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Continuum Mechanics — *Frictionless contact problems for elastic hemitropic solids: Boundary variational inequality approach*, by A. Gachechiladze, R. Gachechiladze and D. Natroshvili, communicated on 22 June 2012.

Dedicated to Professor Gaetano Fichera *on the occasion of 90th anniversary of his birthday.*

Abstract. — The frictionless contact problems for two interacting hemitropic solids with different elastic properties is investigated under the condition of natural impenetrability of one medium into the other. We consider two cases, the so-called coercive case (when elastic media are fixed along some parts of their boundaries), and the semicoercive case (the boundaries of the interacting elastic media are not fixed). Using the potential theory we reduce the problems to the boundary variational inequalities and analyse the existence and uniqueness of weak solutions. In the semicoercive case, the necessary and sufficient conditions of solvability of the corresponding contact problems are written out explicitly.

Key words: Hemitropic solids, Cosserat solids, frictionless contact, boundary variational inequality.

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Introduction

In the present work we consider a frictionless contact of two elastic hemitropic media with different physical properties under the condition of natural impenetrability. Here we consider the model of the theory of elasticity in which, unlike the classical theory, an elementary particle of a body along with displacements undergoes rotation, and hence the condition of mechanical equilibrium of the body is described by means of the three-component displacement vector and three-component micro-rotation vector.

The origin of the rational theories of polar continua goes back to brothers E. and F. Cosserat [CC1], [CC2], who gave a development of the mechanics of continuous media in which each material point has the six degrees of freedom defined by 3 displacement components and 3 micro-rotation components (for the history of the problem see [Min1], [Now1], [KGBB1], [Dy1], and the references therein).

A micropolar continua which is not isotropic with respect to inversion is called *hemitropic, noncentrosymmetric,* or *chiral*. Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules—DNA, as well as on a large scale, as in composites with helical or screw–shaped inclusions, certain types of nanotubes, bone, fabricated structures such as foams, chiral sculptured thin films and twisted fibers. For more details and applications see the references [Er1], [HZ1], [Sh1], [Ro1], [La1], [Mu1], [Mu2], [Now1], [Dy1], [YL1], [LB1] [CC1].

Refined mathematical models describing the hemitropic properties of elastic materials have been proposed by Aero and Kuvshinski [AK1], [AK2]. In the mathematical theory of hemitropic elasticity there are introduced the asymmetric force stress tensor and couple stress tensor, which are kinematically related with the asymmetric strain tensor and torsion (curvature) tensor via the constitutive equations. All these quantities are expressed in terms of the components of the displacement and micro-rotation vectors. In turn, the displacement and micro-rotation vectors satisfy a coupled complex system of second order partial differential equations. We note that the governing equations in this model become very involved and generate 6×6 matrix partial differential operator of second order. Evidently, the corresponding 6×6 matrix boundary differential operators describing the force stress and couple stress vectors have also an involved structure in comparison with the classical case.

In [NGS1], [NGZ1], [NS1], [NGGS1] the fundamental matrices of the associated systems of partial differential equations of statics and steady state oscillations have been constructed explicitly in terms of elementary functions and the basic boundary value and transmission problems of hemitropic elasticity have been studied by the potential method for smooth and non-smooth Lipschitz domains. Particular problems of the elasticity theory of hemitropic continuum have been considered in [EL1], [La1], [LB1], [LVV1], [LVV2], [Now1], [Now2], [NN1], [We1]. Unilateral boundary value problems for hemitropic elastic solids have been studied in [GGN1], while the contact problems were treated in [GaGaNa1] with the help of spatial variational inequality technique.

The main goal of the present paper is the study of frictionless contact problems for hemitropic elastic solids, their mathematical modelling as transmission-boundary value problems with *natural impenetrability conditions* and their analysis with the help of the *boundary variational inequality* technique based on properties of the corresponding potential operators. This approach reduces the dimension of the problem by one which is important for numerical realizations.

Similar unilateral problems of the classical linear elasticity theory with various modifications have been considered in many monographs and papers (see, e.g., [DuLi1], [Fi1], [Fi2], [GaNa1], [HHNL1], [KiOd1], [Ki1], [Rod1], and the references therein). More general problems, including the unilateral problems of nonlinear classical elasticity, are studied in [BBGT].

The work consists of five sections and is organized as follows. First, in Section 1, we collect the basic field equations of statics of the theory of elasticity for hemitropic media in vector and matrix forms, introduce the generalized stress operator and the potential energy quadratic form. Then, in Sections 2 and 3, we formulate the contact problem for two elastic homogeneous hemitropic continua with different elastic properties under the condition of natural impenetrability of one body into the other. We consider the coercive case when the interacting bodies are fixed along some parts of their boundaries. With the help of the potential method the problem is reduced equivalently to the boundary variational inequality. In Section 4, we present a detailed analysis of these inequalities and investigate existence and uniqueness of a weak solution of the original contact problem. Finally, in Section 5, we consider a semicoercive case when the contacting bodies are not fixed along their boundaries. In this case, the corresponding mathematical problem is not solvable, in general. We derive the necessary conditions of solvability and formulate also some sufficient conditions of solvability in explicit form.

1. Basic field equations

Let $\Omega \in \mathbb{R}^3$ be a bounded simply connected domain with a piecewise smooth boundary $S = \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. We assume that Ω is occupied by a homogeneous hemitropic elastic material. Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the *displacement vector* and the *micro-rotation vector*, respectively; here and in what follows the symbol $(\cdot)^\top$ denotes transposition.

