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Mechanics — *Constrained ephemeral continua*, by Gianfranco Capriz, Eliot Fried and Brian Seguin.

Abstract. — The theory of ephemeral continua was proposed to model bodies for which the basic tenet of permanence of material elements fails. The goal of the proposal was, principally, to lessen the impact of critical arguments against the imposition of the principle of material frame-indifference in continuum mechanics. Those arguments were based on the remark that, in any case, any mathematical model of reality is necessarily observer dependent; however, as it happens, they were urged on by noticing that some corollaries in the kinetic theory of gases appear to contradict requirements of frame-indifference. The proposed theory cures that wound via a definition of peculiar velocities which assures their exact observer independence. Besides, that theory offers a wider base, allowing one to breed models for bodies where the processes are irremediably influenced by events at a lower scale, models based so far on a bottom up approach with partly uncertain steps up from the standard case. Here, we proceed via a top down approach, by introducing progressively stronger constraints. Largely, earlier results are confirmed, securer ground is given to some interpretations or alternatives offered. Briefly, the general requirements for any linear constraint to have physical content are stated and their immediate implications are derived. These requirements are applied to study certain special cases of the general constraint. Connections are provided between the consequent simplifications and results available by bottom-up approaches, so to speak—including, in particular, hypocontinua and Navier–Stokes-$\alpha\beta$ continua. To show that the value of our general approach transcends the building of an all-encompassing framework, we conclude by deriving the reduced balance laws consistent with the most general version of the constraint. These laws open an easy approach to obtaining potentially useful special cases.

Key words: Extended mechanics, kinematical constraints, hypocontinua, turbulence models.

Mathematics Subject Classification: 74A25, 74A30, 74A35, 74A60, 76A02, 76A05, 76A99, 76F02.

1. Introduction

A basic tenet of a rigorous account of the classical theory of fields (fully and elegantly provided by Truesdell and Toupin [14] in the homonymous treatise within the *Encyclopaedia of Physics*) is the declaration that 'material elements' (intended to be ground constituents of any physical body) are perfectly identifiable. Each element occupies exclusively at any instant τ a place x in Euclidean point space, its real expanse being ideally collapsed into x.

The alternative to that tenet is pursued by the theory of ephemeral continua (see Capriz [3]). Bodies are imagined as placed over a fixed network of loculi (or representative volume elements) $\mathfrak{e}$, each so small that, at the gross macroscopic level, it is confused again with a place x (as each material element above is).

However, at a deeper investigation, each loculus $\mathfrak{e}(x)$ is expanded into a cube of edge δ, say, and a subset of molecules making up the body is seen to transit at time τ through subplaces y within $\mathfrak{e}(x)$ with velocity $w(x, y, \tau)$. The molecular numerosity is also specified as depending on subplace y within $\mathfrak{e}(x)$ under the only condition that the center of mass coincide with the center of the cube. The molecular subset is not necessarily constrained to obey a sort of allegiance to a permanent population; rather, its members are allowed to roam later into different loculi. Then the kinetic fields entering the theory are at each instant τ born out of statistics over $\mathfrak{e}(x)$: average velocity $v(x, \tau)$, mesoinertia tensor $Y(x, \tau)$, moment of mesomomentum tensor $K(x, \tau)$, and variance tensor $H(x, \tau)$. It is worth emphasizing that we tend to consider the statistical quantities K and H as members of the kinetic world, though, seeing as how these objects have respective dimensions of energy per unit mass per unit time and energy per unit mass, there might be a tendency to view them otherwise. The picture becomes complex leading to a system of balance laws, which formally have a strong similarity with those for multipolar bodies, though there are substantial differences in significance and detail; besides some foreign terms need to be explained.

Basically, even if, to get nearer to an accord with that theory, a phantom material element is, in imagination, created, passing through x at time τ with velocity $v(x, \tau)$, we must, at least, attribute it a variable mass with a rate of change, per unit volume, $\varrho\sigma$ (ϱ, mass density; $\sigma = \mathrm{tr}(L - B)$, with the usual significance of L as $\mathrm{grad}\, v$ and B a mesoscale measure of molecular motion at the locular level so that the standard relation $K = YB^{\top}$ applies). Of course, this discrepancy could be eliminated, at the cost of altering, possibly, the physical picture of events, by introducing the constraint $\sigma = 0$ and by adapting the balance laws consequently to make the constraint acceptable.

In addition to exploring the ramifications of the constraint $\sigma = 0$, we consider those of various stronger constraints that incorporate the requirement that the traces of L and B agree, and, thus, that $\sigma = 0$. The particular cases we examine first are addressed just to evidence new prospects conceivable, rather than the feasibility of a confluence. Even if the field v is, in addition, divergence-free, the mass density need not be constant within a material element (which is here, we insist, a mental construct rather than, necessarily, a physical reality). Conversely, for that condition to apply, it is the trace of B, not that of L, which must vanish.

Still, if appropriate constraints are imposed on the macroscopic and mesoscopic disfigurements, as described, respectively, by L and B, a deeper discrepancy would arise anyway from the essential premise of the theory that, locally, the movement of molecules is not necessarily exactly that of an affine (or, as some prefer to call it, pseudorigid) body. Actually the interest centers on the contrary circumstances when w differs from $v + By$ and the random peculiar velocity $c = w - (v + By)$ does not vanish. Otherwise H would vanish also and most of the peculiarities of ephemerality would be lost.

Another discrepancy is met when the time derivatives of K and H are sought; local time dependence apart, there is the effect of entrainment, not by the macroscopic motion (decided by the field v), but rather by the deeper events so that the derivative must be of 'coshaping' type. This rate hinges on treating each loculus as a Euclidean space with local tangent space decided by mesoscopic effects through a tensor G in analogy with the notion of the usual macroscopic placement gradient F.[1] The term 'coshaping' was coined in similarity with the more common 'corotational.' However, since the essential features of the rate are not substantially different from what has become known as the Truesdell [12] rate, the introduction of the term 'coshaping' might have been avoided. Other constraints may lessen or avoid altogether this latter discrepancy and we explore here some of them.

A comment regarding the balance law for H is appropriate. We argue that such a balance law is buried usually under the wider shelter offered by the first principle of thermodynamics. With some courage one could suggest that, at times, it may, in fact, replace that principle, thus overcoming worrying later restrictions to quasistaticity and reversibility; Vilar and Rubi [15] and Reguera, Rubi and Vilar [11] have advanced a similar suggestion with notable success.

All these remarks tend to clarify points already made, if sometimes only implicitly, in the cited papers. We thought opportune to repeat them here, however, to avoid possible doubts or misunderstandings. The main topic of the paper is, of course, the consequence of the introduction of a class of perfect constraints in ephemeral continua, with the goal being to explore connections with other more or less established theories. The matter calls for a number of difficult choices, the predominant one regards the definition itself of perfection: it is modeled on the usual requirement on the power of internal actions, but it invokes also a restrictive assumption of independence of global effects (measured by the tensor Z) on 'collisions' (measured by H) from L and B, allowing perhaps only an influence of their gradients. But there are other important and debatable choices in the interpretation of consequences. Already Dunn and Serrin [7] had indicated the need, in dealing with complex bodies, to introduce interstitial effects and Capriz [2] showed that terms of that type could be interpreted as the consequences of the existence of a latent substructure. In the present setting, the matter arises emphatically and calls for discussion. It seemed, from the papers just quoted, that the need only arose for the insertion of additional active stresses and powers in the balance laws. Here, we dare to suggest that additions are required also to measure the intensity of external actions associated with kinetic energy. Were it to

[1] Capriz [3], followed by Brocato and Capriz [1] and Capriz and Fried [5], suggested obtaining G by

$$\dot{G}G^{-1} + \frac{1}{2}\sigma I = K^{\top} Y^{-1}$$

instead of simply, and, perhaps, more appropriately, by

$$\dot{G}G^{-1} = K^{\top} Y^{-1}.$$

(Recall that $B = KY^{-1}$.) The former suggestion was an attempt to attribute to G responsibility for changes of volume adequate, when multiplied by ϱ, to account for changes of mass. Rather, the change of volume must be accounted for by the rate of a ratio involving the determinants of F and G. Whereas, in the case of the former suggestion, the mesoinertia tensor is defined by $Y = \delta^2 GG^{\top}$, in the case of the alternative suggestion, we have $Y = \delta^2 \det(GF^{-1})GG^{\top}$.

stop at that, the matter would certainly not be gravely contentious; but, then, in the end, the suggestion would imply that some sort of anisotropic pressure be classified as an inertial effect rather than as a form of stress! On the other hand, our proposal is far less radical than that appearing in Truesdell's [13] paper on hypoelasticity, where the whole stress is vouchsafed some quality of inertia.

The paper is organized as follows. In Section 2, we introduce essential notation. In Section 3, we reprise the theory of ephemeral continua, focusing on its governing balance laws and the most salient of its distinguishing features. In Section 4, we discuss, in general, linear constraints involving the velocity gradient L and the mesodistorion rate B. Aside from exploring the algebraic properties of such constraints, with the goal of providing a complete understanding of the range of admissible values of L and B allowed, we deduce the ramifications of frame-indifference. We are led to two alternative, but equivalent, descriptions of a linear constraint involving L and B, one requiring the provision of two fourth-order tensors $\mathbb{H}$ and $\mathbb{K}$ and the other involving the provision of two isotropic subspaces $\mathscr{U}$ and $\mathscr{W}$ of the space of second-order tensors and a single isotropic fourth-order tensor $\mathbb{N}$. In Section 5, we determine the characteristic attributes of the consequent reactions. In Section 6, we focus on obtaining the reduced forms of the balance laws corresponding to five particular constraints, each addressed in a separate subsection. To elucidate aspects of the theory of ephemeral continua that are independent of suffusion, these particular constraints forbid that effect. In each case, the reduction hinges upon eliminating the reactions associated with terms entering the law expressing moment of mesomomentum balance and our treatment concludes with the relevant set of balance equations. Unsatisfactory endpieces, perhaps; however, the goal here is to expose the kinetic quantities intrinsic to each case, thus deciding limits to fantasy in subsequent attempts to propose constitutive laws. In Section 7, we use the results of Section 5 to determine the general forms of the reduced balance laws for any linear constraint involving L and B. Aside from confirming the endpieces of Section 6, the results of Section 7 are easily applied to obtain other potentially interesting or illuminating specializations of the balance laws.

2. Notation

For clarity, we use the following notational scheme:

- Lower-case Greek letters, $\alpha, \beta, \dots, \omega$, signify scalars.
- Lower-case Latin letters, $a, b, \dots, z$, signify vectors.
- Upper-case Latin letters, $A, B, \dots, Z$, and those upper-case Greek letters, $\Gamma, \Delta, \dots, \Omega$, distinct from upper-case Latin letters signify second-order tensors.
- Lower-case, outlined Latin letters, $\mathbb{a}, \mathbb{b}, \dots, \mathbb{z}$, signify third-order tensors.
- Upper-case, outlined Latin letters, $\mathbb{A}, \mathbb{B}, \dots, \mathbb{Z}$, signify fourth-order tensors.

The space of second-order tensors is denoted by Lin. Also, the symmetric, skew, deviatoric, spherical, and symmetric-deviatoric subspaces of Lin are denoted by

$$
\begin{aligned}
\mathrm{Sym} &= \{A \in \mathrm{Lin} : A^{\top} = A\},\\
\mathrm{Skw} &= \{A \in \mathrm{Lin} : A^{\top} = -A\},\\
\mathrm{Dev} &= \{A \in \mathrm{Lin} : \operatorname{tr} A = 0\},\\
\mathrm{Sph} &= \{A \in \mathrm{Lin} : A = \tfrac{1}{3}(\operatorname{tr} A)I\},
\end{aligned}
$$

where I denotes the second-order identity tensor, and

$$\mathrm{SymDev} = \{A \in \mathrm{Lin} : A^{\top} = A, \operatorname{tr} A = 0\}.$$

Moreover, the group of rotations is denoted by

$$\mathrm{Orth}^{+} = \{Q \in \mathrm{Lin} : Q^{\top}Q = I, \det Q = 1\}.$$

We make regular use of certain fourth-order tensors. Specifically, the fourth-order tensors $\mathbb{S}$, $\mathbb{W}$, $\mathbb{D}$, $\mathbb{T}$, and $\mathbb{S}_0$ are defined such that, given an element A of Lin,

$$
\begin{cases}
\mathbb{S}A = \tfrac{1}{2}(A + A^{\top}), \quad \mathbb{W}A = \tfrac{1}{2}(A - A^{\top}), \quad \mathbb{D}A = A - \tfrac{1}{3}(\operatorname{tr} A)I,\\
\mathbb{T}A = \tfrac{1}{3}(\operatorname{tr} A)I, \quad \mathbb{S}_0 A = \tfrac{1}{2}(A + A^{\top}) - \tfrac{1}{3}(\operatorname{tr} A)I,
\end{cases}
\tag{1}
$$

and, thus, map Lin onto Sym, Skw, Dev, Sph, and SymDev, respectively. As easy consequences of the definitions (1), the fourth-order identity tensor $\mathbb{I}$ admits the decompositions

$$\mathbb{I} = \mathbb{S} + \mathbb{W}, \quad \mathbb{I} = \mathbb{D} + \mathbb{T}, \quad \mathbb{I} = \mathbb{S}_0 + \mathbb{W} + \mathbb{T}. \tag{2}$$

When convenient, we also use $\operatorname{sym} A$, $\operatorname{skw} A$, $\operatorname{dev} A$, and $\operatorname{sph} A$ to denote the symmetric, skew, deviatoric, and spherical components of an element A of Lin. Bearing in mind (1), we therefore have the correspondences:

$$
\begin{cases}
\mathbb{S}A = \operatorname{sym} A, \quad \mathbb{W}A = \operatorname{skw} A, \quad \mathbb{D}A = \operatorname{dev} A,\\
\mathbb{T}A = \operatorname{sph} A, \quad \mathbb{S}_0 A = \operatorname{sym}(\operatorname{dev} A) = \operatorname{dev}(\operatorname{sym} A).
\end{cases}
\tag{3}
$$

Given A and B in Lin, $A \otimes B$ is the fourth-order tensor defined such that, for any C in Lin,

$$(A \otimes B)C = (B \cdot C)A, \tag{4}$$

As a consequence of (4), the fourth-order tensor $\mathbb{T}$ defined in $(1)_4$ admits the representation

$$\mathbb{T} = \tfrac{1}{3} I \otimes I. \tag{5}$$

The major transpose $\mathbb{A}^\top$ of $\mathbb{A}$ is the fourth-order tensor defined such that

$$B \cdot \mathbb{A}^\top C = C \cdot \mathbb{A} B \tag{6}$$

for all second-order tensors B and C. Further,

$$\mathbb{A} \text{ is major-symmetric} \quad \Leftrightarrow \quad \mathbb{A}^\top = \mathbb{A}. \tag{7}$$

