betti numbers as $\mathbb{P}^2$. A fake projective plane is a nonsingular $\mathbb{Q}$-homology projective plane, hence every quotient is again a $\mathbb{Q}$-homology projective plane. An (a,b)-elliptic surface is a relatively minimal elliptic surface over $\mathbb{P}^1$ with $c_2 = 12$ having two multiple fibres of multiplicity a and b respectively. It has Kodaira dimension 1 if and only if $a \geq 2, b \geq 2, a+b \geq 5$. It is an Enriques surface iff $a = b = 2$, and it is rational iff $a = 1$ or $b = 1$. An (a,b)-elliptic surface has $p_g = q = 0$, and by [D] its fundamental group is the cyclic group of order the greatest common divisor of a and b. An (a,b)-elliptic surface is called a Dolgachev surface if a and b are relatively prime integers with $a \geq 2$, $b \geq 2$.

REMARK 0.3. (1) Since X/G has rational singularities only, X/G and its minimal resolution have the same fundamental group. Let $\overline{\Gamma}$ be the maximal arithmetic subgroup of $PU(2,1)$ containing $\pi_1(X)$. There is a subgroup $\tilde{G} \subset \overline{\Gamma}$ such that $\pi_1(X)$ is normal in $\tilde{G}$ and $G = \tilde{G}/\pi_1(X)$. Thus,

$$X/G \cong \mathbf{B}/\tilde{G}.$$

It is well known (cf. [Arm]) that

$$\pi_1(\mathbf{B}/\tilde{G}) \cong \tilde{G}/H,$$

where H is the minimal normal subgroup of $\tilde{G}$ containing all elements acting non-freely on the 2-ball $\mathbf{B}$. In our situation, it can be shown that H is generated by torsion elements of $\tilde{G}$, and Cartwright and Steger have computed, along with their computation of the fundamental groups, the quotient group $\tilde{G}/H$ for each pair (X, G).

- [CS] If $G = C_3$, then

$$\pi_1(X/G) \cong \{1\}, C_2, C_3, C_4, C_6, C_7, C_{13}, C_{14}, C_2^2, C_2 \times C_4, S_3, D_8 \text{ or } Q_8,$$

where S_3 is the symmetric group of order 6, and D_8 and Q_8 are the dihedral and quaternion groups of order 8.

- [CS2] If $G = C_3^2$ or C_7 or $7:3$, then

$$\pi_1(X/G) \cong \{1\} \text{ or } C_2.$$

This eliminates the possibility of $(3,3)$-elliptic surfaces in Theorem 0.2, as $(3,3)$-elliptic surfaces have $\pi_1 = C_3$.

(2) It is interesting to consider all arithmetic ball quotients which have a non-Galois cover by a fake projective plane. Indeed, Cartwright and Steger have considered all subgroups $\tilde{G} \subset PU(2,1)$ such that $\pi_1(X) \subset \tilde{G} \subset \overline{\Gamma}$ for some maximal arithmetic subgroup $\overline{\Gamma}$ and some fake projective plane X, where $\pi_1(X)$ is not necessarily normal in $\tilde{G}$. It turns out [CS2] that, if $\pi_1(X)$ is not normal in $\tilde{G}$, then there is another fake projective plane X' such that $\pi_1(X')$ is normal in $\tilde{G}$, hence $\mathbf{B}/\tilde{G} \cong X'/G'$ where $G' = \tilde{G}/\pi_1(X')$. Thus such a general subgroup $\tilde{G}$ does not produce a new surface.

It is a major step toward a geometric construction of a fake projective plane to construct a $\mathbb{Q}$-homology projective plane satisfying one of the descriptions (1)–(4) from Theorem 0.2. Suppose that one has such a $\mathbb{Q}$-homology projective plane. Then, can one construct a fake projective plane by taking a suitable cover? In other words, does the description (1)–(4) from Theorem 0.2 characterize the quotients of fake projective planes? The answer is affirmative in all cases.

Theorem 0.4. *Let Z be a $\mathbb{Q}$-homology projective plane satisfying one of the descriptions* (1)–(4) *from Theorem* 0.2.

(1) *If Z is a $\mathbb{Q}$-homology projective plane with* 3 *singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 3$, then there is a C_3-cover $X \to Z$ branched exactly at the three singular points of Z such that X is a fake projective plane.*

(2) *If Z is a $\mathbb{Q}$-homology projective plane with* 4 *singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 1$, then there is a C_3-cover $Y \to Z$ branched exactly at three of the four singular points of Z and a C_3-cover $X \to Y$ branched exactly at the three singular points on Y, the pre-image of the remaining singularity on Z, such that X is a fake projective plane. Furthermore, the composite map $X \to Z$ is a C_3^2-cover.*

(3) *If Z is a $\mathbb{Q}$-homology projective plane with* 3 *singular points of type $\frac{1}{7}(1,5)$ and its minimal resolution is a $(2,3)$- or $(2,4)$-elliptic surface, then there is a C_7-cover $X \to Z$ branched exactly at the three singular points of Z such that X is a fake projective plane.*

(4) *If Z is a $\mathbb{Q}$-homology projective plane with* 4 *singular points,* 3 *of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{7}(1,5)$, and its minimal resolution is a $(2,3)$- or $(2,4)$-elliptic surface, then there is a C_3-cover $Y \to Z$ branched exactly at the three singular points of type $\frac{1}{3}(1,2)$ and a C_7-cover $X \to Y$ branched exactly at the three*

singular points, the pre-image of the singularity on Z of type $\frac{1}{7}(1,5)$, such that X is a fake projective plane.

In the case (4), we give a detailed information on the types of singular fibres of the elliptic fibration on the minimal resolution of Z.

THEOREM 0.5. *Let Z be a $\mathbb{Q}$-homology projective plane with 4 singular points, 3 of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{7}(1,5)$. Assume that its minimal resolution $\tilde{Z}$ is a $(2,3)$-elliptic surface. Then*

(1) *the triple cover Y of Z branched at the three singular points of type $\frac{1}{3}(1,2)$ is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{7}(1,5)$;*
(2) *the minimal resolution $\tilde{Y}$ of Y is a $(2,3)$-elliptic surface, where every fibre of the elliptic fibration on $\tilde{Z}$ does not split;*
(3) *the elliptic fibration on $\tilde{Z}$ has 4 singular fibres of type I_3, some of which may have multiplicity 2 or 3;*
(4) *the elliptic fibration on $\tilde{Y}$ has 4 singular fibres, one of type I_9 and 3 of type I_1, and each fibre has the same multiplicity as the corresponding fibre on $\tilde{Z}$.*

The case where $\tilde{Z}$ is a $(2,4)$-elliptic surface was treated in [K11]. The last two assertions of Theorem 0.5 were given without proof in ([K08], Corollary 4.12 and 1.4).