In the hemitropic elasticity theory we have the following constitutive equations for the *force stress tensor* $\{\tau_{pq}\}$ and the *couple stress tensor* $\{\mu_{pq}\}$ [AK1], [NGS1]:

$$\tag{1.1} \begin{aligned}\tau_{pq} = \tau_{pq}(U) &:= (\mu+\alpha)\partial_p u_q + (\mu-\alpha)\partial_q u_p + \lambda\delta_{pq}\operatorname{div} u + \delta\delta_{pq}\operatorname{div}\omega \\ &\quad + (\varkappa+\nu)\partial_p\omega_q + (\varkappa-\nu)\partial_q\omega_p - 2\alpha\varepsilon_{pqk}\omega_k,\end{aligned}$$

$$\tag{1.2} \begin{aligned}\mu_{pq} = \mu_{pq}(U) &:= \delta\delta_{pq}\operatorname{div} u + (\varkappa+\nu)[\partial_p u_q - \varepsilon_{pqk}\omega_k] + \beta\delta_{pq}\operatorname{div}\omega \\ &\quad + (\varkappa-\nu)[\partial_q u_p - \varepsilon_{qpk}\omega_k] + (\gamma+\varepsilon)\partial_p\omega_q + (\gamma-\varepsilon)\partial_q\omega_p,\end{aligned}$$

where $U = (u, \omega)^\top$, δ_{pq} is the Kronecker delta, $\partial = (\partial_1, \partial_2, \partial_3)$ with $\partial_j = \partial/\partial x_j$, ε_{pqk} is the permutation (Levi-Civitá) symbol, and α, β, γ, δ, λ, μ, ν, $\varkappa$, and ε are the material constants. Throughout the paper summation over repeated indexes is meant from one to three if not otherwise stated.

The components of the force stress vector $\tau^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)}, \tau_3^{(n)})^\top$ and the couple stress vector $\mu^{(n)} = (\mu_1^{(n)}, \mu_2^{(n)}, \mu_3^{(n)})^\top$, acting on a surface element with a normal vector $n = (n_1, n_2, n_3)$, read as

$$\tag{1.3} \tau_q^{(n)} = \tau_{pq} n_p, \quad \mu_q^{(n)} = \mu_{pq} n_p, \quad q = 1, 2, 3.$$

Denote by $T(\partial, n)$ the generalized 6×6 matrix differential stress operator [NGS1]

$$\tag{1.4} T(\partial, n) = \begin{bmatrix} T_1(\partial, n) & T_2(\partial, n) \\ T_3(\partial, n) & T_4(\partial, n) \end{bmatrix}_{6\times 6}, \quad T_j = [T_{jpq}]_{3\times 3}, \quad j = \overline{1,4},$$

where

$$
(1.5) \qquad \begin{aligned}
T_{1pq}(\partial, n) &= (\mu+\alpha)\delta_{pq}\partial_n + (\mu-\alpha)n_q\partial_p + \lambda n_p\partial_q,\\
T_{2pq}(\partial, n) &= (\varkappa+\nu)\delta_{pq}\partial_n + (\varkappa-\nu)n_q\partial_p + \delta n_p\partial_q - 2\alpha\varepsilon_{pqk}n_k,\\
T_{3pq}(\partial, n) &= (\varkappa+\nu)\delta_{pq}\partial_n + (\varkappa-\nu)n_q\partial_p + \delta n_p\partial_q,\\
T_{4pq}(\partial, n) &= (\gamma+\varepsilon)\delta_{pq}\partial_n + (\gamma-\varepsilon)n_q\partial_p + \beta n_p\partial_q - 2\nu\varepsilon_{pqk}n_k.
\end{aligned}
$$

Here $\partial_n = \partial/\partial n$ denotes the usual normal derivative.

From formulas (1.1), (1.2) and (1.3) it can be easily checked that

$$
(\tau^{(n)}, \mu^{(n)})^\top = T(\partial, n)U.
$$

The equilibrium equations of statics in the theory of hemitropic elasticity read as [NGS1]

$$
\begin{aligned}
&\partial_p\tau_{pq}(x) + \varrho F_q(x) = 0, \quad q = 1,2,3,\\
&\partial_p\mu_{pq}(x) + \varepsilon_{qlr}\tau_{lr}(x) + \varrho M_q(x) = 0, \quad q = 1,2,3,
\end{aligned}
$$

where ϱ is the mass density of the elastic material, and $F = (F_1, F_2, F_3)^\top$ and $M = (M_1, M_2, M_3)^\top$ are the body force and body couple vectors.

Using the constitutive equations (1.1) and (1.2) we can rewrite the equilibrium equations in terms of the displacement and micro-rotation vectors,

$$
(1.6) \qquad \begin{aligned}
&(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,u(x) + (\varkappa+\nu)\Delta\omega(x)\\
&\qquad + (\delta+\varkappa-\nu)\,\mathrm{grad\,div}\,\omega(x) + 2\alpha\,\mathrm{curl}\,\omega(x) + \varrho F(x) = 0,\\
&(\varkappa+\nu)\Delta u(x) + (\delta+\varkappa-\nu)\,\mathrm{grad\,div}\,u(x) + 2\alpha\,\mathrm{curl}\,u(x)\\
&\qquad + (\gamma+\varepsilon)\Delta\omega(x) + (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\omega(x) + 4\nu\,\mathrm{curl}\,\omega(x)\\
&\qquad - 4\alpha\omega(x) + \varrho M(x) = 0,
\end{aligned}
$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator.

Let us introduce the matrix differential operator generated by the left hand side expressions of the system (1.6):

$$
(1.7) \qquad L(\partial) := \begin{bmatrix} L_1(\partial) & L_2(\partial)\\ L_3(\partial) & L_4(\partial) \end{bmatrix}_{6\times 6},
$$

where

$$
(1.8) \qquad \begin{aligned}
&L_1(\partial) := (\mu+\alpha)\Delta I_3 + (\lambda+\mu-\alpha)Q(\partial),\\
&L_2(\partial) = L_3(\partial) := (\varkappa+\nu)\Delta I_3 + (\delta+\varkappa-\nu)Q(\partial) + 2\alpha R(\partial),\\
&L_4(\partial) := [(\gamma+\varepsilon)\Delta - 4\alpha]I_3 + (\beta+\gamma-\varepsilon)Q(\partial) + 4\nu R(\partial).
\end{aligned}
$$

Here and in the sequel I_k stands for the $k \times k$ unit matrix and

$$
Q(\partial) := [\partial_k\partial_j]_{3\times 3}, \quad R(\partial) := [-\varepsilon_{kjl}\partial_l]_{3\times 3}.
$$

It is easy to see that $R(\partial)u = \operatorname{curl} u$ and $Q(\partial)u = \operatorname{grad}\operatorname{div} u$.