Direct calculations then show that each of the fourth-order tensors $\mathbb{S}$, $\mathbb{W}$, $\mathbb{D}$, $\mathbb{T}$, and $\mathbb{S}_0$ defined in (1) is major-symmetric:

$$\mathbb{S}^\top = \mathbb{S}, \quad \mathbb{W}^\top = \mathbb{W}, \quad \mathbb{D}^\top = \mathbb{D}, \quad \mathbb{T}^\top = \mathbb{T}, \quad \mathbb{S}_0^\top = \mathbb{S}_0. \tag{8}$$

A fourth-order tensor $\mathbb{A}$ is isotropic if and only if

$$\mathbb{A}(QBQ^\top) = Q(\mathbb{A}B)Q^\top, \tag{9}$$

for all B in Lin and all Q in Orth$^+$. A standard representation theorem shows that $\mathbb{A}$ is isotropic if and only if there exist scalars λ_1, λ_2, and λ_3 such that

$$\mathbb{A} = \lambda_1 \mathbb{S}_0 + \lambda_2 \mathbb{W} + \lambda_3 \mathbb{T}, \tag{10}$$

in which case $\mathbb{S}$, $\mathbb{W}$, $\mathbb{D}$, $\mathbb{T}$, and $\mathbb{S}_0$ are isotropic. Moreover, by $(8)_{2,4,5}$ and (10), if a fourth-order tensor is isotropic then it must be major-symmetric:

$$\mathbb{A} \text{ is isotropic} \quad \Rightarrow \quad \mathbb{A}^\top = \mathbb{A}. \tag{11}$$

Given a third-order tensor $\mathbbm{a}$ and a vector b, $\mathbbm{a}b$ is the element of Lin defined such that

$$(\mathbbm{a}b) \cdot (a_1 \otimes a_2) = \mathbbm{a} \cdot (a_1 \otimes a_2 \otimes b) \tag{12}$$

for all vectors a_1 and a_2. Moreover, the left and right transposes ${}^t\mathbbm{a}$ and $\mathbbm{a}^t$ of $\mathbbm{a}$ are the third-order tensors defined such that

$${}^t\mathbbm{a} \cdot (a_1 \otimes a_2 \otimes a_3) = \mathbbm{a} \cdot (a_2 \otimes a_1 \otimes a_3) \tag{13}$$

and

$$\mathbbm{a}^t \cdot (a_1 \otimes a_2 \otimes a_3) = \mathbbm{a} \cdot (a_1 \otimes a_3 \otimes a_2) \tag{14}$$

for all vectors a_1, a_2, and a_3. Further, given a fourth-order tensor $\mathbb{A}$ and a third-order tensor $\mathbbm{b}$, $\mathbb{A}\mathbbm{b}$ is the third-order tensor defined such that

$$(\mathbb{A}\mathbbm{b}) \cdot (a_1 \otimes a_2 \otimes a_3) = (\mathbb{A}^\top(a_1 \otimes a_2)) \cdot (\mathbbm{b}a_3) \tag{15}$$

for all vectors a_1, a_2, and a_3.

The spatial gradient and divergence operators are denoted by grad and div, respectively. The divergence of a third-order tensor field is defined via the divergence of a field with values in Lin. Specifically, given a third-order tensor field $\mathbb{a}$ its divergence $\operatorname{div}\mathbb{a}$ is a field with values in Lin defined such that

$$(\operatorname{div}\mathbb{a})b = \operatorname{div}(\mathbb{a}^t b) \tag{16}$$

for all constant vectors b.

3. Reprise of the theory of ephemeral continua

In Capriz's [3] theory of ephemeral continua, each place x in the region $\mathscr{B}(\tau)$ occupied by a body at a time τ is the mass center of molecules passing through a loculus $\mathfrak{e}(x)$ of subplaces. Aside from the conventional notions of mass density ϱ and velocity v, averages with a suitably defined locular number density give rise to a symmetric and positive-definite mesoinertia tensor Y, an affine mesodistortion tensor G, with an affiliated mesoscale measure of motion B, a tensor moment of mesomomentum K, and a symmetric and positive-semidefinite variance H, all of which may depend on place and time. The spatial fields Y, K, and H are all measured per unit mass. Since B is assumed affine, $B = K^\top Y^{-1}$ or, equivalently,

$$K = YB^\top. \tag{17}$$

Possible discrepancy between the macroscopic and mesoscopic disfigurements described by the velocity gradient

$$L = \operatorname{grad} v \tag{18}$$

and the mesodistortion rate B is accompanied by suffusion of matter between loculi, as characterized by

$$\sigma = \operatorname{tr}(L - B). \tag{19}$$

The theory generates balance laws for mass, moment of mesoinertia, linear momentum, moment of mesomomentum, and mesofluctuations. In pointwise form, these balance laws read

$$\begin{cases} \dot\varrho + \varrho \operatorname{div} v = \sigma\varrho, \\ \varrho(\dot Y + \sigma Y - YB^\top - BY) = 0, \\ \varrho(\dot v + \sigma v) = \varrho b + \operatorname{div} T, \\ \varrho(Y\dot B^\top - H) = \varrho M - A + \operatorname{div}\mathbb{m}, \\ \varrho(\dot H + \sigma H - HB^\top - BH) = \varrho J - Z + \operatorname{div}\mathbb{j}, \end{cases} \tag{20}$$

where a superposed dot denotes the material time-derivative, T is the familiar Cauchy stress tensor, A and Z are second-order tensorial internal supply densities associated, respectively, with the moment of mesomomentum and mesofluctuations, $\mathbb{m}$ and $\mathbb{j}$ are third-order tensors, the former a hyperstress associated with the moment of mesomomentum and the latter a measure of power flux, b and M are applied or noninertial external forces, measured per unit mass, and J is the mesofluctuation supply, also measured per unit mass.

When seeking solutions of the system (20), an explicit knowledge of the field Y is typically indispensable; hence, recourse to equation $(20)_2$ is almost always necessary. It may occur sometimes that the whole field G is required, not only the symmetric and positive-definite component $\sqrt{GG^\top}$ appearing in its polar decomposition, in which case an evolution equation for G supersedes $(20)_2$. Vice versa, when deep constraints are imposed such as that leading to hypocontinua, $(20)_2$ simply determines the macroscopic placement gradient F in terms of the macroscopic velocity gradient L via the usual relation $\dot{F} = LF$.

In view of (17) and $(20)_2$, it follows that $Y\dot{B}^\top = \dot{K} + \sigma K - KB^\top - BK$. The left-hand side of $(20)_4$ can thus be expressed in a way that involves explicitly the coshaping rate of K. Hence, the coshaping rate arises in all three of the tensorial balances $(20)_2$, $(20)_4$, and $(20)_5$ of the theory. In place of the classical requirement that T be symmetric, stipulating that the power of the internal actions, which in the theory of ephemeral continua has the form

$$T \cdot L + A^\top \cdot B + {}^t\mathbb{m} \cdot \operatorname{grad} B + \tfrac{1}{2}\operatorname{tr} Z, \tag{21}$$

be frame-indifferent yields the requirement that T and A satisfy

$$\operatorname{skw} T = \operatorname{skw} A, \tag{22}$$

whereas no condition impinges on the deviator of Z. As in the standard reasoning, the presumption is that here—and, later, in a different context—the dependence of the power of the internal actions on L and B occurs only as shown in the first and second terms of (21).

For a detailed justification of the balance equations (20), see Brocato and Capriz [1]. In view of the goal of this report, we must, however, provide at least a cursory intimation of the ideas underlying those equations. Consider a time τ and a place x in $\mathscr{B}(\tau)$. The value $v(x,\tau)$ of the velocity v at that time and place arises on averaging a mesoscale velocity w over all subplaces in the loculus $\mathfrak{e}(x)$. As such, v and w are justly termed filtered and unfiltered velocities. For the difference $w - v$, statistical mesofiltering is performed: the average being obviously null, the mesoinertia tensor Y, the moment of mesomomentum K, and the variance H are evaluated and their laws of evolution sought. The tensor H is present to account for the intensity of collisions within each loculus, so that ϱH provides a sort of anisotropic pressure—more precisely, its spherical component $\frac{1}{3}\varrho(\operatorname{tr} H)I$ takes the role of pressure while its deviatoric component $\operatorname{dev}(\varrho H) = \varrho\big(H - \frac{1}{3}(\operatorname{tr} H)I\big)$ is a sort of additional stress. Since H is symmetric and positive-semidefinite, it possesses nonnegative eigenvalues η_i, $i = 1, 2, 3$, and a corresponding othonormal eigenbasis $\{h_1, h_2, h_3\}$. Additional insight regarding the nature of H ensues on expressing it in canonical form

$$H = \sum_{i=1}^{3} \eta_i h_i \otimes h_i, \tag{23}$$

which provides a caricature of the primitive definition of H, based on averaging, as the sum of three terms as though the population of molecules, all having the same mass, were spread between three swarms: within the i-th swarm, all molecules move along the line spanned by h_i with speed η_i; each swarm is split evenly into two subswarms but with opposing velocities $\pm\eta_i h_i$. Alternatively, one may imagine all molecules to have not only the same mass but also the same speed, but with the fraction of those moving along the line spanned by h_i being $\eta_i/(\eta_1+\eta_2+\eta_3)$. With this interpretation in mind, it emerges that the square-root $H^{1/2}$ of H regulates the balanced cross-flux of molecules—$H^{1/2}n$ being a measure of the flux of molecules through a plane with unit normal n. Thus, with reference to $(20)_4$, within a loculus, the tensor $\operatorname{sym} A$ should account for coherence opposing suffusion and dispersal, actions promoted, in contrast, by collisions.

4. Linear constraints

4.1. Basic ideas

Here, our interest is in discerning the influence of constraints involving the velocity gradient L and the mesodistortion rate B, as described by the zero set,

$$\{(L,B) : \Phi(L,B) = 0\}, \tag{24}$$

of a mapping Φ from Lin $\times$ Lin into Lin, on the form of the governing equations (20) of the theory of ephemeral continua.

We confine attention to linear constraints. Consistent with this restriction, the mapping Φ entering (24) must obey

$$\Phi(\alpha L, \alpha B) = \alpha\Phi(L,B) \tag{25}$$

for all scalars α and all L and B in Lin and

$$\Phi(L_1 + L_2, B_1 + B_2) = \Phi(L_1, B_1) + \Phi(L_2, B_2) \tag{26}$$

for all L_1, L_2, B_1, and B_2 in Lin. Thus, Φ is linear and the zero set (24) is simply the null space $\operatorname{Null}\Phi$ of Φ.

Granted (25) and (26), there exist fourth-order tensors $\mathbb{H}$ and $\mathbb{K}$ such that

$$\Phi(L,B) = \mathbb{H}L - \mathbb{K}B \tag{27}$$

for all (L,B) in Lin $\times$ Lin and

$$\{(L,B) : \mathbb{H}L = \mathbb{K}B\} = \operatorname{Null}\Phi. \tag{28}$$

Observe that the mapping obtained by multiplying Φ—or, equivalently, $\mathbb{H}$ and $\mathbb{K}$—by any invertible fourth-order tensor has null space identical to that of Φ and, thus, delivers a homologous constraint. When exploring the properties of the constraint furnished by Φ, only $\mathrm{Null}\,\Phi$ is relevant. Importantly, the null spaces $\mathrm{Null}\,\mathbb{H}$ and $\mathrm{Null}\,\mathbb{K}$ of $\mathbb{H}$ and $\mathbb{K}$ do not determine $\mathrm{Null}\,\Phi$ unless the ranges $\mathrm{Rng}\,\mathbb{H}$ and $\mathrm{Rng}\,\mathbb{K}$ of $\mathbb{H}$ and $\mathbb{K}$ share only the zero tensor in common, in which case:

$$(\mathrm{Rng}\,\mathbb{H}) \cap (\mathrm{Rng}\,\mathbb{K}) = \{0\} \quad \Leftrightarrow \quad \mathrm{Null}\,\Phi = (\mathrm{Null}\,\mathbb{H}) \times (\mathrm{Null}\,\mathbb{K}). \tag{29}$$

More generally, if $\mathrm{Rng}\,\mathbb{H}$ and $\mathrm{Rng}\,\mathbb{K}$ have elements other than the zero tensor in common, then $\mathrm{Null}\,\Phi$ contains pairs of second-order tensors that are not provided by $\mathrm{Null}\,\mathbb{H}$ and $\mathrm{Null}\,\mathbb{K}$:

$$(\mathrm{Rng}\,\mathbb{H}) \cap (\mathrm{Rng}\,\mathbb{K}) \neq \{0\} \quad \Leftrightarrow \quad \mathrm{Null}\,\Phi \supsetneq (\mathrm{Null}\,\mathbb{H}) \times (\mathrm{Null}\,\mathbb{K}). \tag{30}$$

A representation of $\mathrm{Null}\,\Phi$ that affords as much information as possible regarding the implication of the constraint on the range of admissible values for L and B is therefore needed. Moreover, any ramifications that might be required by frame-indifference should be deduced. We now address these demands, with the purpose of establishing the essential features of the constraint.

4.2. Algebraic insights

Given fourth-order tensors $\mathbb{H}$ and $\mathbb{K}$, we seek a characterization of $\mathrm{Null}\,\Phi$. In view of (28), let $\mathscr{U}$ be the subspace of Lin for which there is an element B of Lin such that

$$\mathbb{H}L = \mathbb{K}B. \tag{31}$$

Suppose that an element L of $\mathscr{U}$ is given. If $\mathbb{K}$ is invertible, then (31) determines B uniquely via $B = \mathbb{K}^{-1}\mathbb{H}L$. If, more generally, $\mathbb{K}$ is not invertible, then there exist multiple choices of B that satisfy (31); indeed, given B satisfying (31) and a nontrivial element B_N of $\mathrm{Null}\,\mathbb{K}$, then $\mathbb{K}(B + B_N) = \mathbb{K}B + \mathbb{K}B_N = \mathbb{K}B$, whereby $B + B_N$ also satisfies (31). Moreover, any B that satisfies (31) admits a decomposition of the form

$$B = B_N + B_\perp, \tag{32}$$

where B_N belongs to $\mathrm{Null}\,\mathbb{K}$ and where $B_\perp$ is a uniquely determined element of the orthogonal complement $(\mathrm{Null}\,\mathbb{K})^\perp$ of $\mathrm{Null}\,\mathbb{K}$. Importantly, $B_\perp$ is determined uniquely for each L in $\mathscr{U}$. To verify this assertion, suppose that, B aside, there exists another second-order tensor $B' = B'_N + B'_\perp$ that satisfies (31). Then, since B_N and B'_N belong to $\mathrm{Null}\,\mathbb{K}$, $\mathbb{H}L = \mathbb{K}B = \mathbb{K}B_\perp$ and $\mathbb{H}L = \mathbb{K}B' = \mathbb{K}B'_\perp$, from which it follows that

$$\mathbb{K}(B_\perp - B'_\perp) = 0. \tag{33}$$

Since both $B_\perp$ and $B'_\perp$ are in $(\mathrm{Null}\,\mathbb{K})^\perp$, so also is the difference $B_\perp - B'_\perp$, whereby (33) implies that $B'_\perp = B_\perp$, which verifies that $B_\perp$ is determined uniquely for each L in $\mathscr{U}$.