NOTATION

- K_X: a canonical (Weil) divisor of a normal projective variety or a complex manifold X
- $b_i(X) := \dim H^i(X,\mathbb{Q})$ the i-th Betti number of a topological space X
- $e(X)$: the topological Euler number of a complex variety X
- $p_g(X) := \dim H^2(X,\mathcal{O}_X)$, $q(X) := \dim H^1(X,\mathcal{O}_X)$, where X is a compact smooth surface
- $V^G := \{v \in V \mid g(v) = v \text{ for all } g \in G\}$, where a group G acts on V
- a string of type $[n_1, n_2, \ldots, n_l]$: a string of smooth rational curves of self intersection $-n_1, -n_2, \ldots, -n_l$

1. PRELIMINARIES

First, we recall the topological and holomorphic Lefschetz fixed point formulas.

TOPOLOGICAL LEFSCHETZ FIXED POINT FORMULA. *Let M be a compact complex manifold of dimension m admitting a holomorphic map $\sigma : M \to M$. Then the Euler number of the fixed locus M^σ is equal to the alternating sum of the trace of σ^* acting on the cohomology space $H^j(M,\mathbb{Q})$, i.e.,*

$$e(M^\sigma) = \sum_{j=0}^{2m} (-1)^j \, Tr\, \sigma^* \mid H^j(M,\mathbb{Q}).$$

Holomorphic Lefschetz Fixed Point Formula ([AS3], p. 567). *Let M be a compact complex manifold of dimension 2 admitting an automorphism σ. Let $p_1, \ldots, p_l$ be the isolated fixed points of σ and $R_1, \ldots, R_k$ be the 1-dimensional components of the fixed locus S^σ. Then*

$$\sum_{j=0}^{2}(-1)^j \, Tr\,\sigma^* \mid H^j(M, \mathcal{O}_M) = \sum_{j=1}^{l} \frac{1}{\det(I - d\sigma) \mid T_{p_j}} + \sum_{j=1}^{k} \left\{ \frac{1 - g(R_j)}{1 - \xi_j} - \frac{\xi_j R_j^2}{(1 - \xi_j)^2} \right\},$$

where T_{p_j} is the tangent space at p_j, $g(R_j)$ is the genus of R_j and ξ_j is the eigenvalue of the differential $d\sigma$ acting on the normal bundle of R_j in M.

Assume further that σ is of finite and prime order p. Then

$$\frac{1}{p-1}\sum_{i=1}^{p-1}\sum_{j=0}^{2}(-1)^j \, Tr\,\sigma^{i*} \mid H^j(M, \mathcal{O}_M) = \sum_{i=1}^{p-1} a_i r_i + \sum_{j=1}^{k} \left\{ \frac{1 - g(R_j)}{2} + \frac{(p+1)R_j^2}{12} \right\},$$

where r_i is the number of isolated fixed points of σ of type $\frac{1}{p}(1, i)$, and

$$a_i = \frac{1}{p-1}\sum_{j=1}^{p-1} \frac{1}{(1 - \zeta^j)(1 - \zeta^{ij})}$$

with $\zeta = \exp\left(\frac{2\pi\sqrt{-1}}{p}\right)$, e.g., $a_1 = \frac{5-p}{12}$, $a_2 = \frac{11-p}{24}$, etc.

Proposition 1.1. *Let G be a finite group acting on a smooth compact Kähler surface M. Let M/G be the quotient surface and $Y \to M/G$ a minimal resolution. Then the following hold true:*

(1) $q(Y) = \frac{1}{2} b_1(M/G) = \dim H^{0,1}(M)^G$.
(2) *If in addition there is a G-equivariant blowing-up M' of M such that M'/G is isomorphic to a blowing-up of Y, then*

$$p_g(Y) = \dim H^{0,2}(M)^G.$$

(3) *The additional condition of (2) is always satisfied when $|G| \le 3$.*

Proof. (1) By the Hodge decomposition theorem, $H^1(M, \mathbb{C}) \cong H^{0,1}(M) \oplus H^{1,0}(M)$. Thus

$$b_1(M/G) = \dim H^1(M, \mathbb{R})^G = \dim(H^{0,1}(M) \oplus H^{1,0}(M))^G = 2 \dim H^{0,1}(M)^G.$$

Since quotient singularities are rational, van Kampfen's theorem applies to prove

$$\pi_1(Y) \cong \pi_1(M/G),$$

in particular, $b_1(Y) = b_1(M/G)$.

(2)

$$\begin{aligned} p_g(Y) = p_g(M'/G) &= \dim H^0(M', \Omega^2_{M'})^G \\ &= \dim H^0(M, \Omega^2_M)^G = \dim H^{0,2}(M)^G. \end{aligned}$$

(3) Assume $|G| = 3$. For a singular point on M/G of type $\frac{1}{3}(1,1)$, its minimal resolution can be obtained by first blowing up once the corresponding fixed point on M and then taking the quotient by the extended action of G. For a singular point of type $\frac{1}{3}(1,2)$, first blow up three times the corresponding fixed point on M so that the action of G extends to the blowing-up, where the resulting 3 exceptional curves form a string of type $[1,3,1]$, and then take the quotient by the extended action of G, to get a string of type $[3,1,3]$. This gives the blowing-up of Y at the intersection point of the two exceptional curves lying over the singularity. The case with $|G| = 2$ is more simpler. □

For a compact complex manifold M of dimension 2 with $K_M^2 = 3c_2(M) = 9$, it is known that

$$p_g(M) = q(M) \le 2.$$

Indeed, such a surface M has $\chi(\mathcal{O}_M) = 1$, $p_g(M) = q(M)$, and is either isomorphic to $\mathbb{P}^2$ or of general type. (No compact complex smooth surface with $K^2 > 8$ can be birationally isomorphic to a ruled surface or an elliptic surface.) By a result of Miyaoka [Mi], a compact complex smooth surface of general type with $K^2 = 3c_2$ has ample canonical divisor, and hence by [Y] is a ball-quotient. Furthermore, compact complex smooth surfaces with $c_2 < 4$ (such as M) cannot be fibred over a curve of genus ≥ 2 with a general fibre of genus ≥ 2. This can be seen easily by the Euler number formula for fibred surfaces (see e.g. [BHPV], Proposition 11.4). Thus by Castelnuovo-de Franchis theorem $p_g(M) \ge 2q(M) - 3$, which implies $p_g(M) = q(M) \le 3$. The case of $p_g(M) = q(M) = 3$ was eliminated by the classification result of Hacon and Pardini [HP] (see also [Pi] and [CCM]).