Equations (1.6) can be written in matrix form as

$$L(\partial)U(x) + \mathscr{G}(x) = 0 \quad \text{with } U := (u, \omega)^\top,\ \mathscr{G} := (\varrho F, \varrho M)^\top.$$

Note that the operator $L(\partial)$ is formally self-adjoint, i.e., $L(\partial) = [L(-\partial)]^\top$.

1.1. Green's formulas

For real-valued vector functions $U = (u, \omega)^\top$ and $U' = (u', \omega')^\top$ from the class $[C^2(\overline{\Omega})]^6$ the following Green formula holds [NGS1]

$$\int_\Omega [L(\partial)U \cdot U' + E(U, U')]\, dx = \int_S \{T(\partial, n)U\}^+ \cdot \{U'\}^+\, dS, \tag{1.9}$$

where $\{\cdot\}^+$ denotes the trace operator on S from Ω, while $E(\cdot\,,\cdot)$ is the bilinear form defined by the equality:

$$\begin{aligned} E(U, U') &= E(U', U) \\ &= \{(\mu + \alpha)u'_{pq}u_{pq} + (\mu - \alpha)u'_{pq}u_{qp} + (\varkappa + \nu)(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}) \\ &\quad + (\varkappa - \nu)(u'_{pq}\omega_{qp} + \omega'_{pq}u_{qp}) + (\gamma + \varepsilon)\omega'_{pq}\omega_{pq} + (\gamma - \varepsilon)\omega'_{pq}\omega_{qp} \\ &\quad + \delta(u'_{pp}\omega_{qq} + \omega'_{qq}u_{pp}) + \lambda u'_{pp}u_{qq} + \beta\omega'_{pp}\omega_{qq}\}, \end{aligned} \tag{1.10}$$

where u_{pq} and ω_{pq} are the so called *strain* and *torsion* (*curvature*) tensors for hemitropic bodies,

$$u_{pq} = \partial_p u_q - \varepsilon_{pqk}\omega_k, \quad \omega_{pq} = \partial_p \omega_q, \quad p, q = 1, 2, 3. \tag{1.11}$$

Here and in what follows the central dot $a \cdot b$ denotes the usual scalar product of two vectors $a, b \in \mathbb{R}^m$: $a \cdot b = a_j b_j$.

From formulas (1.10) and (1.11) we get

$$\begin{aligned} E(U, U') &= \frac{3\lambda + 2\mu}{3}\Big(\operatorname{div} u + \frac{3\delta + 2\varkappa}{3\lambda + 2\mu}\operatorname{div}\omega\Big)\Big(\operatorname{div} u' + \frac{3\delta + 2\varkappa}{3\lambda + 2\mu}\operatorname{div}\omega'\Big) \\ &\quad + \frac{1}{3}\Big(3\beta + 2\gamma - \frac{(3\delta + 2\varkappa)^2}{3\lambda + 2\mu}\Big)(\operatorname{div}\omega)(\operatorname{div}\omega') \\ &\quad + \Big(\varepsilon - \frac{\nu^2}{\alpha}\Big)\operatorname{curl}\omega \cdot \operatorname{curl}\omega' \\ &\quad + \alpha\Big(\operatorname{curl} u + \frac{\nu}{\alpha}\operatorname{curl}\omega - 2\omega\Big) \cdot \Big(\operatorname{curl} u' + \frac{\nu}{\alpha}\operatorname{curl}\omega' - 2\omega'\Big) \end{aligned} \tag{1.12}$$

$$
\begin{aligned}
&+\frac{\mu}{2}\sum_{k,j=1,k\neq j}^{3}\Big[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\Big)\Big]\\
&\times\Big[\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial \omega'_k}{\partial x_j}+\frac{\partial \omega'_j}{\partial x_k}\Big)\Big]\\
&+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\Big)\Big]\\
&\times\Big[\frac{\partial u'_k}{\partial x_k}-\frac{\partial u'_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial \omega'_k}{\partial x_k}-\frac{\partial \omega'_j}{\partial x_j}\Big)\Big]\\
&+\Big(\gamma-\frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,k\neq j}^{3}\Big[\frac{1}{2}\Big(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\Big)\Big(\frac{\partial \omega'_k}{\partial x_j}+\frac{\partial \omega'_j}{\partial x_k}\Big)\\
&\qquad\qquad+\frac{1}{3}\Big(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\Big)\Big(\frac{\partial \omega'_k}{\partial x_k}-\frac{\partial \omega'_j}{\partial x_j}\Big)\Big].
\end{aligned}
$$

The potential energy density function $E(U,U)$ is a positive definite quadratic form with respect to the variables u_{pq} and ω_{pq}, i.e., there exists a positive constant $c_0>0$ depending only on the material parameters, such that

$$
E(U,U)\geq c_0\sum_{p,q=1}^{3}[u_{pq}^2+\omega_{pq}^2]. \tag{1.13}
$$

The necessary and sufficient conditions for the quadratic form $E(U,U)$ to be positive definite are the following inequalities (see [AK2], [Dy1], [GGN1])

$$
\begin{gathered}
\mu>0,\quad \alpha>0,\quad \gamma>0,\quad \varepsilon>0,\quad \lambda+2\mu>0,\quad \mu\gamma-\varkappa^2>0,\quad \alpha\varepsilon-\nu^2>0,\\
(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2>0,\quad (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0,\\
\mu[(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2]+(\lambda+\mu)(\mu\gamma-\varkappa^2)>0,\\
\mu[(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2]+(3\lambda+2\mu)(\mu\gamma-\varkappa^2)>0.
\end{gathered}
$$

Let us note that, if the condition $3\lambda+2\mu>0$ is fulfilled, which is very natural in the classical elasticity, then the above conditions are equivalent to the following simultaneous inequalities

$$
\begin{gathered}
\mu>0,\quad \alpha>0,\quad \gamma>0,\quad \varepsilon>0,\quad 3\lambda+2\mu>0,\quad \mu\gamma-\varkappa^2>0,\\
\alpha\varepsilon-\nu^2>0,\quad (\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\\
(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0.
\end{gathered} \tag{1.14}
$$

The following assertion describes the null space of the energy quadratic form $E(U,U)$ (see [NGS1]).