In view of the uniqueness of $B_\perp$, there exists a unique linear mapping $\mathbb{N}$ from $\mathscr{U}$ to $(\mathrm{Null}\,\mathbb{K})^\perp$ such that

$$B_\perp = \mathbb{N}L \tag{34}$$

for each L in $\mathscr{U}$. By (32) and (34), an element (L, B) of Lin $\times$ Lin satisfies (31) if and only if L belongs to $\mathscr{U}$ and B has the form

$$B = B_N + \mathbb{N}L, \tag{35}$$

with B_N being some element of Null $\mathbb{K}$. Further, since B_N belongs to Null $\mathbb{K}$, (31) and (35) imply that, for any L in $\mathscr{U}$,

$$\mathbb{H}L = \mathbb{K}\mathbb{N}L \tag{36}$$

and, thus, that $(L, \mathbb{N}L)$ satisfies the constraint.

On extending $\mathbb{N}$ so that $\mathbb{N}L = 0$ for all L in the orthogonal complement $\mathscr{U}^\perp$ of $\mathscr{U}$, whereby

$$\mathbb{N}L = \begin{cases} B_\perp & \text{if } L \in \mathscr{U}, \\ 0 & \text{if } L \in \mathscr{U}^\perp, \end{cases} \tag{37}$$

the foregoing result can be summarized as follows: there exists a unique fourth-order tensor $\mathbb{N}$ such that $\mathscr{U}^\perp$ and Rng $\mathbb{N}$ are subsets of Null $\mathbb{N}$ and $(\mathrm{Null}\,\mathbb{K})^\perp$, respectively, with the property that the constraint (31) holds if and only if L belongs to $\mathscr{U}$ and $B = B_N + \mathbb{N}L$ for some B_N in Null $\mathbb{K}$.

Importantly, the last result possesses a converse of sorts, the proof of which we omit for brevity: given subspaces $\mathscr{U}$ and $\mathscr{W}$ of Lin with dimensions $\dim\mathscr{U}$ and $\dim\mathscr{W}$ satisfying

$$\dim\mathscr{U} + \dim\mathscr{W} \geq \dim\mathrm{Lin} = 9 \tag{38}$$

and a fourth-order tensor $\mathbb{N}$ consistent with

$$\mathrm{Null}\,\mathbb{N} \supseteq \mathscr{U}^\perp \quad \text{and} \quad \mathrm{Rng}\,\mathbb{N} \subseteq \mathscr{W}^\perp, \tag{39}$$

there exist fourth-order tensors $\mathbb{H}$ and $\mathbb{K}$, with Null $\mathbb{K} = \mathscr{W}$, such that L belongs to $\mathscr{U}$ and $B = B_N + \mathbb{N}L$ for some B_N in $\mathscr{W}$ if and only if $\mathbb{H}L = \mathbb{K}B$.

4.3. Consequences of frame-indifference

Our application of frame-indifference presumes that, under a change of observer involving a time-dependent Q in Orth$^+$ and a time-dependent frame spin

$$\Omega = \dot{Q}Q^\top \tag{40}$$

in Skw, B transforms like L, so that

$$L \mapsto QLQ^\top + \Omega \quad \text{and} \quad B \mapsto QBQ^\top + \Omega. \tag{41}$$

Faced with the constraint, we stipulate that $\operatorname{Null}\Phi$ be frame-indifferent in the sense that if (L, B) belongs to $\operatorname{Null}\Phi$, then, given Q in Orth^+ and Ω in Skw, $(QLQ^\top + \Omega, QLQ^\top + \Omega)$ must also belong to $\operatorname{Null}\Phi$. By (28), $(QLQ^\top + \Omega, QLQ^\top + \Omega)$ belongs to $\operatorname{Null}\Phi$ if and only if

$$\mathbb{H}(QLQ^\top) + \mathbb{H}\Omega = \mathbb{K}(QBQ^\top) + \mathbb{K}\Omega. \tag{42}$$

Bearing in mind that Q and Ω in (42) can be chosen independently, let Q coincide with the second-order identity tensor I. Then, since (L, B) belongs to $\operatorname{Null}\Phi$, (42) becomes

$$\mathbb{H}\Omega = \mathbb{K}\Omega. \tag{43}$$

Since (43) holds for any Ω in Skw, it follows that $\{(\Omega, \Omega) : \Omega \in \mathrm{Skw}\} \subset \operatorname{Null}\Phi$. More importantly, using (43) in (42) gives

$$\mathbb{H}(QLQ^\top) = \mathbb{K}(QBQ^\top), \tag{44}$$

which implies that $(QLQ^\top, QBQ^\top)$ belongs to $\operatorname{Null}\Phi$ whenever (L, B) belongs to $\operatorname{Null}\Phi$. Since (44) holds for any Q in Orth^+, a result that might have been anticipated as a consequence of frame-indifference follows: $\operatorname{Null}\Phi$ is an isotropic subspace of $\mathrm{Lin} \times \mathrm{Lin}$. Next, since (L, B) is in $\operatorname{Null}\Phi$, L belongs to $\mathscr{U}$ and we may use (36) in (44) to yield

$$\mathbb{H}(QLQ^\top) = \mathbb{K}(Q(\mathbb{N}L)Q^\top). \tag{45}$$

Reasoning analogously with reference to $(QLQ^\top, QBQ^\top)$, we obtain

$$\mathbb{H}(QLQ^\top) = \mathbb{K}\mathbb{N}(QLQ^\top), \tag{46}$$

which in combination with (45) delivers

$$\mathbb{K}(\mathbb{N}(QLQ^\top) - Q(\mathbb{N}L)Q^\top) = 0. \tag{47}$$

Since $\mathbb{N}(QLQ^\top)$ and $Q(\mathbb{N}L)Q^\top$ are in $(\operatorname{Null}\mathbb{K})^\perp$, so also is the difference $\mathbb{N}(QLQ^\top) - Q(\mathbb{N}L)Q^\top$. Hence, (47) holds if and only if

$$\mathbb{N}(QLQ^\top) = Q(\mathbb{N}L)Q^\top. \tag{48}$$

Bearing in mind that (48) must hold for any choices of L in $\mathscr{U}$ and Q in Orth^+ and that, by (37), $\mathbb{N}L = 0$ for any choice of L in $\mathscr{U}^\perp$, we infer that $\mathbb{N}$ must be an isotropic fourth-order tensor. Thus, by (10) and (11), $\mathbb{N}$ admits a representation of the form

$$\mathbb{N} = \lambda_1 \mathbb{S}_0 + \lambda_2 \mathbb{W} + \lambda_3 \mathbb{T} \tag{49}$$

and, thus, is major-symmetric:

$$\mathbb{N}^\top = \mathbb{N}. \tag{50}$$

To conclude this section, we mention another noteworthy consequence of frame-indifference, namely that both Null $\mathbb{H}$ and Null $\mathbb{K}$ are isotropic subspaces of Lin. It can also be shown that Null Φ if frame-indifferent if and only if $\mathbb{H}\Omega = \mathbb{K}\Omega$ for all Ω in Skw, $\mathbb{N}$ is isotropic, and the subspaces $\mathscr{U}$ and Null $\mathbb{K}$ are isotropic. Notice that, contrary to what one might expect, frame-indifference does not require that $\mathbb{H}$ or $\mathbb{K}$ be isotropic.

4.4. Synopsis

The results of Sections 4.2 and 4.3 suggest two alternative descriptions of a physically viable—*in the sense that the associated null space is frame-indifferent*—linear constraint on the velocity gradient L and the mesodistortion rate B.

One description involves providing fourth-order tensors $\mathbb{H}$ and $\mathbb{K}$ subject to two conditions. First, if $\mathbb{H}L = \mathbb{K}B$, so that L and B satisfy the constraint, then $\mathbb{H}$ and $\mathbb{K}$ must obey

$$\mathbb{H}(QLQ^{\top}) = \mathbb{K}(QBQ^{\top}) \tag{51}$$

for all Q in Orth$^+$ and

$$\mathbb{H}\Omega = \mathbb{K}\Omega, \tag{52}$$

for all Ω in Skw.

The other, somewhat less exigent, description involves providing isotropic subspaces $\mathscr{U}$ and $\mathscr{W}$ of Lin satisfying (38) and an isotropic fourth-order tensor $\mathbb{N}$ consistent with (39). Here, L and B satisfy the constraint if and only if L is in $\mathscr{U}$ and B has the form

$$B = B_{\mathscr{W}} + \mathbb{N}L, \tag{53}$$

where $B_{\mathscr{W}}$ is in $\mathscr{W}$.

Granted that $\mathscr{W}$ is identified with Null $\mathbb{K}$, these two descriptions of the constraint are equivalent.

5. Elementary implications of the constraint

To deduce the implications of a general, frame-indifferent, linear constraint on L and B, we follow the derivations of the theories of hypocontinua and Navier–Stokes-$\alpha\beta$ continua provided by Capriz [3] and Capriz and Fried [5]. This approach rests on considerations involving the power of the internal actions (21).

Following the traditional approach to dealing with constraints, we suppose that the fields T, A, m, and Z split, additively, into active and reactive components,

$$T = T_a + T_r, \quad A = A_a + A_r, \quad \mathbb{m} = \mathbb{m}_a + \mathbb{m}_r, \quad Z = Z_a + Z_r, \tag{54}$$

and we require that the power of the internal actions obey

$$
\begin{aligned}
(55)\qquad T\cdot L + A^{\top}\cdot B + {}^{t}\mathbb{m}\cdot \operatorname{grad} B + \tfrac{1}{2}\operatorname{tr} Z \\
= T_a\cdot L + A_a^{\top}\cdot B + {}^{t}\mathbb{m}_a\cdot \operatorname{grad} B + \tfrac{1}{2}\operatorname{tr} Z_a,
\end{aligned}
$$

for all choices of L, B, and $\operatorname{grad} B$ consistent with the constraint, so that the reactions T_r, A_r, m_r, and Z_r are powerless.

Since the power flux $\mathbb{j}$ associated with mesofluctuations does not enter the power of the internal actions (21), it seems reasonable to assume that it does not react to the imposition of any internal constraint. With this assumption, $\mathbb{j}_r = 0$ and

$$
(56)\qquad \mathbb{j} = \mathbb{j}_a.
$$

Moreover, since the deviatoric component $\operatorname{dev} Z = Z - \frac{1}{3}(\operatorname{tr} Z)I$ of Z is absent from (21), it seems reasonable to assume that it cannot include, under internally constrained circumstances, an additive reactive component. This amounts to assuming that Z_r belongs to Sph:

$$
(57)\qquad Z_r = \tfrac{1}{3}(\operatorname{tr} Z_r)I.
$$

To deduce the consequences of requiring that (55) hold for all L, B, and $\operatorname{grad} B$ consistent with the constraint, we rely on the description of the constraint involving the provision of isotropic subspaces $\mathscr{U}$ and $\mathscr{W}$ of Lin satisfying (38) and an isotropic fourth-order tensor $\mathbb{N}$ consistent with $\operatorname{Null}\mathbb{N} \supseteq \mathscr{U}^{\perp}$ and $\operatorname{Rng}\mathbb{N} \subseteq \mathscr{W}^{\perp}$. Using the decomposition (53) of B and the decompositions (54) of T, A, $\mathbb{m}$, and Z in (55), while bearing in mind from (50) that $\mathbb{N}$ is major-symmetric, we obtain

$$
(58)\qquad (T_r + \mathbb{N}A_r^{\top})\cdot L + A_r^{\top}\cdot B_{\mathscr{W}} + {}^{t}\mathbb{m}_r\cdot \operatorname{grad} B + \tfrac{1}{2}\operatorname{tr} Z_r = 0.
$$

Granted (57) and that T_r, A_r, $\mathbb{m}_r$, and $\operatorname{tr} Z_r$ are independent of L, B, and $\operatorname{grad} B$, (58) is satisfied for all choices of L, B, and $\operatorname{grad} B$ if and only if

$$
(59)\qquad
\begin{gathered}
T_r + \mathbb{N}A_r^{\top} \in \mathscr{U}^{\perp}, \quad A_r^{\top} \in \mathscr{W}^{\perp}, \quad \operatorname{Rng}\mathbb{m}_r \subseteq \mathscr{W}^{\perp}, \\
\mathbb{N}\,{}^{t}\mathbb{m}_r + (\mathbb{N}\,{}^{t}\mathbb{m}_r)^{t} = 0, \quad Z_r = 0,
\end{gathered}
$$

where, in reference to $(59)_3$, $\mathbb{m}_r$ is viewed as a linear mapping from vectors to second-order tensors. Notice that, by (57) and $(59)_5$,

$$
(60)\qquad Z = Z_a.
$$

A direct calculation shows that if $(59)_{1\text{–}5}$ hold, then (58) is satisfied for all choices of L, B, and $\operatorname{grad} B$ consistent with the constraint. To establish the converse, note that since L, $B_{\mathscr{W}}$, and $\operatorname{grad} B$ can be prescribed consistent with the constraint and independently at any given point and time, (58) holds for all choices of L, $B_{\mathscr{W}}$, and $\operatorname{grad} B$ only if each of its terms vanish separately:

$$(61)\qquad (T_r + \mathbb{N}A_r^{\top}) \cdot L = 0, \quad A_r^{\top} \cdot B_{\mathscr{W}} = 0, \quad {}^t\mathbb{m}_r \cdot \operatorname{grad} B = 0, \quad \operatorname{tr} Z_r = 0.$$

Since they must hold for all L in $\mathscr{U}$ and all $B_{\mathscr{W}}$ in $\mathscr{W}$, $(61)_1$ and $(61)_2$ imply $(59)_1$ and $(59)_2$. Next, by (53), $(61)_3$ becomes

$$(62)\qquad {}^t\mathbb{m}_r \cdot \operatorname{grad} B_{\mathscr{W}} + {}^t\mathbb{m}_r \cdot \operatorname{grad}(\mathbb{N}L) = 0.$$

Since $B_{\mathscr{W}}$ and $\mathbb{N}L$ can be chosen independently, (62) holds if and only if its terms vanish separately. We must, therefore, have

$$(63)\qquad {}^t\mathbb{m}_r \cdot \operatorname{grad} B_{\mathscr{W}} = 0 \quad \text{and} \quad {}^t\mathbb{m}_r \cdot \operatorname{grad}(\mathbb{N}L) = 0,$$

for all $B_{\mathscr{W}}$ in $\mathscr{W}$ and all L in $\mathscr{U}$, respectively. Since $\mathscr{W}$ is isotropic and, thus, so is $\mathscr{W}^{\perp}$, $(63)_1$ holds if and only if

$$(64)\qquad \mathbb{m}_r \cdot \operatorname{grad} B_{\mathscr{W}} = 0,$$

which establishes $(59)_3$. Regarding the second requirement, since $\mathbb{N}$ is constant, isotropic, and, by (50), major-symmetric, (63) yields

$$(65)\qquad {}^t\mathbb{m}_r \cdot \operatorname{grad}(\mathbb{N}L) = {}^t\mathbb{m}_r \cdot \mathbb{N} \operatorname{grad} L = \mathbb{N}\,{}^t\mathbb{m}_r \cdot \operatorname{grad} L = 0,$$

which, since $(\operatorname{grad} L)^t = (\operatorname{grad}\operatorname{grad} v)^t = \operatorname{grad}\operatorname{grad} v = \operatorname{grad} L$, implies $(59)_4$. Finally, granted (57), to satisfy $(61)_4$, Z_r must vanish, whereby $(59)_5$ holds. The restrictions (59) are therefore both necessary and sufficient to ensure that the reactions T_r, A_r, $\mathbb{m}_r$, and Z_r are powerless.