Proposition 1.2. *Let M be a complex manifold M of dimension 2 with $K_M^2 = 3c_2(M) = 9$. Then, the following hold true.*

(1) *If M admits an order 7 automorphism σ with isolated fixed points only, then $b_i(M/\langle\sigma\rangle) = b_i(M)$ for $i = 1,2$ and σ fixes exactly 3 points, which yield on the quotient $M/\langle\sigma\rangle$ either 3 singular points of type $\frac{1}{7}(1,5)$ or 2 singular points of type $\frac{1}{7}(1,2)$ and 1 singular point of type $\frac{1}{7}(1,6)$.*
(2) *If M has $p_g(M) = q(M) = 1$ and admits an order 3 automorphism σ with isolated fixed points only, then*

(a) *$b_1(M/\langle\sigma\rangle) = 0$, $b_2(M/\langle\sigma\rangle) = 3$, and $M/\langle\sigma\rangle$ has 6 singular points of type $\frac{1}{3}(1,1)$; or*
(b) *$b_1(M/\langle\sigma\rangle) = 0$, $b_2(M/\langle\sigma\rangle) = 5$, and $M/\langle\sigma\rangle$ has 3 singular points of type $\frac{1}{3}(1,1)$ and 6 singular points of type $\frac{1}{3}(1,2)$; or*
(c) *$b_1(M/\langle\sigma\rangle) = 2$, $b_2(M/\langle\sigma\rangle) = 5$, and $M/\langle\sigma\rangle$ has 3 singular points of type $\frac{1}{3}(1,2)$.*

PROOF. Note that M cannot admit an automorphism of finite order acting freely, because $\chi(\mathcal{O}_M) = 1$ not divisible by any integer ≥ 2.

(1) By the Hodge decomposition theorem,

$$Tr\,\sigma^* \,|\, H^1(M,\mathbb{Z}) = Tr\,\sigma^* \,|\, H^1(M,\mathbb{C}) = Tr\,\sigma^* \,|\, (H^{0,1}(M) \oplus H^{1,0}(M)).$$

Note that this number is an integer. Let $\zeta = \exp\left(\frac{2\pi\sqrt{-1}}{7}\right)$.

Assume that $p_g(M) = q(M) = 2$. Let ζ^i and ζ^j be the eigenvalues of σ^* acting on $H^{0,1}(M)$. Then

$$Tr\,\sigma^* \,|\, H^1(M,\mathbb{Z}) = \zeta^i + \zeta^j + \bar{\zeta}^i + \bar{\zeta}^j,$$

and this is an integer iff $\zeta^i = \zeta^j = 1$. This implies that $Tr\,\sigma^* \,|\, H^{0,1}(M) = 2$ and

$$b_1(M/\langle\sigma\rangle) = \dim H^1(M,\mathbb{R})^{\langle\sigma\rangle} = \frac{1}{|\langle\sigma\rangle|}\sum_{k=1}^{7} Tr\,\sigma^{k*} \,|\, H^1(M,\mathbb{R}) = 4 = b_1(M).$$

By the Topological Lefschetz Fixed Point Formula,

$$e(M^\sigma) = -6 + Tr\,\sigma^* \,|\, H^2(M,\mathbb{Z}), \quad \text{so } 6 < Tr\,\sigma^* \,|\, H^2(M,\mathbb{Z}).$$

Since $b_2(M) = 1 + 4q(M) = 9$ and σ is of order 7, it follows that $Tr\,\sigma^* \,|\, H^2(M,\mathbb{R}) \leq 9 - 7$, unless σ^* acts trivially on $H^2(M,\mathbb{R})$. Thus

$$b_2(M/\langle\sigma\rangle) = \dim H^2(M,\mathbb{R})^{\langle\sigma\rangle} = b_2(M) \quad \text{and} \quad e(M^\sigma) = 3.$$

In particular, σ^* acts trivially on $H^{0,2}(M)$ and $Tr\,\sigma^* \,|\, H^{0,2}(M) = 2$. By the Holomorphic Lefschetz Fixed Point Formula,

$$1 = -\frac{1}{6}r_1 + \frac{1}{6}(r_2 + r_4) + \frac{1}{3}(r_3 + r_5) + \frac{2}{3}r_6,$$

where r_i is the number of isolated fixed points of σ of type $\frac{1}{7}(1,i)$. Since

$$\sum r_i = e(M^\sigma) = 3,$$

we have two solutions:

$$r_3 + r_5 = 3, \quad r_1 = r_2 = r_4 = r_6 = 0; \quad r_2 + r_4 = 2, \quad r_6 = 1, \quad r_1 = r_3 = r_5 = 0.$$

In the former case the quotient $M/\langle\sigma\rangle$ has 3 singular points of type $\frac{1}{7}(1,5)$, and in the latter case 2 singular points of type $\frac{1}{7}(1,2)$ and 1 singular point of type $\frac{1}{7}(1,6)$.

Assume that $p_g(M) = q(M) \le 1$. By the same argument, σ^* acts trivially on $H^1(M,\mathbb{R}) \oplus H^2(M,\mathbb{R})$, and $e(M^\sigma) = 3$.

(2) First note that

$$b_1(M/\langle\sigma\rangle) \le b_1(M) = 2 \quad \text{and} \quad b_2(M/\langle\sigma\rangle) \le b_2(M) = 5.$$

Also note that $\dim H^{1,1}(M) = 1 + 2q(M) = 3$. Since σ^* fixes the class of a fibre of the Albanese fibration $M \to Alb(M)$ and the class of K_M, we have

$$Tr\,\sigma^* \mid H^{1,1}(M) = 2 + \zeta^k \quad \text{where } \zeta = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right).$$

Let ζ^i and ζ^j be the eigenvalues of σ^* acting on $H^{0,1}(M)$ and $H^{0,2}(M)$, respectively.