LEMMA 1.1. *Let* $U = (u, \omega)^\top \in [C^1(\overline{\Omega})]^6$ *and* $E(U, U) = 0$ *in* Ω. *Then*

$$u(x) = [a \times x] + b, \quad \omega(x) = a, \quad x \in \Omega,$$

where a and b are arbitrary three-dimensional constant vectors and symbol $[\cdot \times \cdot]$ *denotes the cross product of two vectors.*

Vectors of type $([a \times x] + b, a)$ are called *generalized rigid displacement vectors*. Note that a generalized rigid displacement vector vanishes identically if it is zero at a single point.

Throughout the paper $L_p(\Omega)$ with $1 \le p < \infty$ and $H^s(\Omega) = H_2^s(\Omega)$ with $s \in \mathbb{R}$ denote Lebesgue and Bessel potential spaces (see, e.g., [LiMa1], [Tr1]). The corresponding norms we denote by symbols $\| \cdot \|_{L_p(\Omega)}$ and $\| \cdot \|_{H^s(\Omega)}$. Denote by $\mathscr{D}(\Omega)$ the class of $C^\infty(\Omega)$ functions with support in the domain Ω. If S^* is an open proper part of the manifold $\partial\Omega$, i.e., $S^* \subset \partial\Omega$, $S^* \ne \partial\Omega$, then by $H^s(S^*)$ we denote the restriction of the space $H^s(\partial\Omega)$ onto S^*,

$$H^s(S^*) := \{ r_{S^*}\varphi : \varphi \in H^s(\partial\Omega) \},$$

where r_{S^*} denotes the restriction operator onto the set S^*. Further, let

$$\tilde{H}^s(S^*) := \{ \varphi \in H^s(\partial\Omega) : \operatorname{supp}\varphi \subset \overline{S^*} \}.$$

From the positive definiteness of the energy form $E(\cdot\,,\cdot)$ with respect to the variables (1.11) (see (1.13)) it follows that

$$B(U, U) := \int_\Omega E(U, U)\, dx \ge 0. \tag{1.15}$$

Moreover, there exist positive constants c_1 and c_2, depending only on the material parameters, such that the inequality

$$B(U, U) \ge c_1 \int_\Omega \left\{ \sum_{p,q=1}^{3} [(\partial_p u_q)^2 + (\partial_p \omega_q)^2] + \sum_{q=1}^{3} [u_q^2 + \omega_q^2] \right\} dx$$
$$- c_2 \int_\Omega \sum_{q=1}^{3} [u_q^2 + \omega_q^2]\, dx$$

holds for an arbitrary real-valued vector function $U \in [C^1(\overline{\Omega})]^6$. By standard limiting arguments we easily conclude that for any $U \in [H^1(\Omega)]^6$ the following Korn's type inequality holds (cf. [Fi1], Part I, §12)

$$B(U, U) \ge c_1 \|U\|^2_{[H^1(\Omega)]^6} - c_2 \|U\|^2_{[L_2(\Omega)]^6}. \tag{1.16}$$

REMARK 1.1. If $U \in [H^1(\Omega)]^6$ and the trace $\{U\}^+$ vanishes on some open subsurface S^* of the boundary $\partial\Omega$, i.e., $r_{S^*}\{U\}^+ = 0$, then we have the strict Korn's inequality

$$B(U, U) \geq c\|U\|^2_{[H^1(\Omega)]^6}$$

with some positive constant $c > 0$ which does not depend on the vector U. The constant c depends on the material parameters and on the geometry of the domain Ω. This follows from (1.15), (1.16) and the fact that in this case $B(U, U) > 0$ for $U \neq 0$ (see, e.g., [Ne1], [Mc1], Ch. 2, Exercise 2.17).

Remark 1.2. By standard limiting arguments Green's formula (1.9) can be extended to Lipschitz domains and to vector functions $U \in [H^1(\Omega)]^6$ with $L(\partial)U \in [L_2(\Omega)]^6$ and $U' \in [H^1(\Omega)]^6$ (see, [Ne1], [LiMa1]),

$$\int_{\Omega} [L(\partial)U \cdot U' + E(U, U')]\, dx = \langle \{T(\partial, n)U\}^+, \{U'\}^+ \rangle_{\partial\Omega}, \tag{1.17}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality between the spaces $[H^{-1/2}(\partial\Omega)]^6$ and $[H^{1/2}(\partial\Omega)]^6$, which generalizes the usual $[L_2(\partial\Omega)]^6$ inner product. By this relation the generalized trace of the stress operator $\{T(\partial, n)U\}^+$ on the boundary $\partial\Omega$ is correctly determined and $\{T(\partial, n)U\}^+ \in [H^{-1/2}(\partial\Omega)]^6$. Note that for arbitrary real valued vector functions $V, V' \in [L_2(\partial\Omega)]^6$ we have

$$\langle V, V' \rangle_{\partial\Omega} = \int_{\partial\Omega} V \cdot V'\, dS.$$

2. Statement of the problems and uniqueness results

Let $\Omega_q \in \mathbb{R}^3$, $q = 1, 2$, be simply connected bounded Lipschitz domains with piecewise smooth, simply connected boundaries $S_q := \partial\Omega_q$. Further, let Ω_1 and Ω_2 be filled with hemitropic materials possessing different elastic properties. The elastic constants corresponding to the elastic medium occupying the domain Ω_q are denoted by $\alpha^{(q)}$, $\beta^{(q)}$, $\gamma^{(q)}$, $\delta^{(q)}$, $\lambda^{(q)}$, $\mu^{(q)}$, $\nu^{(q)}$, $\varkappa^{(q)}$, and $\varepsilon^{(q)}$, $q = 1, 2$. Analogously, $u^{(q)} = (u_1^{(q)}, u_2^{(q)}, u_3^{(q)})^\top$ and $\omega^{(q)} = (\omega_1^{(q)}, \omega_2^{(q)}, \omega_3^{(q)})^\top$ denote the displacement and micro-rotation vectors in the domain Ω_q, $E^{(q)}(U^{(q)}, U^{(q)})$ designates the corresponding potential energy density, $L^{(q)}(\partial)$ and $T^{(q)}(\partial, n^{(q)})$ are the corresponding differential operators given by formulas (1.7), (1.8) and (1.4), (1.5) respectively.