6. Constraints that prohibit suffusion

The theory of ephemeral continua is distinguished by its allowance for the exchange of molecules between material points. To expose other differences and offer some detail regarding remarks already made in the introduction, we explore the implications of various constraints consistent with the requirement,

$$(66)\qquad \sigma = 0,$$

that the suffusion vanish. Though very trivial, the examples we consider are nevertheless illuminating instances of constraints relevant when a special physical hypothesis is accepted; more general occurrences will be the theme of another paper.

6.1. *Purely suffusionless continua*

In view of the definition (19) of σ, the weakest possible linear constraint between L and B that agrees with (66) takes the form

$$\operatorname{tr} L = \operatorname{tr} B. \tag{67}$$

We refer to media governed by the equations obtained by reducing the balance laws (20) of the theory of ephemeral continua in accord with (67) as 'purely suffusionless continua.''

The constraint defining suffusionless continua corresponds to imposing (31) with

$$\mathbb{H} = \mathbb{K} = \mathbb{T}, \tag{68}$$

where $\mathbb{T}$, defined via $(1)_4$, is the fourth-order tensor that maps Lin onto Sph. Since $\mathbb{T}$ is isotropic, the choice (68) of $\mathbb{H}$ and $\mathbb{K}$ is consistent with (51). Moreover, since $\mathbb{T}\Omega = \frac{1}{3}(\operatorname{tr}\Omega)I = 0$ for any Ω in Skw, the choice (68) of $\mathbb{H}$ and $\mathbb{K}$ is consistent with (52). Thus, (67) is frame-indifferent.

To determine the reactions T_r, A_r, and $\mathbb{m}_r$ via $(59)_{1-4}$, we next recast the constraint (67) in terms of isotropic subspaces $\mathscr{U}$ and $\mathscr{W}$ of Lin satisfying (38) and an isotropic fourth-order tensor $\mathbb{N}$ consistent with $\operatorname{Null}\mathbb{N} \supseteq \mathscr{U}^\perp$ and $\operatorname{Rng}\mathbb{N} \subseteq \mathscr{W}^\perp$. Since L in (67) is unrestricted, $\mathscr{U}$ must coincide with all of Lin:

$$\mathscr{U} = \mathrm{Lin}. \tag{69}$$

Further, since $\mathscr{W}$ can be identified with $\operatorname{Null}\mathbb{K}$, (68) implies that $\mathscr{W}$ must coincide with the deviatoric subspace Dev of Lin:

$$\mathscr{W} = \mathrm{Dev}. \tag{70}$$

From (69) and (70), we deduce that

$$\mathscr{U}^\perp = \{0\} \quad \text{and} \quad \mathscr{W}^\perp = \mathrm{Sph}. \tag{71}$$

In view of $(2)_3$, (4), (67), (70), and (71), we next find that

$$B = B_{\mathscr{W}} + \mathbb{T}L, \tag{72}$$

which, when compared with (53), allows us to identify $\mathbb{N}$ as

$$\mathbb{N} = \mathbb{T}. \tag{73}$$

Since $\operatorname{Null}\mathbb{T} = \mathrm{Dev}$ and $\operatorname{Rng}\mathbb{T} = \mathrm{Sph}$, (71) and (73) imply that $\operatorname{Null}\mathbb{N} = \operatorname{Null}\mathbb{T} = \mathrm{Dev} \supseteq \{0\} = \mathscr{U}^\perp$ and $\operatorname{Rng}\mathbb{N} = \operatorname{Rng}\mathbb{T} = \mathrm{Sph} \subseteq \mathrm{Sph} = \mathscr{W}^\perp$. The choice (73) therefore satisfies (39).

On using (71) in $(59)_{1-4}$, we find that

$$T_r + \mathbb{N}A_r^\top \in \{0\}, \quad A_r^\top \in \mathrm{Sph}, \quad \mathrm{Rng}\,\mathbb{m}_r \subseteq \mathrm{Sph}, \quad \mathbb{T}\,{}^t\mathbb{m}_r + (\mathbb{T}\,{}^t\mathbb{m}_r)^t = 0. \tag{74}$$

Further, by $(74)_3$,

$$\mathbb{m}_r = \mathbb{T}\mathbb{m}_r = {}^t(\mathbb{T}\mathbb{m}_r) = \mathbb{T}\,{}^t\mathbb{m}_r = {}^t\mathbb{m}_r, \tag{75}$$

whereby $(74)_4$ becomes

$$\mathbb{m}_r + \mathbb{m}_r^t = 0. \tag{76}$$

By (74)–(76), the reactions T_r, A_r, and $\mathbb{m}_r$ must therefore take the particular forms:

$$T_r = -\tfrac{1}{3}(\mathrm{tr}\,A_r)I, \quad A_r = \tfrac{1}{3}(\mathrm{tr}\,A_r)I, \quad \text{and} \quad \mathbb{m}_r = 0. \tag{77}$$

Whereas $(77)_1$ and $(77)_2$ are immediate consequence of $(74)_1$ and $(74)_2$, the remaining result $(77)_3$ is a consequence of noting that, by (13) and (14), the only third-order tensor $\mathbb{a}$ that satisfies ${}^t\mathbb{a} = \mathbb{a} = -\mathbb{a}^t$ is the zero third-order tensor.

In view of $(54)_{2,3}$ and $(77)_{2,3}$, the spherical component of the balance $(20)_4$ of moment of mesomomentum yields an expression for $\mathrm{tr}\,A_r$ which, by $(77)_{1,2}$, determines $T_r = -A_r$ in the form

$$T_r = -\tfrac{1}{3}[\mathrm{tr}(\varrho H + \varrho M - A_a + \mathrm{div}\,\mathbb{m}_a - \varrho Y\dot{B}^\top)]I. \tag{78}$$

By (19) and (67), all terms with factors of σ must vanish from (20). Further, since the spherical component of $(20)_4$ determines the reaction A_r via (78), only the deviatoric component of that balance survives. Thus, on recalling (56) and (60), under the constraint (66) characterizing purely suffusionless continua, the balance laws (20) for ephemeral continua specialize to

$$\begin{cases} \dot{\varrho} + \varrho\,\mathrm{div}\,v = 0, \\ \varrho(\dot{Y} - YB^\top - BY) = 0, \\ \varrho\dot{v} - \tfrac{1}{3}\,\mathrm{grad}[\varrho(Y\cdot\dot{B} - \mathrm{tr}\,H)] = \varrho f + \mathrm{div}\,T_a + \tfrac{1}{3}\,\mathrm{grad}[\mathrm{tr}(A_a - \mathrm{div}\,\mathbb{m}_a)], \\ \varrho\,\mathrm{dev}(Y\dot{B}^\top - H) = \varrho\,\mathrm{dev}\,M - \mathrm{dev}\,A_a + \mathrm{dev}\,\mathrm{div}\,\mathbb{m}_a, \\ \varrho(\dot{H} - HB^\top - BH) = \varrho J - Z_a + \mathrm{div}\,\mathbb{j}_a, \end{cases} \tag{79}$$

where L and B must satisfy $\mathrm{tr}\,L = \mathrm{tr}\,B$ and the effective specific external body force f entering $(79)_3$ is defined by

$$f = b - \tfrac{1}{3}\mathrm{grad}\,\mathrm{tr}\,M - \tfrac{1}{3}(\mathrm{tr}\,M)\,\mathrm{grad}\ln\varrho. \tag{80}$$

By $(77)_{1,2}$, the basic relation (22) requires that T_a and A_a satisfy

$$\operatorname{skw} T_a = \operatorname{skw} A_a. \tag{81}$$

Ostensibly, (79) and (81) allow for the possibility that T_a might not be symmetric. At the same time, only the deviatoric components of terms enter the law of moment of mesomomentum balance, whereas the influence of the traces of $Y\dot{B}^{\top}$, A_a, and $\mathbb{m}_a$ is transferred to the law of linear momentum balance.

A closer relation between the set of balance laws above and those ruling multipolar continua, as developed by Green and Rivlin [10], might have been expected. Such expectation is, however, largely misplaced. The radical difference explaining much of the discrepancy is in that the molecules belonging at a certain instant in a loculus are supposed, in a multipolar theory, to lay always within the same material element, however far they may possibly move later from their center of mass, without needing any constraint to conform to that discipline. Another basic difference between the system (79) and the equations governing multipolar continua is that in (79) the multipolarity is compacted entirely in the tensor moment K and the variance H.

6.2. *Fully-incompressible mesostretch continua*

Consistent with but somewhat stronger than the requirement (67) that the suffusion vanish is the constraint

$$\operatorname{skw} L = \operatorname{skw} B, \quad \operatorname{tr} L = \operatorname{tr} B = 0, \tag{82}$$

which, while slaving the mesospin $\operatorname{skw} B$ to the macrospin $W = \operatorname{skw} L$, affords the possibility that the mesostretching $\operatorname{sym} B$ may evolve independently of the macrostretching $D = \operatorname{sym} L$. Moreover, (82) requires that the macroscopic and mesoscopic disfigurements be isochoric and, thus, imposes incompressibility at both levels. This second requirement ensures that no suffusion occurs. We refer to media governed by the balance laws obtained by reducing the equations (20) of the theory of ephemeral continua in accord with (82) as 'fully-incompressible mesostretch continua.'

It is not particularly easy to formulate the constraint (82) in terms of fourth-order tensors $\mathbb{H}$ and $\mathbb{K}$. We, therefore, choose the alternative approach involving isotropic subspaces $\mathscr{U}$ of $\mathscr{W}$ of Lin satisfying (38) and an isotropic fourth-order tensor $\mathbb{N}$ consistent with $\operatorname{Null}\mathbb{N} \supseteq \mathscr{U}^{\perp}$ and $\operatorname{Rng}\mathbb{N} \subseteq \mathscr{W}^{\perp}$. Specifically, since $\operatorname{tr} L = 0$, $\mathscr{U}$ must coincide with the deviatoric subspace Dev of Lin:

$$\mathscr{U} = \mathrm{Dev}. \tag{83}$$

Consistent with (82), we find also that

$$B = \tfrac{1}{2}(B + B^{\top}) - \tfrac{1}{3}(\operatorname{tr} B)I + \mathbb{W}B = B_{\mathscr{W}} + \mathbb{W}L, \tag{84}$$

where $\mathbb{W}$, defined via $(1)_2$, is the fourth-order tensor that maps Lin onto Skw. On comparing (84) to (53), we identify $\mathbb{N}$ with $\mathbb{W}$:

$$\mathbb{N} = \mathbb{W}. \tag{85}$$

Further, on the basis of (84), we recognize that

$$\mathscr{W} = \mathrm{SymDev}. \tag{86}$$

From (83) and (86), we deduce that

$$\mathscr{U}^{\perp} = \mathrm{Sph} \quad \text{and} \quad \mathscr{W}^{\perp} = (\mathrm{SymDev})^{\perp} = \mathrm{span}(\mathrm{Skw} \cup \mathrm{Sph}). \tag{87}$$

Since $\mathrm{Null}\,\mathbb{W} = \mathrm{Sym}$ and $\mathrm{Rng}\,\mathbb{W} = \mathrm{Skw}$, (87) and (85) imply that $\mathrm{Null}\,\mathbb{N} = \mathrm{Null}\,\mathbb{W} = \mathrm{Sym} \supseteq \mathrm{Sph} = \mathscr{U}^{\perp}$ and $\mathrm{Rng}\,\mathbb{N} = \mathrm{Rng}\,\mathbb{W} = \mathrm{Skw} \subseteq \mathrm{span}(\mathrm{Skw} \cup \mathrm{Sph}) = \mathscr{W}^{\perp}$. The choice (85) therefore satisfies (39).

On using (87) in $(59)_{1\text{–}3}$, we find that

$$T_r + \mathbb{N}A_r^{\top} \in \mathrm{Sph}, \quad A_r^{\top} \in \mathrm{span}(\mathrm{Skw} \cup \mathrm{Sph}), \quad \mathrm{Rng}\,\mathbb{m}_r \subseteq \mathrm{span}(\mathrm{Skw} \cup \mathrm{Sph}), \tag{88}$$

and, thus, that

$$\mathbb{S}_0 T_r = \mathrm{skw}\,A_r, \quad \mathbb{S}_0 A_r = 0, \quad \mathbb{S}_0 \mathbb{m}_r = 0, \tag{89}$$

where $\mathbb{S}_0$, defined via $(1)_5$, is the fourth-order tensor that maps Lin onto SymDev. Further, by $(59)_4$ and (85), $\mathbb{W}\,{}^t\mathbb{m}_r + (\mathbb{W}\,{}^t\mathbb{m}_r)^t = 0$ or, equivalently,

$$\mathbb{m}_r + \mathbb{m}_r^t = {}^t\mathbb{m}_r + ({}^t\mathbb{m}_r)^t. \tag{90}$$

As an immediate but useful consequence of $(89)_3$, we have

$$\mathbb{S}_0\,\mathrm{div}\,\mathbb{m}_r = 0, \tag{91}$$

while, from (90), we find that

$$\mathrm{div}\,\mathrm{div}(\mathbb{W}\mathbb{m}_r) = \mathrm{div}(\mathbb{W}\,\mathrm{div}\,\mathbb{m}_r) = 0. \tag{92}$$