Assume that $\zeta^i \ne 1$ and $\zeta^j \ne 1$. Then

$$Tr\,\sigma^* \mid H^1(M,\mathbb{Z}) = Tr\,\sigma^* \mid (H^{0,1}(M) \oplus H^{1,0}(M)) = \zeta^i + \bar{\zeta}^i = -1,$$
$$Tr\,\sigma^* \mid (H^{0,2}(M) \oplus H^{2,0}(M)) = \zeta^j + \bar{\zeta}^j = -1.$$

The latter implies that $Tr\,\sigma^* \mid H^{1,1}(M)$ is an integer, hence $\zeta^k = 1$ and $Tr\,\sigma^* \mid H^{1,1}(M) = 3$. Thus

$$b_1(M/\langle\sigma\rangle) = 0 \quad \text{and} \quad b_2(M/\langle\sigma\rangle) = 3.$$

Now by the Topological Lefschetz Fixed Point Formula,

$$e(M^\sigma) = 6,$$

and by the Holomorphic Lefschetz Fixed Point Formula,

$$1 = \frac{1}{6}r_1 + \frac{1}{3}r_2,$$

where r_i is the number of isolated fixed points of σ of type $\frac{1}{3}(1,i)$. Since $r_1 + r_2 = e(M^\sigma) = 6$, we have a unique solution: $r_1 = 6$, $r_2 = 0$. This gives (a).

Assume $\zeta^i \ne 1$ and $\zeta^j = 1$. Then

$$Tr\,\sigma^* \mid H^1(M,\mathbb{Z}) = Tr\,\sigma^* \mid (H^{0,1}(M) \oplus H^{1,0}(M)) = \zeta^i + \bar{\zeta}^i = -1,$$
$$Tr\,\sigma^* \mid (H^{0,2}(M) \oplus H^{2,0}(M)) = 1 + 1 = 2.$$

The latter implies that $Tr\,\sigma^* \mid H^{1,1}(M)$ is an integer, hence $Tr\,\sigma^* \mid H^{1,1}(M) = 3$. Thus

$$b_1(M/\langle\sigma\rangle) = 0 \quad \text{and} \quad b_2(M/\langle\sigma\rangle) = 5.$$

By the Topological Lefschetz Fixed Point Formula, $e(M^\sigma) = 9$, and by the Holomorphic Lefschetz Fixed Point Formula,

$$\frac{1}{2}\{(1-\zeta^i+1)+(1-\zeta^{2i}+1)\} = \frac{5}{2} = \frac{1}{6}r_1 + \frac{1}{3}r_2.$$

Since $r_1 + r_2 = 9$, we have a unique solution: $r_1 = 3$, $r_2 = 6$. This gives (b).

Assume that $\zeta^i = \zeta^j = 1$. Then

$$Tr\,\sigma^* \mid (H^{0,1}(M) \oplus H^{1,0}(M)) = Tr\,\sigma^* \mid (H^{0,2}(M) \oplus H^{2,0}(M)) = 2,$$

$Tr\,\sigma^* \mid H^{1,1}(M) = 3$ and $e(M^\sigma) = 3$. By the Holomorphic Lefschetz Fixed Point Formula,

$$1 = \frac{1}{6}r_1 + \frac{1}{3}r_2.$$

Since $r_1 + r_2 = 3$, we have a unique solution: $r_1 = 0$, $r_2 = 3$. This gives (c).

Assume that $\zeta^i = 1$ and $\zeta^j \neq 1$. Then

$$Tr\,\sigma^* \mid (H^{0,1}(M) \oplus H^{1,0}(M)) = 2,$$
$$Tr\,\sigma^* \mid (H^{0,2}(M) \oplus H^{2,0}(M)) = \zeta^j + \bar{\zeta}^j = -1,$$

$Tr\,\sigma^* \mid H^{1,1}(M) = 3$ and $e(M^\sigma) = 0$. Thus σ acts freely, a contradiction. □

PROPOSITION 1.3. *Let M be an abelian surface. Assume that it admits an order 3 automorphism σ such that $H^{2,0}(M)^{\langle\sigma\rangle} = 0$. Then $b_2(M/\langle\sigma\rangle) = 4$ or 2.*

PROOF. First note that $p_g(M) = 1$ and rank $H^{1,1}(M) = 4$. Let $\zeta = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$.

Let ζ^k be the eigenvalue of σ^* acting on $H^{0,2}(M)$. Since $H^{2,0}(M)^{\langle\sigma\rangle} = 0$, we have $\bar{\zeta}^k \neq 1$, hence

$$Tr\,\sigma^* \mid (H^{0,2}(M) \oplus H^{2,0}(M)) = \zeta^k + \bar{\zeta}^k = -1.$$

It implies that $Tr\,\sigma^* \mid H^{1,1}(M)$ is an integer, hence is equal to 4, 1 or -2. The last possibility can be ruled out, as there is a σ-invariant ample divisor yielding a σ^*-invariant vector in $H^{1,1}(M)$. Finally note that $b_2(M/\langle\sigma\rangle) = \dim H^{1,1}(M)^{\langle\sigma\rangle}$. □

REMARK 1.4. If in addition $H^{1,0}(M)^{\langle\sigma\rangle} = 0$, then either

(1) $r_2 = 0$, $r_1 - \sum R_j^2 = 9$, $b_2(M/\langle\sigma\rangle) = 4$; or
(2) $r_2 = 3$, $r_1 - \sum R_j^2 = 3$, $b_2(M/\langle\sigma\rangle) = 2$.

Here r_i is the number of isolated fixed points of type $\frac{1}{3}(1,i)$, and $\bigcup R_j$ is the 1-dimensional fixed locus of σ.

PROPOSITION 1.5. *Let M be a surface of general type with $p_g(M) = q(M) = 2$. Assume that it admits an order 3 automorphism σ with isolated fixed points only such that $p_g(M/\langle\sigma\rangle') = q(M/\langle\sigma\rangle') = 0$ where $M/\langle\sigma\rangle'$ is a minimal resolution of $M/\langle\sigma\rangle$. Let $\bar{a} : M/\langle\sigma\rangle \to Alb(M)/\langle\sigma\rangle$ be the map induced by the Albanese map $a : M \to Alb(M)$. Then $\bar{a}$ cannot factor through a surjective map $M/\langle\sigma\rangle \to N$ to a normal projective surface N with Picard number* 1.