Let the boundaries $S_q := \partial\Omega_q$ fall into three mutually disjoint open portions S_q^D, S_q^N and S_c, such that $\overline{S_q^D} \cup \overline{S_q^N} \cup \overline{S}_c = S_q$, $\overline{S_q^D} \cap \overline{S}_c = \emptyset$ and $S_c \in C^{2,\alpha'}$, $\alpha' \in (0; 1)$. Denote by $n^{(q)}(x)$ the unit, outward with respect to Ω_q, normal at the point $x \in S_q$ (see Fig. 1). We assume that the elastic hemitropic solids occupying the domains Ω_1 and Ω_2 are fixed along the subsurfaces S_1^D and S_2^D, along the subsurfaces S_1^N and S_2^N there are prescribed some stresses, while the two bodies under consideration are in frictionless contact along the subsurface $\overline{S}_c := S_1 \cap S_2$.

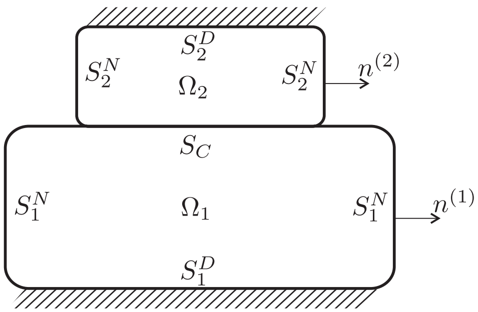

Figure 1. Geometry of the contacting solids

Below we describe mathematically this contact problem by means of the usual mixed Dirichlet-Neumann type boundary conditions on the sub-manifolds S_q^D and S_q^N, while on S_c the frictionless contact is modeled with the help of the so called *natural non-penetration conditions*.

2.1. Formulation of the problem

Consider the equation in the domain Ω_q, $q = 1, 2$,

$$(2.1)\quad L^{(q)}(\partial)U^{(q)} + \mathscr{G}^{(q)} = 0, \quad \mathscr{G}^{(q)} := (\varrho^{(q)}F^{(q)}, \varrho^{(q)}M^{(q)})^\top \in [L_2(\Omega_q)]^6,$$

where $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top$ are the unknown vectors, $L^{(q)}(\partial)$ is the matrix differential operator given by formulas (1.7) and (1.8), $\varrho^{(q)}$ are the mass densities of the elastic materials under consideration, $F^{(q)} = (F_1^{(q)}, F_2^{(q)}, F_3^{(q)})^\top$ and $M^{(q)} = (M_1^{(q)}, M_2^{(q)}, M_3^{(q)})^\top$ are the corresponding body force and body couple vectors. In the sequel, we will be concerned with weak solutions of the corresponding differential equations. By definition, the vector function $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top \in [H^1(\Omega_q)]^6$ is called a weak solution of equation (2.1) in the domain Ω_q, if for every $\Phi \in [\mathscr{D}(\Omega_q)]^6$

$$B^{(q)}(U^{(q)}, \Phi) = \int_{\Omega_q} \mathscr{G}^{(q)} \cdot \Phi \, dx, \quad q = 1, 2,$$

where the bilinear form $B^{(q)}(U^{(q)}, \Phi)$ is defined by the formula

$$B^{(q)}(U^{(q)}, \Phi) := \int_{\Omega_q} E^{(q)}(U^{(q)}, \Phi) \, dx$$

with $E^{(q)}(U^{(q)}, \Phi)$ given by (1.12).

Below, for the force and couple stress vectors we use the notation

$$\mathscr{T}^{(q)}U^{(q)} = T_1^{(q)}u^{(q)} + T_2^{(q)}\omega^{(q)}, \quad \mathscr{M}^{(q)}U^{(q)} = T_3^{(q)}u^{(q)} + T_4^{(q)}\omega^{(q)},$$

where the boundary operators $T_k^{(l)}$, $k = 1, 2, 3, 4$, $l = 1, 2$, are given by formulas (1.5). For the normal and tangential components of the force stress vector we will use, respectively, the following notation

$$(\mathscr{T}^{(q)} U^{(q)})_{n^{(q)}} := (\mathscr{T}^{(q)} U^{(q)}) \cdot n^{(q)},$$
$$(\mathscr{T}^{(q)} U^{(q)})_t := \mathscr{T}^{(q)} U^{(q)} - n^{(q)} (\mathscr{T}^{(q)} U^{(q)})_{n^{(q)}}.$$

Let

$$\Psi^{(q)} = (\Psi_1^{(q)}, \Psi_2^{(q)})^\top \in [\tilde{H}^{-1/2}(S_q^N)]^6,$$
$$\Psi_l^{(q)} = (\Psi_{l1}^{(q)}, \Psi_{l2}^{(q)}, \Psi_{l3}^{(q)})^\top \in [\tilde{H}^{-1/2}(S_q^N)]^3, \quad q, l = 1, 2,$$

and consider the following boundary-contact problem.