In view of $(54)_{2,3}$ and $(89)_{2,3}$, the spherical and skew components of the balance $(20)_4$ of moment of mesomomentum specialize to

$$\varrho\,\mathrm{tr}(Y\dot{B}^{\top} - H) = \varrho\,\mathrm{tr}\,M - \mathrm{tr}(A_a + A_r) + \mathrm{tr}\,\mathrm{div}(\mathbb{m}_a + \mathbb{m}_r) \tag{93}$$

and, bearing in mind that H is in Sym,

$$\varrho\,\mathrm{skw}(Y\dot{B}^{\top}) = \varrho\,\mathrm{skw}\,M - \mathrm{skw}(A_a + A_r) + \mathrm{skw}\,\mathrm{div}(\mathbb{m}_a + \mathbb{m}_r), \tag{94}$$

the first of which determines a gauge relation for $\mathrm{tr}(A_r - \mathrm{div}\,\mathbb{m}_r)$ and the second of which, by $(89)_1$, determines $T_r = \mathrm{skw}\,A_r + \frac{1}{3}(\mathrm{tr}\,T_r)I$ in the form

$$T_r = -\varpi I + \varrho\,\mathrm{skw}\,M - \varrho\,\mathrm{skw}(Y\dot{B}^{\top}) - \mathrm{skw}\,A_a + \mathrm{skw}\,\mathrm{div}(\mathbb{m}_a + \mathbb{m}_r), \tag{95}$$

where we have introduced the reactive pressure

$$\varpi = -\tfrac{1}{3}(\operatorname{tr} T_r)I. \tag{96}$$

By (19) and $(82)_2$, the suffusion vanishes for the constraint under consideration and all terms with factors of σ drop out of (20). Further, since the symmetric, deviatoric component (94) of $(20)_4$ determines the reaction A_r via (78), only the skew component of $(20)_4$ survives. Thus, on recalling (56) and (60) while bearing in mind (91) and (92), under the constraint (82) characterizing fully-incompressible mesostretch continua, the balance laws (20) for ephemeral continua specialize to

$$\begin{cases} \dot{\varrho} = 0, \\ \varrho(\dot{Y} - YB^{\top} - BY) = 0, \\ \varrho\dot{v} + \operatorname{div}[\varrho\operatorname{skw}(Y\dot{B}^{\top})] = \varrho f - \operatorname{grad}\varpi + \operatorname{div}(\operatorname{sym} T_a + \operatorname{skw}\operatorname{div}\mathbb{m}_a), \\ \varrho\operatorname{sym}[\operatorname{dev}(Y\dot{B}^{\top} - H)] = \varrho\operatorname{sym}(\operatorname{dev} M) - \operatorname{sym}(\operatorname{dev} A_a) + \operatorname{sym}(\operatorname{dev}\operatorname{div}\mathbb{m}_a), \\ \varrho(\dot{H} - HB^{\top} - BH) = \varrho J - Z_a + \operatorname{div}\mathbb{j}_a, \end{cases} \tag{97}$$

where L and B must satisfy $\operatorname{skw} L = \operatorname{skw} B$ and $\operatorname{tr} L = \operatorname{tr} B = 0$ and the effective specific external body force f entering $(97)_3$ is defined by

$$f = b + \operatorname{div}(\operatorname{skw} M) + (\operatorname{skw} M)\operatorname{grad}\ln\varrho. \tag{98}$$

By (22), $(54)_1$, and $(89)_{1,2}$, the basic relation (22) requires that T_a and A_a satisfy

$$\operatorname{skw} T_a = \operatorname{skw} A_a. \tag{99}$$

As was the case for purely suffusionless continua, (97) and (99) seem to allow for the possibility that T_a might not be symmetric. However, since A_a enters (97) only through its symmetric and deviatoric component, generality would not be lost by insisting that A_a be symmetric and traceless, in which case (99) would yield $\operatorname{skw} T_a = 0$ or, equivalently,

$$T_a = T_a^{\top}. \tag{100}$$

The mass balance $(97)_1$ requires the mass density ϱ to be constant along particle trajectories. For homogeneous fluids, this requirement is tantamount to the stipulation that the mass density be constant.

Consistent with the assertion about the preponderance of the constraint (82) defining the class of fully incompressible mesostretch continua relative to the constraint (67) defining purely suffusionless continua, the linear momentum balance $(97)_3$ is more strongly influenced by terms associated with the moment of mesomomentum balance than is its counterpart $(79)_3$. Moreover, whereas the terms entering the moment of mesomomentum balance $(79)_4$ take values in Dev, an 8-dimensional subspace of Lin, those entering $(97)_4$ take values in the 5-dimensional subspace SymDev. Effects associated with moments of mesomomenta are therefore somewhat more easily characterized for fully incompressible mesostretch continua than for purely suffusionless continua.

6.3. *Mesospin continua*

Another constraint that is consistent with the requirement (66) that the suffusion vanish but somewhat stronger than not only (67) but also (82) is the constraint

$$\operatorname{sym} L = \operatorname{sym} B, \tag{101}$$

which, while slaving the mesostretching $\operatorname{sym} B$ to the macrostretching $D = \operatorname{sym} L$, affords the possibility that the mesospin $\operatorname{skw} B$ may evolve independently of the macrospin $W = \operatorname{skw} L$. We refer to media governed by the equations obtained by reducing the balance laws (20) of the theory of ephemeral continua in accord with (101) as 'mesospin continua.'

The constraint defining mesospin continua corresponds to imposing (31) with

$$\mathbb{K} = \mathbb{H} = \mathbb{S}, \tag{102}$$

where $\mathbb{S}$, defined via $(1)_1$, is the fourth-order tensor that maps Lin onto Sym. Since $\mathbb{S}$ is isotropic, the choice (102) of $\mathbb{H}$ and $\mathbb{K}$ is consistent with (51). Moreover, since $\mathbb{S}\Omega = 0$ for any Ω in Skw, the choice (68) of $\mathbb{H}$ and $\mathbb{K}$ is consistent with (52). Thus, (101) is frame-indifferent.

To determine the reactions T_r, A_r, and $\mathbb{m}_r$ via $(59)_{1-4}$, we next recast the constraint (101) in terms of isotropic subspaces $\mathscr{U}$ and $\mathscr{W}$ of Lin satisfying (38) and an isotropic fourth-order tensor $\mathbb{N}$ consistent with $\operatorname{Null} \mathbb{N} \supseteq \mathscr{U}^\perp$ and $\operatorname{Rng} \mathbb{N} \subseteq \mathscr{W}^\perp$. Since L in (101) is unrestricted, $\mathscr{U}$ must coincide with all of Lin:

$$\mathscr{U} = \mathrm{Lin}. \tag{103}$$

Further, since $\mathscr{W}$ can be identified with $\operatorname{Null} \mathbb{K}$, (102) implies that $\mathscr{W}$ must coincide with the skew subspace Skw of Lin:

$$\mathscr{W} = \mathrm{Skw}. \tag{104}$$

From (103) and (104), we deduce that

$$\mathscr{U}^\perp = \{0\} \quad \text{and} \quad \mathscr{W}^\perp = \mathrm{Sym}. \tag{105}$$

In view of $(3)_1$, (101), (104), and (105), we next find that

$$B = B_{\mathscr{W}} + \mathbb{S}L, \tag{106}$$

which, when compared with (53), allows us to identify $\mathbb{N}$ as

$$\mathbb{N} = \mathbb{S}. \tag{107}$$

Since $\operatorname{Null}\mathbb{S} = \operatorname{Skw}$ and $\operatorname{Rng}\mathbb{S} = \operatorname{Sym}$, $(105)_1$ and (107) imply that $\operatorname{Null}\mathbb{N} = \operatorname{Null}\mathbb{S} = \operatorname{Skw} \supseteq \{0\} = \mathscr{U}^{\perp}$ and $\operatorname{Rng}\mathbb{N} = \operatorname{Rng}\mathbb{S} = \operatorname{Sym} \subseteq \operatorname{Sym} = \mathscr{W}^{\perp}$. The choice (107) therefore satisfies (39).

On using (105) in $(59)_{1-3}$, we find that

$$(108)\qquad T_r + \mathbb{N}A_r^{\top} \in \{0\}, \quad A_r^{\top} \in \operatorname{Sym}, \quad \operatorname{Rng}\mathbbm{m}_r \subseteq \operatorname{Sym},$$

and, thus, that

$$(109)\qquad T_r = -A_r, \quad A_r = \operatorname{sym} A_r, \quad \mathbbm{m}_r = {}^{t}\mathbbm{m}_r.$$

Notice that $(109)_2$ has been used to simplify the direct result $T_r = -\operatorname{sym} A_r$ of $(108)_1$ to the form $T_r = -A_r$ appearing in $(109)_2$. Further, by $(59)_4$, (107), and $(109)_3$,

$$(110)\qquad \mathbbm{m}_r^{t} = -\mathbbm{m}_r,$$

from which it follows that $\operatorname{div}\operatorname{div}\mathbbm{m}_r = -\operatorname{div}\operatorname{div}(\mathbbm{m}_r^{t}) = -\operatorname{div}\operatorname{div}\mathbbm{m}_r$ and, thus, that

$$(111)\qquad \operatorname{div}\operatorname{div}\mathbbm{m}_r = 0.$$

In view of $(54)_{2,3}$ and $(109)_{2,3}$, the symmetric and skew components of the balance $(20)_4$ of moment of mesomomentum specialize to

$$(112)\qquad \varrho\operatorname{sym}(Y\dot{B}^{\top} - H) = \varrho\operatorname{sym} M - \operatorname{sym}(A_a + A_r) + \operatorname{sym}\operatorname{div}(\mathbbm{m}_a + \mathbbm{m}_r)$$

and

$$(113)\qquad \varrho\operatorname{skw}(Y\dot{B}^{\top}) = \varrho\operatorname{skw} M - \operatorname{skw} A_a + \operatorname{skw}\operatorname{div}\mathbbm{m}_a,$$

the first of which, by $(109)_1$, determines $T_r = -A_r$ in the form

$$(114)\qquad T_r = \varrho\operatorname{sym}(Y\dot{B}^{\top} - H) - \varrho\operatorname{sym} M + \operatorname{sym} A_a - \operatorname{sym}\operatorname{div}(\mathbbm{m}_a + \mathbbm{m}_r).$$

By (19) and (101), the suffusion vanishes for the constraint under consideration and all terms with factors of σ drop out of (20). Further, since the symmetric component (112) of $(20)_4$ determines the reaction A_r via (114), only the skew component (113) of that balance survives. Thus, on recalling (56) and (60) while bearing in mind (111) and that H is symmetric, under the constraint (101) characterizing mesospin continua, the balance laws (20) for ephemeral continua specialize to

$$(115)\qquad \begin{cases} \dot{\varrho} + \varrho\operatorname{div} v = 0, \\ \varrho(\dot{Y} - YB^{\top} - BY) = 0, \\ \varrho\dot{v} - \operatorname{div}[\varrho(\operatorname{sym}(Y\dot{B}^{\top}) - H)] = \varrho f + \operatorname{div}(\operatorname{sym} T_a + A_a - \operatorname{sym}\operatorname{div}\mathbbm{m}_a), \\ \varrho\operatorname{skw}(Y\dot{B}^{\top}) = \varrho\operatorname{skw} M - \operatorname{skw} A_a + \operatorname{skw}\operatorname{div}\mathbbm{m}_a, \\ \varrho(\dot{H} - HB^{\top} - BH) = \varrho J - Z_a + \operatorname{div}\mathbbm{j}_a, \end{cases}$$

where L and B must satisfy $\operatorname{sym} L = \operatorname{sym} B$ and the effective specific external body force f entering $(115)_3$ is defined by

$$f = b - \operatorname{div}(\operatorname{sym} M) - (\operatorname{sym} M)\operatorname{grad}\ln\varrho. \tag{116}$$

In view of (106), given a second-order tensor field U, the rate

$$\dot{U} - UB^{\top} - BU = \dot{U} - UL^{\top} - LU + (W - \operatorname{skw} B)U - U(W - \operatorname{skw} B) \tag{117}$$

is the sum of the Oldroyd rate $\dot{U} - UL^{\top} - LU$ of U and 'corotational' terms involving the difference between the macrospin W and the mesospin $\operatorname{skw} B$. By $(54)_1$ and $(109)_{1,2}$, the basic relation (22) requires that T_a and A_a satisfy

$$\operatorname{skw} T_a = \operatorname{skw} A_a. \tag{118}$$

Only the symmetric part of $T_a + A_a$, which might be viewed as an effective measure of stress at the macroscale, appears in the linear momentum balance $(115)_3$.

Observations analogous to those appearing the final paragraph of the previous section apply here as well: the linear momentum balance $(115)_3$ is more strongly influenced by terms associated with the moment of mesomomentum balance than is its counterpart $(97)_3$ and, since the terms entering the momentum of mesomomentum balance $(115)_3$ take values in the 3-dimensional subpace Skw of Lin, the associated effects are more easily characterized for mesospin continua than for either fully incompressible mesostretch continua or purely suffusionless continua.

6.4. Hypocontinua

The strongest linear constraint between the velocity gradient L and mesodistortion rate B that forbids suffusion slaves the latter to the former, so that

$$L = B. \tag{119}$$

Following Capriz [4], we refer to media governed by the equations obtained by reducing the balance laws (20) of the theory of ephemeral continua in accord with (119) as 'hypocontinua.'

Regardless of which of the two alternative evolution equations for obtaining G mentioned in Footnote 1 is used, the constraint (119) implies that $\dot{G} = LG$. Since the macroscopic deformation gradient F obeys $\dot{F} = LF$, provided that G and F agree initially, it follows that $G = F$ thereafter.[2] Under these circumstances, $Y = \delta^2 GG^{\top} = \delta^2 FF^{\top}$, where δ is the locular edge length, from which, bearing in mind (119), it follows that

$$\dot{Y} - YB^{\top} - BY = \delta^2(\dot{F} - LF)F^{\top} + \delta^2 F(\dot{F} - LF)^{\top} = 0. \tag{120}$$

[2] Since (119) rules out suffusion and $\dot{F} = LF$, the alternatives in Footnote 1 both give $\dot{G} = LG$. Moreover, since $G = F$, $\det(GF^{-1}) = 1$ and $Y = \delta^2 GG^{\top} = \delta^2 FF^{\top}$ for either alternative.

For hypocontinua, the mesoinertia balance $(20)_2$ is thus simply an embodiment of the kinematical identity $\dot{F} = LF$. Our discussion of hypocontinua therefore makes no further mention of that balance.

The constraint defining hypocontinua corresponds to imposing (31) with

$$\mathbb{H} = \mathbb{K} = \mathbb{I}. \tag{121}$$

That the choice (121) of $\mathbb{H}$ and $\mathbb{K}$ satisfies (51) and (52) follows immediately.

To determine the reactions T_r, A_r, and $\mathbb{m}_r$ via $(59)_{1-4}$, we next recast the constraint (119) in terms of isotropic subspaces $\mathscr{U}$ and $\mathscr{W}$ of Lin satisfying (38) and an isotropic fourth-order tensor $\mathbb{N}$ consistent with $\operatorname{Null}\mathbb{N} \supseteq \mathscr{U}^{\perp}$ and $\operatorname{Rng}\mathbb{N} \subseteq \mathscr{W}^{\perp}$. Since L in (119) is unrestricted, $\mathscr{U}$ must coincide with all of Lin:

$$\mathscr{U} = \mathrm{Lin}. \tag{122}$$

Further, since $\mathscr{W}$ can be identified with $\operatorname{Null}\mathbb{K}$, (121) implies that $\mathscr{W}$ must coincide with the subspace $\{0\}$ of Lin:

$$\mathscr{W} = \{0\}. \tag{123}$$

From (122) and (123), we deduce that

$$\mathscr{U}^{\perp} = \{0\} \quad \text{and} \quad \mathscr{W}^{\perp} = \mathrm{Lin}. \tag{124}$$

Next, (119) allows us to immediately identify $\mathbb{N}$ as

$$\mathbb{N} = \mathbb{I}. \tag{125}$$

Since $\operatorname{Null}\mathbb{I} = \{0\}$ and $\operatorname{Rng}\mathbb{I} = \mathrm{Lin}$, (124) and (125) imply that $\operatorname{Null}\mathbb{N} = \operatorname{Null}\mathbb{I} = \{0\} \supseteq \{0\} = \mathscr{U}^{\perp}$ and $\operatorname{Rng}\mathbb{N} = \operatorname{Rng}\mathbb{I} = \mathrm{Lin} \subseteq \mathrm{Lin} = \mathscr{W}^{\perp}$. Thus, the choice (125) satisfies (39).