PROOF. Suppose that $\bar{a}$ factors through a surjective map $M/\langle\sigma\rangle \to N$ to a normal projective surface N with Picard number 1, i.e.,

$$\bar{a} : M/\langle\sigma\rangle \to N \to Alb(M)/\langle\sigma\rangle.$$

Let $b : N \to Alb(M)/\langle\sigma\rangle$ be the second map. Since a normal projective surface with Picard number 1 cannot be fibred over any curve, the map b is surjective. Since $p_g(M/\langle\sigma\rangle') = q(M/\langle\sigma\rangle') = 0$ and the map $M/\langle\sigma\rangle' \to Alb(M)/\langle\sigma\rangle$ is a surjection, we have

$$p_g(Alb(M)/\langle\sigma\rangle') = q(Alb(M)/\langle\sigma\rangle') = 0,$$

where $Alb(M)/\langle\sigma\rangle'$ is a minimal resolution of $Alb(M)/\langle\sigma\rangle$. Since $Alb(M)/\langle\sigma\rangle'$ has $p_g = q = 0$, we have

$$\mathrm{Pic}(Alb(M)/\langle\sigma\rangle') \cong H^2(Alb(M)/\langle\sigma\rangle', \mathbb{Z}).$$

It follows that the Picard number of $Alb(M)/\langle\sigma\rangle$ is equal to $b_2(Alb(M)/\langle\sigma\rangle)$, which is, by Proposition 1.1 and 1.3, equal to 4 or 2. This is a contradiction, as a normal projective surface with Picard number 1 cannot be mapped surjectively onto a surface with Picard number ≥ 2. $\square$

Let S be a normal projective surface with quotient singularities and

$$f : S' \to S$$

be a minimal resolution of S. It is well-known (e.g., [Ka] or [S]) that quotient singularities are log-terminal singularities. Thus one can write the adjunction formula,

$$K_{S'} \underset{num}{\equiv} f^*K_S - \sum_{p \in Sing(S)} \mathscr{D}_p,$$

where $\mathscr{D}_p = \sum(a_j A_j)$ is an effective $\mathbb{Q}$-divisor with $0 \leq a_j < 1$ supported on $f^{-1}(p) = \bigcup A_j$ for each singular point p. It implies that

$$K_S^2 = K_{S'}^2 - \sum_p \mathscr{D}_p^2 = K_{S'}^2 + \sum_p \mathscr{D}_p K_{S'}.$$

The coefficients of the $\mathbb{Q}$-divisor $\mathscr{D}_p$ can be obtained by solving the equations

$$\mathscr{D}_p A_j = -K_{S'} A_j = 2 + A_j^2$$

given by the adjunction formula for each exceptional curve $A_j \subset f^{-1}(p)$.

The computation of $\mathscr{D}_p^2$ is given in [HK], Lemma 3.6 and 3.7.

2. The Proof of Theorem 0.4

2.1. *The case: Z has 3 singular points of type $\frac{1}{3}(1,2)$*

Let p_1, p_2, p_3 be the three singular points of Z of type $\frac{1}{3}(1,2)$, and $\tilde{Z} \to Z$ be the minimal resolution.

Lemma 2.1. *There is a C_3-cover $X \to Z$ branched exactly at the three singular points of Z.*

Proof. We use a lattice theoretic argument. Consider the cohomology lattice

$$H^2(\tilde{Z}, \mathbb{Z})_{free} := H^2(\tilde{Z}, \mathbb{Z})/(torsion)$$

which is unimodular of signature $(1,6)$ under intersection pairing. Since Z is a $\mathbb{Q}$-homology projective plane, $p_g(\tilde{Z}) = q(\tilde{Z}) = 0$ and hence $\mathrm{Pic}(\tilde{Z}) = H^2(\tilde{Z}, \mathbb{Z})$. Let $\mathscr{R}_i \subset H^2(\tilde{Z}, \mathbb{Z})_{free}$ be the sublattice spanned by the numerical classes of the components A_{i1}, A_{i2} of $f^{-1}(p_i)$. Consider the sublattice $\mathscr{R} := \mathscr{R}_1 \oplus \mathscr{R}_2 \oplus \mathscr{R}_3$. Its discriminant group $\mathscr{R}^*/\mathscr{R}$ is generated by three order 3 elements e_1, e_2, e_3, where e_i is the generator of $\mathscr{R}_i^*/\mathscr{R}_i$ of the form

$$e_i = \frac{A_{i1} + 2A_{i2}}{3}.$$

Since $\mathscr{R}$ is of co-rank 1, we see that $\overline{\mathscr{R}}/\mathscr{R}$ is a non-zero subgroup of $\mathscr{R}^*/\mathscr{R}$, where $\overline{\mathscr{R}}$ is the primitive closure of $\mathscr{R}$. Thus there is an element $D \in \overline{\mathscr{R}} \backslash \mathscr{R}$ such that

$$D = a_1 e_1 + a_2 e_2 + a_3 e_3 \text{ modulo } \mathscr{R}.$$

Since $e_i^2 = -\frac{2}{3}$, none of the a_i's is equal to 0 modulo 3; otherwise D^2 would not be an integer. Note that $-e_i = 2e_i = \frac{2A_{i1}+A_{i2}}{3}$ modulo $\mathscr{R}$. Thus we may assume that $a_1 = a_2 = a_3 = 1$, hence

$$D = \frac{A_{11} + 2A_{12}}{3} + \frac{A_{21} + 2A_{22}}{3} + \frac{A_{31} + 2A_{32}}{3} + R \quad \text{for some } R \in \mathscr{R}.$$

It follows that there is a divisor class $L \in \mathrm{Pic}(\tilde{Z})$ such that

$$3L = B + \tau$$

for some torsion divisor τ, where $B = A_{11} + 2A_{12} + A_{21} + 2A_{22} + A_{31} + 2A_{32}$ an integral divisor supported on the six (-2)-curves contracted to the points p_1, p_2, p_3 by the map $\tilde{Z} \to Z$.

If $\tau = 0$, L gives a C_3-cover of $\tilde{Z}$ branched along B and un-ramified outside B, hence yields a C_3-cover $X \to Z$ branched exactly at the three points p_1, p_2, p_3. Since the local fundamental group of the punctured germ of p_i is cyclic of order 3, the covering of the punctured germ is either trivial or the standard one. Since the C_3-cover $X \to Z$ is branched at each p_i, the latter case should occur. Thus X is a nonsingular surface.