Problem (A): Find vector functions $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top \in [H^1(\Omega_q)]^6$, $q = 1, 2$, which are weak solutions of equations (2.1) and satisfy:

(i) the Dirichlet type conditions

$$r_{S_q^D} \{U^{(q)}\}^+ = 0 \quad \text{on } S_q^D, \quad q = 1, 2;$$

(ii) the Neumann type conditions

$$r_{S_q^N} \{T^{(q)}(\partial, n^{(q)}) U^{(q)}\}^+ = r_{S_q^N} \Psi^{(q)} \quad \text{on } S_q^N, \quad q = 1, 2;$$

(iii) the frictionless non-penetration conditions on the contact subsurface S_c

$$r_{S_c} \{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+ \le 0 \quad \text{on } S_c,$$
$$r_{S_c} \{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+ = r_{S_c} \{(\mathscr{T}^{(2)} U^{(2)})_{n^{(2)}}\}^+ \le 0 \quad \text{on } S_c,$$
$$\langle r_{S_c} \{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+, r_{S_c} \{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+ \rangle_{S_c} = 0 \quad \text{on } S_c,$$
$$r_{S_c} \{(\mathscr{T}^{(q)} U^{(q)})_t\}^+ = 0 \quad \text{on } S_c, \quad q = 1, 2,$$
$$r_{S_c} \{\mathscr{M}^{(q)} U^{(q)}\}^+ = 0 \quad \text{on } S_c, \quad q = 1, 2.$$

To reduce this problem to a boundary variational inequality we need first to reduce the nonhomogeneous equation (2.1) to the homogeneous one. To this purpose consider the following auxiliary mixed boundary value problem: Find a vector-function $U_0^{(q)} = (u_0^{(q)}, \omega_0^{(q)})^\top \in [H^1(\Omega_q)]^6$ which is a weak solution of equation (2.1) and satisfies the following mixed boundary conditions:

$$r_{S_q^D} \{U_0^{(q)}\}^+ = 0, \quad r_{S_q \setminus \overline{S_q^D}} \{T^{(q)}(\partial, n^{(q)}) U_0^{(q)}\}^+ = 0. \tag{2.2}$$

This problem possesses a unique solution (see [NGS1]). Clearly, if $W^{(q)} \in [H^1(\Omega_q)]^6$ is a solution of the Problem (A) and $U_0^{(q)} = (u_0^{(q)}, \omega_0^{(q)})^\top \in [H^1(\Omega_q)]^6$

is a solution of the above auxiliary mixed boundary value problem (2.2), then the difference $U^{(q)} = W^{(q)} - U_0^{(q)}$ will solve the following problem.

Problem (A_0): Find vector functions $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top \in [H^1(\Omega_q)]^6$, $q = 1, 2$, which are weak solutions of the equations

$$L^{(q)}(\partial)U^{(q)} = 0 \quad \text{in } \Omega_q \tag{2.3}$$

and satisfy the following boundary and contact conditions:

$$r_{S_q^D}\{U^{(q)}\}^+ = 0 \quad \text{on } S_q^D, \quad q = 1, 2, \tag{2.4}$$

$$r_{S_q^N}\{T^{(q)}(\partial, n^{(q)})U^{(q)}\}^+ = r_{S_q^N}\Psi^{(q)} \quad \text{on } S_q^N, \quad q = 1, 2, \tag{2.5}$$

$$r_{S_c}\{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+ \le \varphi_0 \quad \text{on } S_c, \tag{2.6}$$

$$r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+ = r_{S_c}\{(\mathscr{T}^{(2)}U^{(2)})_{n^{(2)}}\}^+ \le 0 \quad \text{on } S_c, \tag{2.7}$$

$$\langle r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+, \quad r_{S_c}\{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+ - \varphi_0\rangle_{S_c} = 0 \quad \text{on } S_c, \tag{2.8}$$

$$r_{S_c}\{(\mathscr{T}^{(q)}U^{(q)})_t\}^+ = 0 \quad \text{on } S_c, \quad q = 1, 2, \tag{2.9}$$

$$r_{S_c}\{\mathscr{M}^{(q)}U^{(q)}\}^+ = 0 \quad \text{on } S_c, \quad q = 1, 2, \tag{2.10}$$

where $\Psi^{(q)}$ is the same as in the formulation of Problem (A) and

$$\varphi_0 = -r_{S_c}\{u_0^{(1)} \cdot n^{(1)} + u_0^{(2)} \cdot n^{(2)}\}^+. \tag{2.11}$$

Below we will investigate the Problem (A_0). Clearly, if a pair $(U^{(1)}, U^{(2)})^\top$ solves the Problem (A_0), the sum $(W^{(1)}, W^{(2)})^\top := (U^{(1)} + U_0^{(1)}, U^{(2)} + U_0^{(2)})^\top$ solves then the Problem (A).

2.2. *Uniqueness theorem*

Here we prove the following uniqueness theorem.

Theorem 2.1. *Problem (A_0) has at most one solution.*

Proof. Let $U = (U^{(1)}, U^{(2)})^\top$ with $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top$ and $W = (W^{(1)}, W^{(2)})^\top$ with $W^{(q)} = (v^{(q)}, w^{(q)})^\top$ be two distinct solutions of Problem (A_0). Then the difference $\tilde{U} = (\tilde{U}^{(1)}, \tilde{U}^{(2)})^\top := U - W$ will satisfy the conditions (2.3), (2.4), (2.9), (2.10), and the condition (2.5) with $\Psi^{(q)} = 0$. From condition (2.7) we have

$$r_{S_c}\{(\mathscr{T}^{(1)}\tilde{U}^{(1)})_{n^{(1)}}\}^+ = r_{S_c}\{(\mathscr{T}^{(2)}\tilde{U}^{(2)})_{n^{(2)}}\}^+.$$

Using Green's formula (1.17) and taking into account the above conditions, we have