On using (124) in $(59)_{1-3}$, we conclude that

$$T_r + \mathbb{N}A_r^{\top} \in \{0\}, \quad A_r^{\top} \in \mathrm{Lin}, \quad \operatorname{Rng}\mathbb{m}_r \subseteq \mathrm{Lin}, \tag{126}$$

the first of which gives

$$T_r = -A_r^{\top} \tag{127}$$

and the second and third of which leave A_r and m_r unrestricted. By $(59)_4$ and (125),

$${}^{t}\mathbb{m}_r + ({}^{t}\mathbb{m}_r)^{t} = 0, \tag{128}$$

from which it follows that

$$\operatorname{div}[(\operatorname{div}\mathbb{m}_r)^{\top}] = \operatorname{div}\operatorname{div}({}^{t}\mathbb{m}_r) = 0. \tag{129}$$

In view of $(54)_{2,3}$, (119), and (127), the transpose of the balance $(20)_4$ of mesomomentum determines $T_r = -A_r^\top$ in the form

$$T_r = \varrho(\dot{L}Y - H) - \varrho M^\top + A_a^\top - [\operatorname{div}(\mathbb{m}_a + \mathbb{m}_r)]^\top. \tag{130}$$

By (19) and (119), the suffusion vanishes for the constraint under consideration and all terms with factors of σ drop out of (20). Further, aside from determining the reaction T_r via (130), the balance $(20)_4$ is irrelevant to the theory. Thus, on recalling (56) and (60) while bearing in mind the discussion in the paragraph including (120) and the identity (129), under the constraint (119) characterizing hypocontinua, the balance laws (20) for ephemeral continua specialize to

$$\begin{cases} \dot{\varrho} + \varrho \operatorname{div} v = 0, \\ \varrho\dot{v} - \operatorname{div}[\varrho(\dot{L}Y - H)] = \varrho f + \operatorname{div}[\operatorname{sym}(T_a + A_a) - (\operatorname{div} \mathbb{m}_a)^\top], \\ \varrho(\dot{H} - HL^\top - LH) = \varrho J - Z_a + \operatorname{div} \mathbb{j}_a, \end{cases} \tag{131}$$

where the effective specific external body force f entering $(131)_2$ is defined by

$$f = b - \operatorname{div}(M^\top) - M^\top \operatorname{grad} \ln \varrho. \tag{132}$$

By $(54)_1$ and (127), the basic relation (22) requires that T_a and A_a satisfy

$$\operatorname{skw} T_a = \operatorname{skw} A_a. \tag{133}$$

Perhaps unsuprisingly, under the constraint (119), B is completely absent from the final balance laws (131) and the effects of $Y\dot{B}^\top$, A_a, and $\mathbb{m}_a$ are transferred to influence the law of linear momentum balance. As was the case for mesospin continua, only the symmetric part of the erstwhile effective measure of stress at the macroscale $T_a + A_a$ appears in the linear momentum balance $(131)_2$. Notice that the Oldroyd rate appears in $(131)_3$ and, thus, emerges naturally in the theory of hypocontinua.

6.5. *Compressible Navier–Stokes-αβ continua*

In contrast to the previously considered reductions of the balances (20) for ephemeral continua, the moment of mesomomentum balance does not appear among the balance laws (131) governing hypocontinua. Interestingly, (119) is not the only constraint with this property. To illustrate this point, we consider the consequences of insisting that velocity gradient L and the mesodistortion rate B obey the constraint

$$\operatorname{skw} L + \tfrac{1}{3}(\operatorname{tr} L)I = B, \tag{134}$$

which is somewhat weaker than (119). Following Capriz and Fried [5], we refer to media governed by equations obtained by reducing the balance laws (20) of the theory of ephemeral continua in accord with (134) as 'compressible Navier–Stokes-$\alpha\beta$ continua.'

The constraint defining compressible Navier–Stokes-$\alpha\beta$ continua corresponds to imposing (31) with

$$\mathbb{H} = \mathbb{W} + \mathbb{T} \quad \text{and} \quad \mathbb{K} = \mathbb{I}. \tag{135}$$

Since $\mathbb{I}$, $\mathbb{W}$, and $\mathbb{T}$ are isotropic, the choice (135) of $\mathbb{H}$ and $\mathbb{K}$ is consistent with (51). Moreover, since $(\mathbb{W} + \mathbb{T})\Omega = \Omega = \mathbb{I}\Omega$ for any Ω in Skw, the choice (135) of $\mathbb{H}$ and $\mathbb{K}$ is consistent with (52). Thus, (135) is frame-indifferent.

To determine the reactions T_r, A_r, and $\mathbb{m}_r$ via $(59)_{1-4}$, we next recast the constraint (134) in terms of isotropic subspaces $\mathscr{U}$ and $\mathscr{W}$ of Lin satisfying (38) and an isotropic fourth-order tensor $\mathbb{N}$ consistent with $\operatorname{Null}\mathbb{N} \supseteq \mathscr{U}^{\perp}$ and $\operatorname{Rng}\mathbb{N} \subseteq \mathscr{W}^{\perp}$. Since L in (134) is unrestricted, $\mathscr{U}$ must coincide with all of Lin:

$$\mathscr{U} = \text{Lin}. \tag{136}$$

Further, since $\mathscr{W}$ can be identified with $\operatorname{Null}\mathbb{K}$, (135) implies that $\mathscr{W}$ must coincide with the subspace $\{0\}$ of Lin:

$$\mathscr{W} = \{0\}. \tag{137}$$

From (103) and (104), we deduce that

$$\mathscr{U}^{\perp} = \{0\} \quad \text{and} \quad \mathscr{W}^{\perp} = \text{Lin}. \tag{138}$$

In view of $(3)_4$, (134), (137), and (138), we next find that

$$B = (\mathbb{W} + \mathbb{T})L, \tag{139}$$

which, when compared with (53), allows us to identify $\mathbb{N}$ as

$$\mathbb{N} = \mathbb{W} + \mathbb{T}. \tag{140}$$

Since $\operatorname{Null}(\mathbb{W} + \mathbb{T}) = \text{SymDev}$ and $\operatorname{Rng}(\mathbb{W} + \mathbb{T}) = \operatorname{span}(\text{Skw} \cup \text{Sph})$, (138) and (140) imply that $\operatorname{Null}\mathbb{N} = \operatorname{Null}(\mathbb{W} + \mathbb{T}) = \text{SymDev} \supseteq \{0\} = \mathscr{U}^{\perp}$ and $\operatorname{Rng}\mathbb{N} = \operatorname{Rng}(\mathbb{W} + \mathbb{T}) = \operatorname{span}(\text{Skw} \cup \text{Sph}) \subseteq \text{Lin} = \mathscr{W}^{\perp}$. The choice (140) therefore satisfies (39).

On using (138) in $(59)_{1-3}$, we find that

$$T_r + \mathbb{N}A_r^{\top} \in \{0\}, \quad A_r^{\top} \in \text{Lin}, \quad \operatorname{Rng}\mathbb{m}_r \subseteq \text{Lin}, \tag{141}$$

the first of which gives

$$T_r = \operatorname{skw} A_r - \tfrac{1}{3}(\operatorname{tr} A_r)I \tag{142}$$

and, in a repetition of what occurs in the case of hypocontinua, the second and third of which leave A_r and m_r unrestricted. Observe that, as consequences of (142), T_r and A_r must obey

$$\text{(143)} \qquad \operatorname{skw} T_r = \operatorname{skw} A_r \quad \text{and} \quad \operatorname{tr} T_r = -\operatorname{tr} A_r,$$

whereby (142) can be expressed as

$$\text{(144)} \qquad T_r = \operatorname{skw} T_r + \tfrac{1}{3}(\operatorname{tr} T_r)I.$$

Also, by $(59)_4$ and (140), $(\mathbb{W} + \mathbb{T})^t\mathbb{m}_r + [(\mathbb{W} + \mathbb{T})^t\mathbb{m}_r]^t = 0$, whereby

$$\text{(145)} \qquad \operatorname{div}[(\mathbb{T} - \mathbb{W}) \operatorname{div} \mathbb{m}_r] = 0.$$

Next, we use $(54)_{2,3}$ and (134) to specialize the balance $(20)_4$ of mesomomentum. We then extract the symmetric and deviatoric, skew, and spherical components of the resulting equation. The first of these components determines a gauge relation,

$$\text{(146)} \qquad \begin{aligned} \varrho \operatorname{sym}[\operatorname{dev}(Y\dot{B}^\top - H)] = \varrho \operatorname{sym}(\operatorname{dev} M) &- \operatorname{sym}(\operatorname{dev} A_a) \\ &- \operatorname{sym}(\operatorname{dev} A_r) + \operatorname{sym}[\operatorname{dev}\operatorname{div}(\mathbb{m}_a + \mathbb{m}_r)], \end{aligned}$$

for $\operatorname{sym}[\operatorname{dev}(A_r - \operatorname{div} \mathbb{m}_r)]$. With (143), the second and third components yield expressions,

$$\text{(147)} \qquad \operatorname{skw} T_r = -\varrho \operatorname{skw}(Y\dot{B}^\top) + \varrho \operatorname{skw} M - \operatorname{skw} A_a + \operatorname{skw}\operatorname{div}(\mathbb{m}_a + \mathbb{m}_r),$$

and

$$\text{(148)} \qquad \operatorname{tr} T_r = \varrho \operatorname{skw}(Y\dot{B}^\top) - \varrho \operatorname{skw} M + \operatorname{skw} A_a - \operatorname{skw}\operatorname{div}(\mathbb{m}_a + \mathbb{m}_r),$$

which, when used in (144), determine T_r.

Thus, as in the theory of hypocontinua, apart from determining T_r, the moment of mesomomentum balance is irrelevant in the theory of compressible Navier–Stokes-$\alpha\beta$ continua. In view of this observation, recalling (56) and (60), while making use of (144), (145), (147), and (148), we find that, under the constraint (134) characterizing Navier–Stokes-$\alpha\beta$ continua, the balance laws (20) for ephemeral continua specialize to

$$\text{(149)} \qquad \begin{cases} \dot{\varrho} + \varrho \operatorname{div} v = 0, \\ \varrho\big(\dot{Y} + YW - WY - \tfrac{2}{3}(\operatorname{div} v) Y\big) = 0, \\ \varrho\dot{v} - \operatorname{div}[\varrho(\operatorname{skw}(Y\dot{W})] + \tfrac{1}{3}\operatorname{grad}[(\overset{\bullet}{\overline{\operatorname{div} v}}) \operatorname{tr} Y + \operatorname{tr} H] \\ \qquad = \varrho f + \operatorname{div}(\operatorname{sym} T_a + \operatorname{skw}\operatorname{div} \mathbb{m}_a) + \tfrac{1}{3}\operatorname{grad}[\operatorname{tr}(A_a - \operatorname{div} \mathbb{m}_a)], \\ \varrho\big(\dot{H} + HW - WH - \tfrac{2}{3}(\operatorname{div} v) W\big) = \varrho J - Z_a + \operatorname{div} \mathbb{j}_a, \end{cases}$$

where we have used conventional notation,

$$D = \operatorname{sym} L \quad \text{and} \quad W = \operatorname{skw} L, \tag{150}$$

for the symmetric and skew components of L and the trivial identity $\operatorname{tr} L = \operatorname{div} v$ and where the effective specific external body force f entering $(149)_3$ is defined by

$$f = b + \varrho(\operatorname{div}(\operatorname{skw} M) - \tfrac{1}{3}\operatorname{grad}\operatorname{tr} M) + \varrho(\operatorname{skw} M - \tfrac{1}{3}(\operatorname{tr} M)I)\operatorname{grad}\ln\varrho, \tag{151}$$

and, as with hypocontinua, T_a and A_a must satisfy (133). While it is unsurprising that the constraint (119) leads to final balance laws that do not involve B, it is somewhat surprising that the same holds true for (134)—a constraint that leaves the symmetric and deviatoric part of B indeterminate. Consistent with results of Capriz and Fried [5], the corotational rate (adjusted to accommodate compressibility) enters $(149)_{2,4}$ (and, by (17) and $(20)_2$, implicitly in $(149)_3$) emerges as the preferred frame-indifferent rate in the theory of compressible Navier–Stokes-$\alpha\beta$ continua.

7. General corollaries of the constraint

When discussing constraints and their implications in Sections 4 and 5, a general approach was adopted, with some formal complexities which might seem largely avoidable if interest were restricted to relatively simple occurrences of the type discussed in Section 6. Actually, the events contemplated there were furnished to evidence features, curious or crucial, typical of the theory and to establish links with more standard theories. To show that the opening generality offers much more than a formal backdrop, we now tender an all-inclusive analysis of the consequences of linear constraints involving the velocity gradient L and the mesodistortion rate B, opening an easy approach to desired choices.