If $\tau \neq 0$, let m denote the order of τ. Write $m = 3^t m'$ with m' not divisible by 3. By considering $3(m'L) = m'B + m'\tau$, and by putting $B' = m'B$(modulo 3), $\tau' = m'\tau$, we may assume that τ has order 3^t. The torsion bundle τ gives an un-ramified cyclic cover of degree 3^t

$$p : V \to \tilde{Z}.$$

Let g be the corresponding automorphism of V. Pulling $3L = B + \tau$ back to V, we have

$$3p^*L = p^*B.$$

Obviously, g can be linearized on the line bundle p^*L, hence gives an automorphism of order 3^t of the total space of p^*L. Let $V' \to V$ be the C_3-cover given by p^*L. We regard V' as a subvariety of the total space of p^*L. Since g leaves invariant the set of local defining equations for V', g restricts to an automorphism of V' of order 3^t. Thus we have a C_3-cover

$$V'/\langle g \rangle \to \tilde{Z}.$$

This yields a C_3-cover $X \to Z$ branched exactly at the three points p_1, p_2, p_3. Similarly, X is a nonsingular surface. □

Since Z has only rational double points, the adjunction formula gives $K_Z^2 = K_{\tilde{Z}}^2 = 3$. Hence $K_X^2 = 3K_Z^2 = 9$. The smooth part Z^0 of Z has Euler number $e(Z^0) = e(\tilde{Z}) - 9 = 0$, so $e(X) = 3e(Z^0) + 3 = 3$. This shows that X is a ball quotient with $p_g(X) = q(X)$. It is known that such a surface has $p_g(X) = q(X) \leq 2$. (See the paragraph before Proposition 1.2.) In our situation X admits an order 3 automorphism, and Proposition 1.2 eliminates the possibility of $p_g(X) = q(X) = 1$.

It remains to exclude the possibility of $p_g(X) = q(X) = 2$. Suppose that $p_g(X) = q(X) = 2$. Consider the Albanese map $a : X \to Alb(X)$. It induces a map $\bar{a} : Z = X/\langle \sigma \rangle \to Alb(X)/\langle \sigma \rangle$, where σ is the order 3 automorphism of X corresponding to the C_3-cover $X \to Z$. Since Z has Picard number 1 and $p_g(\tilde{Z}) = q(\tilde{Z}) = 0$, Proposition 1.5 gives a contradiction. Thus, $p_g(X) = q(X) = 0$ and X is a fake projective plane.

2.2. *The case: Z has 4 singular points of type $\frac{1}{3}(1,2)$*

Let p_1, p_2, p_3, p_4 be the four singular points of Z, and $f : \tilde{Z} \to Z$ the minimal resolution.

LEMMA 2.2. *If there is a C_3-cover $Y \to Z$ branched exactly at three of the four singular points of Z, then the minimal resolution $\tilde{Y}$ of Y has $K^2_{\tilde{Y}} = 3$, $e(\tilde{Y}) = 9$ and $p_g(\tilde{Y}) = q(\tilde{Y}) = 0$.*

PROOF. We may assume that the three points are p_2, p_3, p_4. Note that Y has 3 singular points of type $\frac{1}{3}(1,2)$, the pre-image of p_1. Let $\tilde{Y} \to Y$ be the minimal resolution. It is easy to see that $K^2_{\tilde{Y}} = 3$, $e(\tilde{Y}) = 9$ and $p_g(\tilde{Y}) = q(\tilde{Y})$.

Suppose that $p_g(\tilde{Y}) = q(\tilde{Y}) = 1$. Consider the Albanese fibration $\tilde{Y} \to Alb(\tilde{Y})$. It induces a fibration $Y \to Alb(\tilde{Y})$. Let σ be the order 3 automorphism of Y corresponding to the C_3-cover $Y \to Z$. It induces a fibration $\phi : \tilde{Z} \to Alb(\tilde{Y})/\langle\sigma\rangle$. Since $q(\tilde{Z}) = 0$, we have $Alb(\tilde{Y})/\langle\sigma\rangle \cong \mathbb{P}^1$. The eight (-2)-curves of $\tilde{Z}$ are contained in a union of fibres of ϕ. It follows that $\tilde{Z}$ has Picard number $\geq 8 + 2 = 10$, a contradiction.

Suppose that $p_g(\tilde{Y}) = q(\tilde{Y}) = 2$. The Albanese map $a : \tilde{Y} \to Alb(\tilde{Y})$ contracts the six (-2)-curves of $\tilde{Y}$, hence the induced map $\bar{a} : \tilde{Y}/\langle\sigma\rangle \to Alb(\tilde{Y})/\langle\sigma\rangle$ factors through a surjective map $\tilde{Y}/\langle\sigma\rangle \to Z$, where σ is the order 3 automorphism of $\tilde{Y}$ corresponding to the C_3-cover $Y \to Z$. Since Z has Picard number 1 and $\tilde{Z}$, being the minimal resolution of $\tilde{Y}/\langle\sigma\rangle$, has $p_g(\tilde{Z}) = q(\tilde{Z}) = 0$, Proposition 1.5 gives a contradiction.

The possibility of $p_g(\tilde{Y}) = q(\tilde{Y}) \geq 3$ can be ruled out by considering a C_3-cover $X \to Y$ branched at the three singular points of Y. See the paragraph below Lemma 2.3. □

LEMMA 2.3. *There is a C_3-cover $Y \to Z$ branched exactly at three of the four singular points of Z, and a C_3-cover $X \to Y$ branched exactly at the three singular points of Y. The composite map $X \to Z$ is a C_3^2-cover.*

PROOF. The existence of two C_3-covers can be proved by a lattice theoretic argument. Note that $\mathrm{Pic}(\tilde{Z}) = H^2(\tilde{Z}, \mathbb{Z})$. We know that $H^2(\tilde{Z}, \mathbb{Z})_{free}$ is a unimodular lattice of signature $(1, 8)$ under intersection pairing. Let $\mathscr{R}_i \subset H^2(\tilde{Z}, \mathbb{Z})_{free}$ be the sublattice spanned by the numerical classes of the components A_{i1}, A_{i2} of $f^{-1}(p_i)$. Consider the sublattice $\mathscr{R} := \mathscr{R}_1 \oplus \mathscr{R}_2 \oplus \mathscr{R}_3 \oplus \mathscr{R}_4$. Its discriminant group $\mathscr{R}^*/\mathscr{R}$ is 3-elementary of length 4, generated by four order 3 elements e_1, e_2, e_3, e_4, where e_i is the generator of $\mathscr{R}_i^*/\mathscr{R}_i$ of the form $e_i = \frac{A_{i1}+2A_{i2}}{3}$. Since the orthogonal complement $\mathscr{R}^\perp$ is of rank 1, we see that $\overline{\mathscr{R}}/\mathscr{R}$ is a subgroup of order 9 of $\mathscr{R}^*/\mathscr{R}$. As we have seen in the proof of Lemma 2.1, every non-zero element of $\overline{\mathscr{R}}/\mathscr{R}$ must be of the form $\pm e_i \pm e_j \pm e_k$. Thus, up to a permutation of e_i's and modulo $\mathscr{R}$, $\overline{\mathscr{R}}/\mathscr{R}$ is generated by the two order 3 elements

$$e_2 + e_3 + e_4 \quad \text{and} \quad e_1 - e_3 + e_4.$$

As in the proof of Lemma 2.1, we infer that there are two divisor classes $L_1, L_2 \in \mathrm{Pic}(\tilde{Z})$ such that

$$3L_1 = B_1 + \tau_1, \quad 3L_2 = B_2 + \tau_2$$

for some torsion divisors τ_i, where B_i is an integral divisor supported on the six (-2)-curves contained in $\bigcup_{j \neq i} f^{-1}(p_j)$ and each coefficient in B_i is 1 or 2.