$$\begin{aligned}
\sum_{q=1}^{2}\int_{\Omega_q} & E^{(q)}(\tilde U^{(q)},\tilde U^{(q)})\,dx \\
&= \sum_{q=1}^{2}\langle\{T^{(q)}(\partial,n^{(q)})\tilde U^{(q)}\}^+,\{\tilde U^{(q)}\}^+\rangle_{S_q} \\
&= \sum_{q=1}^{2}\langle r_{S_c}\{(\mathscr{T}^{(q)}\tilde U^{(q)})_{n^{(q)}}\}^+, r_{S_c}\{\tilde u^{(q)}\cdot n^{(q)}\}^+\rangle_{S_c} \\
&= \langle r_{S_c}\{(\mathscr{T}^{(1)}\tilde U^{(1)})_{n^{(1)}}\}^+, r_{S_c}\{\tilde u^{(1)}\cdot n^{(1)}+\tilde u^{(2)}\cdot n^{(2)}\}^+\rangle_{S_c} \\
&= \langle r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+ - r_{S_c}\{(\mathscr{T}^{(1)}W^{(1)})_{n^{(1)}}\}^+, \\
&\qquad r_{S_c}\{u^{(1)}\cdot n^{(1)} - v^{(1)}\cdot n^{(1)} + u^{(2)}\cdot n^{(2)} - v^{(2)}\cdot n^{(2)}\}^+\rangle_{S_c} \\
&= \langle r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+ - r_{S_c}\{(\mathscr{T}^{(1)}W^{(1)})_{n^{(1)}}\}^+, \\
&\qquad r_{S_c}\{u^{(1)}\cdot n^{(1)} + u^{(2)}\cdot n^{(2)}\}^+ - \varphi_0 \\
&\qquad - r_{S_c}\{v^{(1)}\cdot n^{(1)} + v^{(2)}\cdot n^{(2)}\}^+ + \varphi_0\rangle_{S_c} \\
&= -\langle r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+, r_{S_c}\{v^{(1)}\cdot n^{(1)} + v^{(2)}\cdot n^{(2)}\}^+ - \varphi_0\rangle_{S_c} \\
&\qquad - \langle r_{S_c}\{(\mathscr{T}^{(1)}W^{(1)})_{n^{(1)}}\}^+, r_{S_c}\{u^{(1)}\cdot n^{(1)} + u^{(2)}\cdot n^{(2)}\}^+ - \varphi_0\rangle_{S_c} \le 0.
\end{aligned}$$

Bearing in mind that the quadratic form $E^{(q)}(\tilde U^{(q)},\tilde U^{(q)})$ is positive definite (see (1.13)), we have

$$E^{(q)}(\tilde U^{(q)},\tilde U^{(q)}) = 0, \quad q = 1,2.$$

By Lemma 1.1

$$\tilde U^{(q)} = ([a^{(q)}\times x] + b^{(q)}, a^{(q)})^\top, \quad q = 1,2.$$

Since $r_{S_q^D}\{\tilde U^{(q)}\}^+ = 0$ we conclude $a^{(q)} = b^{(q)} = 0$. Thus $\tilde U^{(q)} = 0$, $q = 1,2$. □

2.3. Reduction of Problem (A_0) to the boundary variational inequality

To reduce Problem (A_0) to the boundary variational inequality equivalently we recall that a solution vector $U^{(q)} = (u^{(q)},\omega^{(q)})^\top \in [H^1(\Omega_q)]^6$ to equation (2.3) satisfying the Dirichlet boundary condition

$$\{U^{(q)}\}^+ = h^{(q)}$$

with $h^{(q)} \in [H^{1/2}(S_q)]^6$, can be uniquely represented as a single layer potential (see [NGS1])

$$U^{(q)}(x) = V^{(q)}([\mathscr{H}^{(q)}]^{-1}h^{(q)})(x) = \int_{S_q}\Gamma^{(q)}(x-y)([\mathscr{H}^{(q)}]^{-1}h^{(q)})(y)d_yS,$$

where $\Gamma^{(q)}$ is the matrix of fundamental solutions of the operator $L^{(q)}(\partial)$ and $\mathscr{H}^{(q)}$ is the boundary integral operator generated by the single layer potential on the boundary S_q (see the explicit expression for $\Gamma^{(q)}$ in [GGN1], [NGS1]):

$$\mathscr{H}^{(q)}h^{(q)}(x) = \lim_{\Omega_q \ni z \to x \in S_q} \int_{S_q} \Gamma^{(q)}(z-y)h^{(q)}(y)d_yS = \{V^{(q)}(h^{(q)})\}^+.$$

Note that the single layer potential operator $V^{(q)}$ and the integral operator $\mathscr{H}^{(q)}$ have the following mapping properties (see [NGS1]):

$$\begin{aligned} V^{(q)} &: [H^{-1/2}(S_q)]^6 \to [H^1(\Omega_q)]^6, \\ \mathscr{H}^{(q)} &: [H^{-1/2}(S_q)]^6 \to [H^{1/2}(S_q)]^6. \end{aligned} \tag{2.12}$$

These operators are continuous. Moreover, the operator $\mathscr{H}^{(q)}$ is continuously invertible and

$$[\mathscr{H}^{(q)}]^{-1} : [H^{1/2}(S_q)]^6 \to [H^{-1/2}(S_q)]^6. \tag{2.13}$$

For arbitrary $h^{(q)} \in [H^{-1/2}(S_q)]^6$ there hold the following jump relations

$$\{T^{(q)}(\partial, n^{(q)})V^{(q)}(h^{(q)})\}^+ = (-2^{-1}I_6 + \mathscr{K}^{(q)})h^{(q)} \quad \text{on } S_q, \quad q = 1,2, \tag{2.14}$$

where $\mathscr{K}^{(q)}$ is a singular integral operator,

$$\mathscr{K}^{(q)}h^{(q)}(x) := \int_{S_q} [T^{(q)}(\partial, n^{(q)})\Gamma^{(q)}(x-y)]h^{(q)}(y)d_yS.$$

Note that the operator

$$-2^{-1}I_6 + \mathscr{K}^{(q)} : [H^{-1/2}(S_q)]^6 \to [H^{-1/2}(S_q)]^6 \tag{2.15}$$

is a continuous singular integral operator of normal type with zero index (for details see [NGS1]).