7.1. *Reduction of the balance laws*

Consider isotropic subspaces $\mathscr{U}$ and $\mathscr{W}$ of Lin satisfying (38) and an isotropic fourth-order tensor

$$\mathbb{N} = \lambda_1\mathbb{S}_0 + \lambda_2\mathbb{W} + \lambda_3\mathbb{T} \tag{152}$$

consistent with (39). Let $\mathbb{P}_{\mathscr{U}}$ and $\mathbb{P}_{\mathscr{W}}$ denote the projectors onto $\mathscr{U}$ and $\mathscr{W}$, respectively. Suppose that (L, B) satisfies the constraint described by $\mathscr{U}$, $\mathscr{W}$, and $\mathbb{N}$. Then, since

$$L = \mathbb{P}_{\mathscr{U}}L \quad \text{and} \quad B_{\mathscr{W}} = \mathbb{P}_{\mathscr{W}}B, \tag{153}$$

(19) and (53) can be used to show that any suffusion permitted under the constraint must be given by

$$\sigma = \operatorname{tr}(\mathbb{P}_{\mathscr{U}}L - \mathbb{P}_{\mathscr{W}}B - \mathbb{N}L). \tag{154}$$

As isotropic fourth-order tensors, $\mathbb{P}_{\mathscr{U}}$ and $\mathbb{P}_{\mathscr{W}}$ must admit representations of the form

$$\mathbb{P}_{\mathscr{U}} = \alpha_{\mathscr{U}}\mathbb{S}_0 + \beta_{\mathscr{U}}\mathbb{W} + \gamma_{\mathscr{U}}\mathbb{T} \tag{155}$$

and

$$\mathbb{P}_{\mathscr{W}} = \alpha_{\mathscr{W}}\mathbb{S}_0 + \beta_{\mathscr{W}}\mathbb{W} + \gamma_{\mathscr{W}}\mathbb{T}. \tag{156}$$

Additionally, as projectors, $\mathbb{P}_{\mathscr{U}}$ and $\mathbb{P}_{\mathscr{W}}$ must obey $\mathbb{P}_{\mathscr{U}}^2 = \mathbb{P}_{\mathscr{U}}$ and $\mathbb{P}_{\mathscr{W}}^2 = \mathbb{P}_{\mathscr{W}}$, which with (155) and (156) yield

$$\begin{cases} \alpha_{\mathscr{U}}^2 = \alpha_{\mathscr{U}}, \quad \beta_{\mathscr{U}}^2 = \beta_{\mathscr{U}}, \quad \gamma_{\mathscr{U}}^2 = \gamma_{\mathscr{U}}, \\ \alpha_{\mathscr{U}}^2 = \alpha_{\mathscr{U}}, \quad \beta_{\mathscr{U}}^2 = \beta_{\mathscr{U}}, \quad \gamma_{\mathscr{U}}^2 = \gamma_{\mathscr{U}}. \end{cases} \tag{157}$$

As a consequence of (157) each coefficient in the list $\alpha_{\mathscr{U}}$, $\beta_{\mathscr{U}}$, $\gamma_{\mathscr{U}}$, $\alpha_{\mathscr{W}}$, $\beta_{\mathscr{W}}$, and $\gamma_{\mathscr{W}}$ must equal either zero or unity. The projectors $\mathbb{P}_{\mathscr{U}}^{\perp}$ and $\mathbb{P}_{\mathscr{W}}^{\perp}$ onto the orthogonal complements $\mathscr{U}^{\perp}$ and $\mathscr{W}^{\perp}$ of $\mathscr{U}$ and $\mathscr{W}$ can be expressed via

$$\mathbb{P}_{\mathscr{U}}^{\perp} = \mathbb{I} - \mathbb{P}_{\mathscr{U}} \quad \text{and} \quad \mathbb{P}_{\mathscr{W}}^{\perp} = \mathbb{I} - \mathbb{P}_{\mathscr{W}}. \tag{158}$$

Notice that, when specialized according to (152), (155), and (156), the relation (154) for any suffusion allowed under the constraint takes the form

$$\sigma = (\gamma_{\mathscr{U}} - \lambda_3)\operatorname{tr} L - \gamma_{\mathscr{W}} \operatorname{tr} B, \tag{159}$$

from which it follows that the constraint described by $\mathscr{U}$, $\mathscr{W}$, and $\mathbb{N}$ rules out suffusion if and only if the coefficients λ_3, $\gamma_{\mathscr{U}}$, and $\gamma_{\mathscr{W}}$ entering the representations (152), (155), and (156) of $\mathbb{N}$, $\mathbb{P}_{\mathscr{U}}$, and $\mathbb{P}_{\mathscr{W}}$ obey $\gamma_{\mathscr{U}} = \lambda_3$ and $\gamma_{\mathscr{W}} = 0$.

The representations (155) and (156) can be used to show that $\mathscr{U}$ and $\mathscr{W}$ obey (38) if and only if

$$\alpha_{\mathscr{U}} + \alpha_{\mathscr{W}} + 5(\beta_{\mathscr{U}} + \beta_{\mathscr{W}}) + 3(\gamma_{\mathscr{U}} + \gamma_{\mathscr{W}}) \geq 9. \tag{160}$$

Similarly, the representations (152) and (155)–(156) can be used to show that $\mathbb{N}$ obeys $(39)_1$ and $(39)_2$ if and only if

$$\begin{cases} (\alpha_{\mathscr{U}} = 0 \quad \Rightarrow \quad \lambda_1 = 0) \quad \Leftrightarrow \quad \alpha_{\mathscr{U}}\lambda_1 = \lambda_1, \\ (\beta_{\mathscr{U}} = 0 \quad \Rightarrow \quad \lambda_2 = 0) \quad \Leftrightarrow \quad \beta_{\mathscr{U}}\lambda_2 = \lambda_2, \\ (\gamma_{\mathscr{U}} = 0 \quad \Rightarrow \quad \lambda_3 = 0) \quad \Leftrightarrow \quad \gamma_{\mathscr{U}}\lambda_3 = \lambda_3, \end{cases} \tag{161}$$

and

$$\begin{cases} (\alpha_{\mathscr{W}} = 1 \quad \Rightarrow \quad \lambda_1 = 0) \quad \Leftrightarrow \quad (1-\alpha_{\mathscr{W}})\lambda_1 = \lambda_1, \\ (\beta_{\mathscr{W}} = 1 \quad \Rightarrow \quad \lambda_2 = 0) \quad \Leftrightarrow \quad (1-\beta_{\mathscr{W}})\lambda_2 = \lambda_2, \\ (\gamma_{\mathscr{W}} = 1 \quad \Rightarrow \quad \lambda_3 = 0) \quad \Leftrightarrow \quad (1-\gamma_{\mathscr{W}})\lambda_3 = \lambda_3, \end{cases} \tag{162}$$

respectively.

Notice that, with the definitions (155) and (156), the general requirements $(59)_{1-4}$ that the reactions T_r, A_r, and $\mathbb{m}_r$ must satisfy are equivalent to

$$\mathbb{P}_{\mathscr{U}}(T_r + \mathbb{N}A_r^\top) = 0, \quad \mathbb{P}_{\mathscr{W}}A_r^\top = 0, \quad \mathbb{P}_{\mathscr{W}}\mathbb{m}_r = 0, \quad \mathbb{N}^t\mathbb{m}_r + (\mathbb{N}^t\mathbb{m}_r)^t = 0. \tag{163}$$

By $(54)_1$, (155), $(158)_1$, and $(163)_1$,

$$\operatorname{skw} T = \operatorname{skw} T_a + \beta_{\mathscr{U}}\lambda_2 \operatorname{skw} A_r + (1-\beta_{\mathscr{U}}) \operatorname{skw} T_r, \tag{164}$$

while, by $(54)_2$, (156), $(158)_2$, and $(163)_2$,

$$\operatorname{skw} A = \operatorname{skw} A_a + (1-\beta_{\mathscr{W}}) \operatorname{skw} A_r. \tag{165}$$

On combining (22), (161), and (164)–(165), we obtain

$$\begin{aligned} T = {} & \operatorname{sym} T_a + \operatorname{skw} A_a + [(1-\alpha_{\mathscr{U}})\mathbb{S}_0 + (1-\gamma_{\mathscr{U}})\mathbb{T}]T_r \\ & - (\lambda_1\mathbb{S}_0 - (1-\beta_{\mathscr{W}})\mathbb{W} + \lambda_3\mathbb{T})A_r. \end{aligned} \tag{166}$$

Next, by $(158)_2$ and $(163)_3$,

$$\mathbb{P}_{\mathscr{W}}^{\perp} \operatorname{div} \mathbb{m}_r = \operatorname{div} \mathbb{m}_r, \tag{167}$$

while, by $(163)_4$,

$$\operatorname{div}\operatorname{div}(\mathbb{N}^t\mathbb{m}_r) = 0. \tag{168}$$

Next, isolating the components of the moment of mesomomentum balance $(20)_4$ belonging to SymDev, Skw, and Sph, defining

$$\Sigma = \varrho(Y\dot{B}^\top - H) - \varrho M + A_a - \operatorname{div} \mathbb{m}_a, \tag{169}$$

a lengthy but straightforward calculation allows us to rewrite (166) in the form

$$\begin{aligned} T = {} & \operatorname{sym} T_a + \operatorname{skw} A_a + (\lambda_1\mathbb{S}_0 - \mathbb{W} + \lambda_3\mathbb{T})\Sigma \\ & + [(1-\alpha_{\mathscr{U}})\mathbb{S}_0 + (1-\gamma_{\mathscr{U}})\mathbb{T}]T_r \\ & + (1-\beta_{\mathscr{W}} - \lambda_2)\mathbb{W} \operatorname{div} \mathbb{m}_r - \mathbb{N} \operatorname{div}({}^t\mathbb{m}_r). \end{aligned} \tag{170}$$

Bearing in mind (168) and (169), we use (170) to write the linear momentum balance $(20)_3$ as

$$
\begin{aligned}
\varrho(\dot{v}+\sigma v) = \varrho b &+ \operatorname{div}(\operatorname{sym} T_a + \operatorname{skw} A_a + (\lambda_1 \mathbb{S}_0 - \mathbb{W} + \lambda_3 \mathbb{T})\Sigma) \\
&+ \operatorname{div}[(1-\alpha_{\mathscr{U}})\mathbb{S}_0 T_r + (1-\gamma_{\mathscr{U}})\mathbb{T} T_r \\
&\qquad + (1-\beta_{\mathscr{W}}-\lambda_2)\mathbb{W} \operatorname{div} \mathbb{m}_r].
\end{aligned} \tag{171}
$$

Moreover, the only potentially nontrivial component of the moment of mesomomentum balance $(20)_4$ is that involving terms with values in $\mathscr{W}$, which, in view of (156) and (169), can be expressed as

$$
\mathbb{P}_{\mathscr{W}}\Sigma = 0. \tag{172}
$$

At first glance, (171) and (172) appear to be deficient to the extent that they contain the reactions T_r and $\mathbb{m}_r$. However, a multitude of constraints, as described by particular choices of $\mathscr{U}$, $\mathscr{W}$, and $\mathbb{N}$, can be read off from (171) and (172). Thus, (171) and (172) provide a platform for developing a broad range of theories for constrained ephemeral continua. For simplicity, we have not indicated the general form of the effective specific body force germane to (171).

7.2. Mesospin continua revisited

The general results (159), (171), and (172) may be easily used to specialize the balance laws (20) of ephemeral continua in accord with the particular constraints considered in Section 6.

Consider, for example, the constraint (102) giving rise to mesospin continua. In this case, (103) and (104) yield $\mathbb{P}_{\mathscr{U}} = \mathbb{I}$ and $\mathbb{P}_{\mathscr{W}} = \mathbb{W}$. Thus, since, by (107), $\mathbb{N} = \mathbb{S}$, (152), (155), and (156) imply that

$$
\lambda_1 = \lambda_3 = 1, \quad \lambda_2 = 0, \quad \alpha_{\mathscr{U}} = \beta_{\mathscr{U}} = \gamma_{\mathscr{U}} = 1, \quad \alpha_{\mathscr{W}} = \gamma_{\mathscr{W}} = 0, \quad \beta_{\mathscr{W}} = 1. \tag{173}
$$

Since $\gamma_{\mathscr{U}} = \lambda_3 = 1$ and $\gamma_{\mathscr{W}} = 0$, (159) yields $\sigma = 0$, confirming that mesospin continua cannot support suffusion and, moreover, that the terms in (20) that have σ as a coefficient must be absent from the evolution equations for mesospin continua. Thus, recalling (56) and (60), the balances $(20)_{1,2,3}$ coincide with $(115)_{1,2,5}$ and it remains only to show that (171)–(172) specialize to $(115)_{3–4}$. Since $\lambda_1 = \lambda_3 = 1$, we next find that $\lambda_1\mathbb{S}_0 - \mathbb{W} + \lambda_3\mathbb{T} = \mathbb{S}_0 + \mathbb{T} - \mathbb{W} = \mathbb{S} - \mathbb{W}$, which, on invoking the definition (169) of Σ and the symmetry of H, delivers

$$
\begin{aligned}
&\operatorname{sym} T_a + \operatorname{skw} A_a + (\alpha\mathbb{S}_0 - \mathbb{W} + \gamma\mathbb{T})\Sigma \\
&\qquad = \operatorname{sym}(T_a + A_a) + \varrho(\dot{B}Y - H) - \varrho M^\top - (\operatorname{div} \mathbb{m}_a)^\top.
\end{aligned} \tag{174}
$$

Since $\alpha_{\mathscr{U}} = \gamma_{\mathscr{U}} = \beta_{\mathscr{W}} = 1$ and $\lambda_2 = 0$, we have

$$
(1-\alpha_{\mathscr{U}})\mathbb{S}_0 T_r + (1-\gamma_{\mathscr{U}})\mathbb{T} T_r + (1-\beta_{\mathscr{W}}-\lambda_2)\mathbb{W} \operatorname{div} \mathbb{m}_r = 0. \tag{175}
$$

On using (174) and (175) in the general linear momentum balance (171), we obtain

$$\varrho\dot{v} - \operatorname{div}[\varrho(\dot{B}Y - H)] = \varrho b + \operatorname{div}(\operatorname{sym}(T_a + A_a) - \varrho M^{\top} - (\operatorname{div}\mathbb{m}_a)^{\top}). \tag{176}$$

Further, since $\mathbb{P}_{\mathscr{W}} = \mathbb{W}$ for a mesospin continuum, we obtain

$$\varrho\,\operatorname{skw}(Y\dot{B}^{\top}) = \varrho\,\operatorname{skw} M - \operatorname{skw} A_a + \operatorname{skw}\operatorname{div}\mathbb{m}_a, \tag{177}$$

which is the momemt of mesomomentum balance $(115)_4$ for a mesospin continuum.

To establish the consistency between (176) and the previously obtained version $(115)_3$ the balance of linear momentum for a mesospin continuum, notice that, by (177),

$$\operatorname{div}[\varrho\,\operatorname{skw}(Y\dot{B}^{\top})] = \operatorname{div}(\varrho\,\operatorname{skw} M - \operatorname{skw} A_a + \operatorname{skw}\operatorname{div}\mathbb{m}_a), \tag{178}$$

which when added to (176) yields

$$\varrho\dot{v} - \operatorname{div}[\varrho(\operatorname{sym}(Y\dot{B}^{\top}) - H)] = \varrho f + \operatorname{div}(\operatorname{sym} T_a + A_a - \operatorname{sym}\operatorname{div}\mathbb{m}_a) \tag{179}$$

which, with the definition (116) of the specific external body force f, is precisely $(115)_3$.

The remaining cases considered in Section 6 can be treated similarly.

7.3. Deviatoric hypocontinua

We now utilize the general results (159), (171), and (172) to explore the implications of slightly weakening the constraint (119) leading to the theory of hypocontinua by choosing

$$\mathscr{U} = \mathrm{Lin}, \quad \mathscr{W} = \{0\}, \quad \mathbb{N} = \mathbb{S}_0 + \mathbb{W}, \tag{180}$$

whereby

$$\mathbb{P}_{\mathscr{U}} = \mathbb{I} \quad \text{and} \quad \mathbb{P}_{\mathscr{W}} = 0. \tag{181}$$

We refer to media governed by equations obtained by reducing the balance laws (20) of the theory of ephemeral continua in accord with (187) as 'deviatoric hypocontinua.'

Of these choices involved in (180), only that involving $\mathbb{N}$ differs from the choice $\mathbb{N} = \mathbb{I}$ arising in the theory of hypocontinua. By (53) and $(180)_{2,3}$, we have

$$\operatorname{dev} L = B, \tag{182}$$

which should be compared with the condition $L = B$ upon which the theory of hypocontinua is based.