By the same argument as in Lemma 2.1, we can take a C_3-cover $Y \to Z$ branched exactly at the three points p_2, p_3, p_4. Then Y has 3 singular points of type $\frac{1}{3}(1,2)$, the pre-image of p_1. This can be done by using the line bundle L_1 if $\tau_1 = 0$. Otherwise, we first take an un-ramified cover $p : V \to \tilde{Z}$ corresponding to τ_1 and then lift the covering automorphism g to the C_3-cover $V' \to V$ given by p^*L_1, then take the quotient $V'/\langle g \rangle$.

Let Y' be the minimal resolution of the fibred product $Y \times_Z \tilde{Z}$, and $\psi : Y' \to \tilde{Z}$ be the C_3-cover corresponding to the C_3-cover $Y \to Z$. Then $Y' \to Y$ is a resolution, hence it factors through a surjection $f' : Y' \to \tilde{Y}$. Now

$$3f'_*(\psi^* L_2) = f'_*(\psi^* B_2) + f'_*(\psi^* \tau_2)$$

and $f'_*(\psi^* B_2)$ is an integral divisor supported on the exceptional locus of $\tilde{Y} \to Y$ with coefficients greater than 0 and less than 3. Now by the same argument as in the proof of Lemma 2.1, there is a C_3-cover $X \to Y$ with X nonsingular.

It remains to show that the composite map $X \to Z$ is a C_3^2-cover. Let σ be the order 3 automorphism of $\tilde{Y}$ corresponding to the C_3-cover $Y \to Z$. It preserves each of the three divisors, $f'_*(\psi^* L_2)$, $f'_*(\psi^* B_2)$, $f'_*(\psi^* \tau_2)$, hence lifts to an automorphism σ' of X, which normalizes the order 3 automorphism μ of X corresponding to the C_3-cover $X \to Y$. The fixed locus $X^{\sigma'}$ is not contained in the fixed locus X^{μ}. Thus $\mu \neq \sigma'^3$, hence the group generated by σ' and μ is isomorphic to C_3^2. □

It is easy to see that $K_X^2 = 9$, $e(X) = 3$ and $p_g(X) = q(X)$. Such a surface has $p_g(X) = q(X) \leq 2$. (See the paragraph before Proposition 1.2.) By Proposition 1.1, $p_g(\tilde{Y}) \leq p_g(X)$ and $q(\tilde{Y}) \leq q(X)$, which completes the proof of Lemma 2.2.

By Lemma 2.2, $p_g(\tilde{Y}) = q(\tilde{Y}) = 0$, so Y has Picard number 1 and contains three singular points of type $\frac{1}{3}(1,2)$. Then by the previous subsection, $p_g(X) = q(X) = 0$, hence X is a fake projective plane.

2.3. *The case: Z has 3 singular points of type $\frac{1}{7}(1,5)$*

Let p_1, p_2, p_3 be the three singular points of Z of type $\frac{1}{7}(1,5)$. Then there is a C_7-cover $X \to Z$ branched at the three points. In the case of $\pi_1(Z) = \{1\}$, this was proved in [K06], p922. In our general situation, we consider the lattice $\mathrm{Pic}(\tilde{Z})/(\mathrm{torsion})$, where $\tilde{Z} \to Z$ is the minimal resolution. Then by the same lattice theoretic argument as in [K06], there is a divisor class $L \in \mathrm{Pic}(\tilde{Z}) = H^2(\tilde{Z}, \mathbb{Z})$ such that $7L = B + \tau$ for some torsion divisor τ, where B is an integral divisor supported on the exceptional curves of the map $\tilde{Z} \to Z$. Here every coefficient of B is not equal to 0 modulo 7. If $\tilde{Z}$ is a $(2,4)$-elliptic surface and if $\tau \neq 0$, then $2\tau = 0$. By considering $7(2L) = 2B$, and by putting $L' = 2L$ and $B' = 2B$, we get $7L' = B'$. This implies the existence of a C_7-cover $X \to Z$ branched exactly at the three points p_1, p_2, p_3. As in the proof of Lemma 2.1, it can be shown that X is nonsingular.

Note that $K_{\tilde{Z}}^2 = 0$. So by the adjunction formula, $K_Z^2 = \frac{9}{7}$. It is easy to see that $K_X^2 = 9$, $e(X) = 3$ and $p_g(X) = q(X)$. Such a surface has $p_g(X) = q(X) \leq 2$. (See the paragraph before Proposition 1.2.) Now by Proposition 1.2, $p_g(X) = q(X) = 0$.

2.4. The case: Z has 3 singular points of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{7}(1,5)$

Let $\tilde{Z} \to Z$ be the minimal resolution, which is a $(2,3)$- or $(2,4)$-elliptic surface. It contains 9 exceptional curves whose dual diagram is given as follows:

$$(-2) - (-2) \quad (-2) - (-2) \quad (-2) - (-2) \quad (-2) - (-2) - (-3).$$

Here the last three smooth rational curves forming a string of type $[2,2,3]$ are lying over the singular point of type $\frac{1}{7}(1,5)$. This can be seen by computing the Hirzebruch-Jung continued fraction of $\frac{7}{5}$,

$$\frac{7}{5} = 2 - \frac{1}{2 - \frac{1}{3}}.$$

In particular, $\tilde{Z}$ contains a (-3)-curve. By the canonical bundle formula (see [BHPV], Theorem 12.1), the canonical class of a $(2,3)$- (resp. $(2,4)$)-elliptic surface is numerically equivalent to $\frac{1}{6}F$ (resp. $\frac{1}{4}F$), where F is the class of a fibre. Thus a (-3)-curve is a 6-section (resp. 4-section) of a $(2,3)$- (resp. $(2,4)$)-elliptic surface.

Let

$$\phi : \tilde{Z} \to \mathbb{P}^1$$

be the elliptic fibration. Note that every (-2)-curve on an elliptic surface is contained in a fiber. Thus the eight (-2)-curves above are contained in a union of fibres. Let $Z' \to Z$ be the minimal resolution of the singular point of type $\frac{1}{7}(1,5)$. Then $\phi : \tilde{Z} \to \mathbb{P}^1$ induces an elliptic fibration

$$\phi' : Z' \to \mathbb{P}^1.$$

Lemma 2.4. (1) *There is a C_3-cover $Y \to Z$ branched exactly at the three points of type $\frac{1}{3}(1,2)$. The cover Y has 3 singular points of type $\frac{1}{7}(1,5)$.*

(2) *The minimal resolution $\tilde{Y}$ of Y is a $(2,3)$- or $(2,4)$-elliptic surface. Every fibre of $\tilde{Z}$ does not split in $\tilde{Y}$, and every fibre of $\tilde{Y}$ has the same multiplicity as the corresponding fibre of $\tilde{Z}$.*

Proof. We may assume that $\tilde{Z}$ is a $(2,3)$-elliptic surface. The case of $(2,4)$-elliptic surfaces was proved in [K11].

(1) The existence of the triple cover can be proved in the same way as in [K06], p920–921. Note that Y has 3 singular points of type $\frac{1}{7}(1,5)$, the pre-image of the singular point of Z of type $\frac{1}{7}(1,5)$.

(2) Consider the C_3-cover $\tilde{Y} \to Z'$ branched at the three singular points of Z'. The elliptic fibration $\phi' : Z' \to \mathbb{P}^1$ induces an elliptic fibration $\psi : \tilde{Y} \to \mathbb{P}^1$. Denote by E the (-3)-curve in Z' lying over the singularity of type $\frac{1}{7}(1,5)$. It does not pass through any of the 3 singular points of Z', hence it splits in $\tilde{Y}$ to give three (-3)-curves E_1, E_2, E_3.

Suppose that a general fibre of Z' splits into 3 fibres in $\tilde{Y}$. Since E is a 6-section, each E_i will be a 2-section of the elliptic fibration $\psi : \tilde{Y} \to \mathbb{P}^1$. Thus, the map from E_i to the base curve $\mathbb{P}^1$ is of degree 2. It implies that $\tilde{Y}$ has at most 2 multiple fibres and the multiplicity of every multiple fibre is 2. Thus each multiple fibre of Z' does not split in $\tilde{Y}$. (Otherwise, it will give 3 multiple fibres of the same multiplicity, a contradiction.) The fibre with multiplicity 3 in Z' does not split, hence it gives a non-multiple fibre in $\tilde{Y}$. But the fibre with multiplicity 2 in Z' must split into 3 fibres in $\tilde{Y}$. This is a contradiction, and we have proved that every fibre of Z' does not split in $\tilde{Y}$. It implies that the multiplicity of a fibre in $\tilde{Y}$ is the same as that of the corresponding fibre in $\tilde{Z}$. Thus $\tilde{Y}$ is an elliptic surface over $\mathbb{P}^1$ having 2 multiple fibres with multiplicity 2 and 3, resp. Since $K_{\tilde{Z}}^2 = 0$ and Z' has only rational double points, the adjunction formula gives $K_{Z'}^2 = K_{\tilde{Z}}^2 = 0$. Hence $K_{\tilde{Y}}^2 = 3K_{Z'}^2 = 0$. In particular, $\tilde{Y}$ is minimal. The smooth part Z^0 of Z' has Euler number $e(Z^0) = e(\tilde{Z}) - 9 = 3$, so $e(\tilde{Y}) = 3e(Z^0) + 3 = 12$. This shows that $\tilde{Y}$ is a $(2,3)$-elliptic surface. □

Now by the previous subsection, there is a C_7-cover $X \to Y$ branched at the three singular points such that X is a fake projective plane.

3. Proof of Theorem 0.5

The first two assertions of Theorem 0.5 were proved in Lemma 2.4.

(3) We know that the eight (-2)-curves on $\tilde{Z}$ are contained in a union of fibres. This is possible only if the union of fibres is one of the following three cases. Here, each fibre of type I_3 may be a multiple fibre with multiplicity 2 or 3.

$$\text{(a) } IV^* + I_3, \quad \text{(b) } IV^* + IV, \quad \text{(c) } I_3 + I_3 + I_3 + I_3.$$

Recall that every fibre in $\tilde{Z}$ does not split in $\tilde{Y}$, and the (-3)-curve in $\tilde{Z}$ is a 6-section. We will eliminate the first two cases. Let $Z' \to Z$ be the minimal resolution of the singular point of type $\frac{1}{7}(1,5)$.

Case (a): $IV^* + I_3$. In this case, the surface $\tilde{Z}$ has a singular fibre of type I_1, which may be multiple. Since the (-3)-curve in $\tilde{Z}$ is a 6-section, it intersects with multiplicity 2 the central component of the IV^*-fibre. Thus the six components of the IV^*-fibre except the central component are the six (-2)-curves contracted by the map $\tilde{Z} \to Z'$, hence both the I_3-fibre and the I_1-fibre are disjoint from the branch points of the C_3-cover $\tilde{Y} \to Z'$. It is easy to see that these two fibres will give a I_9-fibre and a I_3-fibre in $\tilde{Y}$, so $\tilde{Y}$ has Picard number ≥ 12, a contradiction.

Case (b): $IV^* + IV$. Again, the (-3)-curve intersects with multiplicity 2 the central component of the IV^*-fibre, hence the six components of the IV^*-fibre except the central component are the six (-2)-curves contracted by the map $\tilde{Z} \to Z'$. The IV-fibre on $\tilde{Z}$ is disjoint from the branch points of the C_3-cover $\tilde{Y} \to Z'$. But there is no un-ramified connected triple cover of a IV-fibre, a contradiction.

Thus $\tilde{Z}$ has four I_3-fibres.

(4) If the image in Z' of a I_3-fibre contains a singular point of Z', then it will give a I_1-fibre in $\tilde{Y}$. If it does not, then it will give a I_9-fibre in $\tilde{Y}$. Thus $\tilde{Y}$ has one I_9-fibre and three I_1-fibres.

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