By (152), (155), (156), $(180)_3$, and (181), the coefficients entering (159), (171), and (172) are

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 0, \quad \alpha_{\mathscr{U}} = \beta_{\mathscr{U}} = \gamma_{\mathscr{U}} = 1, \quad \alpha_{\mathscr{W}} = \beta_{\mathscr{W}} = \gamma_{\mathscr{W}} = 0. \tag{183}$$

Since $\gamma_{\mathscr{U}} = 1$, $\lambda_3 = 0$, and $\gamma_{\mathscr{W}} = 0$, (159) gives

$$\sigma = \operatorname{tr} L = \operatorname{div} v, \tag{184}$$

from which it follows that, in contrast to (119), the constraint (182) allows for suffusion as determined by the macroscopic rate of dilation. Since $\mathbb{P}_{\mathscr{W}} = 0$, the moment of mesomomentum balance is, as in the theories for hypocontinua and compressible Navier–Stokes-$\alpha\beta$ continua, inconsequential to the theory. Since B is fully determined by (182), this outcome is, perhaps, not too surprising. On using (183) to specialize the linear momentum balance (171) and bearing in mind (184), we find that, under the constraint (182) characterizing deviatoric hypocontinua, the balance laws (20) for ephemeral continua specialize to

$$\begin{cases} \dot{\varrho} = 0, \\ \varrho\big(\dot{Y} - YL^{\top} - LY + \tfrac{1}{3}(\operatorname{div} v)Y\big) = 0, \\ \varrho(\dot{v} + (\operatorname{div} v)v) - \operatorname{div}\big(\varrho \operatorname{dev}(\dot{L}Y - H) - \tfrac{1}{3}\varrho(\overset{\cdot}{\overline{\operatorname{div} v}}) \operatorname{dev} Y\big) \\ \qquad = \varrho f + \operatorname{div}(\operatorname{sym} T_a + \operatorname{sym}\operatorname{dev} A_a - \operatorname{dev}(\operatorname{div} \mathbb{m}_a)^{\top}) \\ \varrho\big(\dot{H} - HL^{\top} - LH + \tfrac{1}{3}(\operatorname{div} v)H\big) = \varrho J - Z_a + \operatorname{div} \mathring{\mathbb{j}}_a, \end{cases} \tag{185}$$

where the effective specific external body force f entering $(185)_3$ is defined by

$$f = b - \operatorname{div}(M^{\top} - \tfrac{1}{3}(\operatorname{tr} M)I) - (M^{\top} - \tfrac{1}{3}(\operatorname{tr} M)I) \operatorname{grad} \log \varrho. \tag{186}$$

The reduced mass balance $(185)_1$ shows, in contrast to what occurs for hypocontinua, during the motion of a deviatoric hypocontinua, the mass density must be preserved along particle trajectories.

7.4. Fully-incompressible continua without mesostretch. Navier–Stokes-$\alpha\beta$ equations for an incompressible fluid

As a final illustration of the general results (159), (171), and (172), we consider the constraint described by choosing

$$\mathscr{U} = \mathrm{Dev}, \quad \mathscr{W} = \mathrm{Skw}, \quad \text{and} \quad \mathbb{N} = 0, \tag{187}$$

whereby

$$\mathbb{P}_{\mathscr{U}} = \mathbb{S}_0 + \mathbb{W} \quad \text{and} \quad \mathbb{P}_{\mathscr{W}} = \mathbb{W}. \tag{188}$$

We refer to media governed by equations obtained by reducing the balance laws (20) of the theory of ephemeral continua in accord with (187) as 'fully-incompressible continua without mesostretch.'

By (53) and $(188)_{2,3}$, we have

$$\operatorname{sym} B = 0 \quad \text{and} \quad \operatorname{tr} L = 0. \tag{189}$$

Further, by (152), (155), (156), $(187)_3$, and (188),

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \alpha_{\mathscr{U}} = \beta_{\mathscr{U}} = 1, \quad \gamma_{\mathscr{U}} = 0, \quad \alpha_{\mathscr{W}} = 0, \quad \beta_{\mathscr{W}} = 1, \quad \gamma_{\mathscr{W}} = 0. \tag{190}$$

Since $\gamma_{\mathscr{U}} = \lambda_3 = 0$ and $\gamma_{\mathscr{W}} = 0$, (159) yields $\sigma = 0$. Thus, purely incompressible continua without mesostretch cannot support suffusion. This and $(189)_1$ imply that, for the constraint described by (187), the coshaping rate of an element U of Lin reduces in accord with:

$$\dot{U} + \sigma U - UB^{\top} - BU = \dot{U} + UB - BU. \tag{191}$$

Bearing in mind (171), (172), $(188)_1$, and (190), we find that, under the constraint (189) characterizing fully-incompressible continua without mesostretch, the balance laws (20) for ephemeral continua specialize to

$$\begin{cases} \dot{\varrho} = 0, \\ \varrho(\dot{Y} + YB - BY) = 0, \\ \varrho\dot{v} = \varrho b - \operatorname{grad}\varphi + \operatorname{div}(\operatorname{sym} T_a + \operatorname{skw} A_a), \\ \varrho \operatorname{skw}(Y\dot{B}) = -\varrho \operatorname{skw} M + \operatorname{skw} A_a - \operatorname{skw}\operatorname{div} \mathbb{m}_a, \\ \varrho(\dot{H} + HB - BH) = \varrho J - Z_a + \operatorname{div} \mathbb{j}_a, \end{cases} \tag{192}$$

where L and B must satisfy $\operatorname{tr} L = 0$ and $\operatorname{sym} B = 0$, and where

$$\varphi = -\tfrac{1}{3}(\operatorname{tr} T_r)I. \tag{193}$$

is a reactive pressure. Mass balance and linear momentum balance aside, the foregoing balances are identical to their counterparts, $(115)_{2,4,5}$, from the theory of mesospin continua. Using $(192)_4$ to eliminate $\operatorname{skw} A_a$ from $(192)_3$ yields a version,

$$\varrho\dot{v} - \operatorname{div}[\varrho \operatorname{skw}(Y\dot{B})] = \varrho f + \operatorname{div}(\operatorname{sym} T_a + \operatorname{skw}\operatorname{div} \mathbb{m}_a), \tag{194}$$

of the linear momentum balance that is more easily contrasted with its counterpart, $(115)_3$, from the theory for mesospin continua. Perhaps most significantly, the variance H enters $(115)_3$ but not (194). Additionally, whereas (194) includes a reactive pressure term needed to ensure that the portion of the constraint associated with incompressibility is met, $(115)_3$ does not. Notice, also, that the effective specific external body force f entering (194) is defined by

$$f = b + \operatorname{div}(\operatorname{skw} M) + (\operatorname{skw} M)\operatorname{grad}\ln\varrho, \tag{195}$$

which differs substantially from the expression, (116), arising in the theory of mesospin continua.

The mass balance $(192)_1$ requires the mass density ϱ to be constant along particle trajectories. Hereafter, we restrict attention to homogeneous fluids, in which case

$$\varrho = \text{constant}. \tag{196}$$

Observe that the moment of mesoinertia balance $(192)_2$ admits a solution of the form

$$Y = \alpha^2 I, \quad \alpha = \text{constant}. \tag{197}$$

On the basis of this observation, we specialize the balance laws (192) for fully-incompressible continua without mesostretch in accord with the assumption that the moment of mesoinertia tensor is isotropic. Moreover, motivated by the absence of a term involving H in the moment of mesomomentum balance $(192)_4$, we forego further consideration of the mesofluctuation balance $(192)_5$. Since each term of $(192)_4$ is skew, we define vectors u, m, and a by

$$u\times = B = \operatorname{skw} B, \quad m\times = -\operatorname{skw} M, \quad \text{and} \quad a\times = -\operatorname{skw} A_a, \tag{198}$$

where, given a vector c, $c\times$ is the element of Skw defined such that $(c\times)d = c \times d$ for any vector d, and an element S of Lin by

$$S\times = -\tfrac{1}{2}(\mathbb{m}_a - {}^t\mathbb{m}_a), \tag{199}$$

where, given U in Lin, $U\times$ is the third-order tensor defined such that $(U\times)c = (Uc)\times$ for any vector c. Bearing in mind (197)–(199), the identities

$$\operatorname{div}(a\times) = -\operatorname{curl} a, \quad \operatorname{div}(S\times) = (\operatorname{div} S)\times, \tag{200}$$

and that $(192)_5$ has been dropped from consideration, the balance laws (192) specialize to

$$\begin{cases} \varrho\dot{v} = \varrho b - \operatorname{grad}\varphi + \operatorname{div}(\operatorname{sym} T_a) + \operatorname{curl} a, \\ \varrho\alpha^2\dot{u} = \varrho m - a + \operatorname{div} S. \end{cases} \tag{201}$$

Alternatively, on solving $(201)_2$ for a and replacing the resulting expression in $(201)_1$ and using (196), we have the system

$$\begin{cases} \varrho(\dot{v} + \alpha^2 \operatorname{curl}\dot{u}) = \varrho f - \operatorname{grad}\varphi + \operatorname{div}(\operatorname{sym} T_a) + \operatorname{curl}\operatorname{div} S, \\ \varrho\alpha^2\dot{u} = \varrho m - a + \operatorname{div} S, \end{cases} \tag{202}$$

where the effective specific external body force f is now defined by

$$f = b + \operatorname{curl} m. \tag{203}$$

If we formally neglect moment of mesoinertia by setting $\varrho\alpha^2\dot{u} = 0$, (202) yield

$$\varrho\dot{v} = \varrho f - \operatorname{grad}\varphi + \operatorname{div}(\operatorname{sym} T_a) + \operatorname{curl}\operatorname{div} S, \tag{204}$$

which, setting aside notational conventions, coincides with the balance upon which Fried and Gurtin [8, 9] based their derivation of the Navier–Stokes-$\alpha\beta$ equations for an incompressible fluid. Specifically, with the choices

$$\operatorname{sym} T_a = 2\varrho(\nu D + \alpha^2 \overset{\circ}{D}) \quad \text{and} \quad S = 2\varrho\nu\beta^2 \operatorname{skw}(\operatorname{grad}\operatorname{curl} v), \tag{205}$$

where α and β carry dimensions of length, (204) specializes to

$$\varrho\dot{v} - 2\varrho\alpha^2 \operatorname{div}\overset{\circ}{D} = \varrho f - \operatorname{grad}\varpi + \varrho\nu(1 - \beta^2\triangle)\triangle v, \tag{206}$$

where a superposed circle denotes the corotational rate, so that, for any element U of Lin,

$$\overset{\circ}{U} = \dot{U} + WU - UW, \tag{207}$$

and the effective pressure ϖ is given by

$$\varpi = \varphi + \alpha^2 \operatorname{tr}(L^2). \tag{208}$$

Capriz and Fried [5] argue that the term $2\varrho\alpha^2 \operatorname{div}\overset{\circ}{D}$ entering the Navier–Stokes-$\alpha\beta$ equation (206) should stem from higher-order inertial effects. However, the steps leading from (204) to (206) identify that term as kinetic and thereby introduce some doubt about the physical legitimacy of the foregoing derivation.

An alternative derivation of (206), stemming directly from the system (202), is nevertheless possible. To achieve this, we first record the commutator identity

$$\operatorname{curl}\dot{u} - \dot{\overline{\operatorname{curl} u}} = \operatorname{div}[(u\times)L^{\top}] \tag{209}$$

distinguishing the difference between the curl of the material time-derivative of u and the material time-derivative of the curl of u. Next, we assume that the vector expression u for the mesodistorion rate $B = \operatorname{skw} B$ is equal to the vorticity $\operatorname{curl} v$:

$$u = \operatorname{curl} v. \tag{210}$$

Then, since

$$(\operatorname{curl} v)\times = 2W = L - L^{\top} \tag{211}$$

and, by $(189)_2$,

$$\operatorname{curl}\operatorname{curl} v = \operatorname{grad}\operatorname{div} v - \triangle v = -\triangle v, \tag{212}$$

(209) yields, after a tedious but straightforward calculation,

$$
\begin{aligned}
\operatorname{curl} \dot{v} &= -\dot{\overline{\triangle v}} - \operatorname{div}(LL^{\top}) + \operatorname{div}(L^{\top}L^{\top}) \\
&= -\dot{\overline{\triangle v}} - \operatorname{div}(LL^{\top}) + \frac{1}{2}\operatorname{grad}\operatorname{tr}(L^2) \\
&= -2\operatorname{div}\dot{W} \\
&= -2\operatorname{div}\overset{\circ}{D} + \operatorname{grad}\operatorname{tr}(L^2).
\end{aligned}
\tag{213}
$$

Thus, granted that u is determined by v by (210), the linear momentum balance $(202)_1$ specializes to

$$
\varrho(\dot{v} - \alpha^2 \operatorname{div}\overset{\circ}{D}) = \varrho f - \operatorname{grad}\varphi + \operatorname{div}(\operatorname{sym} T_a) + \operatorname{curl}\operatorname{div} S,
\tag{214}
$$

which on assuming that, instead of being given by $(205)_1$, $\operatorname{sym} T_a$, which, with reference to Dunn and Fosdick [6], is recognized as the generic expression for the extra stress of a non-Newtonian fluid of second grade, is given by the Newtonian relation

$$
\operatorname{sym} T_a = 2\varrho\nu D
\tag{215}
$$

and that $\mathbb{m}_a$ is, as previously, given by $(205)_2$, yields the Navier–Stokes-$\alpha\beta$ equation (206).

The relation (210) between u and v is perhaps most aptly viewed as a constraint. Since, by $(198)_1$, $u\times = B = \operatorname{skw} B$ and $(\operatorname{curl} v)\times = 2W$, this amounts to the requirement that

$$
B = 2W.
\tag{216}
$$

Thus, when supplemented by (210), the constraint described by (187) differs from that imposed by Capriz and Fried [5] by only a factor of 2 arising in the link between L and B.

8. Criticism and outlook

In many ways our paper opens more questions than it answers. A comprehensive study of the kinematics of ephemeral continua and the illustration of simple possible flows seems essential. Constitutive laws adequate to represent the behavior of some classes of continua must be proposed and justified, be it by exploring the underlying physical motives which suggest them, or be it by proffering some ensuing simple flows. In general, an accurate study of appropriate initial and boundary conditions must still be performed; the presence of inertial terms involving spatial gradients is provocative and, no doubt, behavior at the boundary requires again constitutive decisions. The physical chemistry of surfaces is a vast field, heavily conditioned by the ubiquitous impositions of the divergence theorem. Even here the proposed form of the balance laws prejudices, partly, the issue. However, in deriving consequences of that structure, an essential role is played by Euler cuts, imaginary macro actions which sever a portion of the body from the rest. They can be imagined as smooth as convenient; even so, the question arises here about the consequences at the meso level: going through a macro point they divide also the loculus! So, the interplay between the two scales must be transferred in some modelling repercussions, perhaps simply reflected in a smart choice of constitutive laws. Setting aside imaginary internal boundaries, the matter is distinctly different for an actual physical boundary. Smoothness may not be assured; besides, part of each loculus there belongs to the exterior and the external half might not belong to the same tribe of continua and its reactions might be foreign. An analysis of boundary constitutive laws, involving perhaps also deep geometrical properties, down to the meso level, besides physical instances, might be required. One needs only recall the standard addition of surface tension, absent in bulk, at the boundary between liquid and vapour and the complexities at the border between austenitic and martensitic phases in some solids. The panorama is extraordinarily rich; we hope that it will inspire interest.

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