Further, let us introduce the so-called *Green's operator for the Dirichlet problem*

$$G^{(q)} : [H^{1/2}(S_q)]^6 \to [H^1(\Omega_q)]^6$$

defined by the formula

$$G^{(q)}h^{(q)} = V^{(q)}([\mathscr{H}^{(q)}]^{-1}h^{(q)}). \tag{2.16}$$

It is clear that $L^{(q)}(\partial)(G^{(q)}h^{(q)}) = 0$ in Ω_q and $\{G^{(q)}h^{(q)}\}^+ = h^{(q)}$ on S_q. From the properties of the trace operator and mapping properties of the single layer potential operator it follows that there exist positive numbers C_1 and C_2 such that for all $h^{(q)} \in [H^{1/2}(S_q)]^6$

$$C_1\|h^{(q)}\|_{[H^{1/2}(S_q)]^6} \le \|G^{(q)}h^{(q)}\|_{[H^1(\Omega_q)]^6} \le C_2\|h^{(q)}\|_{[H^{1/2}(S_q)]^6}. \tag{2.17}$$

Now we introduce the generalized Steklov-Poincaré type operator

$$
\begin{aligned}
(2.18) \qquad \mathscr{A}^{(q)} h^{(q)} &:= \{T^{(q)}(\partial, n^{(q)})(G^{(q)} h^{(q)})\}^+ \\
&= \{T^{(q)}(\partial, n^{(q)}) V^{(q)}([\mathscr{H}^{(q)}]^{-1} h^{(q)})\}^+ \\
&= (-2^{-1} I_6 + \mathscr{K}^{(q)})[\mathscr{H}^{(q)}]^{-1} h^{(q)}.
\end{aligned}
$$

Denote by $\Lambda^{(q)}(S_q)$ the set of restrictions on S_q of rigid displacement vectors,

$$
(2.19) \qquad \Lambda^{(q)}(S_q) = \{\chi^{(q)}(x) = ([a^{(q)} \times x] + b^{(q)}, a^{(q)})^\top, x \in S_q\},
$$

where $a^{(q)}$ and $b^{(q)}$ are arbitrary three-dimensional constant vectors. With the help of Green's formula (1.17) with $\Omega = \Omega_q$ and $U = U' = U^{(q)} = V^{(q)}([\mathscr{H}^{(q)}]^{-1} h^{(q)})$, the relations (2.14), (2.18), (2.19) and the uniqueness theorem for the Dirichlet BVP, we infer that $\ker \mathscr{A}^{(q)} = \Lambda^{(q)}(S_q)$. The properties of the Steklov-Poincaré operator is described by the following lemma.

LEMMA 2.1. *Let $h, \eta \in [H^{1/2}(S_q)]^6$ and $g \in [\tilde{H}^{1/2}(S_q^*)]^6$, where S_q^* is an open proper part of the boundary $S_q = \partial\Omega_q$. Then*

(a) $\langle \mathscr{A}^{(q)} h, \eta \rangle_{S_q} = \langle \mathscr{A}^{(q)} \eta, h \rangle_{S_q}$,

(b) $\mathscr{A}^{(q)} : [H^{1/2}(S_q)]^6 \to [H^{-1/2}(S_q)]^6$ *is a continuous operator*,

(c) $\langle \mathscr{A}^{(q)} h, h \rangle_{S_q} \geq C_1 \|h\|^2_{[H^{1/2}(S_q)]^6} - C_2 \|h\|^2_{[L_2(S_q)]^6}$,

(d) $\langle \mathscr{A}^{(q)} g, g \rangle_{S_q} \geq C \|g\|^2_{[H^{1/2}(S_q)]^6}$,

(e) $\langle \mathscr{A}^{(q)} h, h \rangle_{S_q} \geq C \|h - Ph\|^2_{[H^{1/2}(S_q)]^6}$,

where P is the orthogonal projection (in the sense of $L_2(S_q)$) of the space $[H^{1/2}(S_q)]^6$ onto the space $\Lambda^{(q)}(S_q)$; the positive constants C, C_1 and C_2 depend on the material parameters and on the geometry of the surface S_q and do not depend on h and g.

PROOF. Let $h, \eta \in [H^{1/2}(S_q)]^6$. Taking into account that the vector $G^{(q)} h$ solves the homogeneous equation $L^{(q)}(\partial) G^{(q)} h = 0$, from Green's formula (1.17) we get:

$$
\begin{aligned}
\langle \mathscr{A}^{(q)} h, \eta \rangle_{S_q} &= \langle \{T^{(q)}(\partial, n^{(q)})(G^{(q)} h)\}^+, \{G^{(q)} \eta\}^+ \rangle_{S_q} \\
&= B^{(q)}(G^{(q)} h, G^{(q)} \eta) = B^{(q)}(G^{(q)} \eta, G^{(q)} h) \\
&= \langle \{T^{(q)}(\partial, n^{(q)})(G^{(q)} \eta)\}^+, \{G^{(q)} h\}^+ \rangle_{S_q} = \langle \mathscr{A}^{(q)} \eta, h \rangle_{S_q},
\end{aligned}
$$

whence the item (a) follows.

The item (b) is evident, since $\mathscr{A}^{(q)}$ is the composition of the continuous operators $[\mathscr{H}^{(q)}]^{-1}$ and $-2^{-1} I_6 + \mathscr{K}^{(q)}$ (see (2.14) and (2.15)).

To prove the item (c) we proceed as follows. For arbitrary $h \in [H^{1/2}(S_q)]^6$ with the help of (1.16) we derive

$$\begin{aligned}\langle \mathscr{A}^{(q)} h, h \rangle_{S_q} &= B^{(q)}(V^{(q)}([\mathscr{H}^{(q)}]^{-1}h), V^{(q)}([\mathscr{H}^{(q)}]^{-1}h)) \\ &\geq c_1 \| V^{(q)}([\mathscr{H}^{(q)}]^{-1}h) \|^2_{[H^1(\Omega_q)]^6} - c_2 \| V^{(q)}([\mathscr{H}^{(q)}]^{-1}h) \|^2_{[L_2(\Omega_q)]^6}.\end{aligned}$$

By formulas (2.16) and (2.17) we have

$$\| V^{(q)}([\mathscr{H}^{(q)}]^{-1}h) \|_{[H^1(\Omega_q)]^6} \geq C_1 \|h\|_{[H^{1/2}(S_q)]^6}.$$

On the other hand, since $[L_2(S_q)]^6$ is compactly embedded into $[H^{-1/2}(S_q)]^6$, by virtue of continuity of the operators (2.12) and (2.13) we have:

$$\begin{aligned}\| V^{(q)}([\mathscr{H}^{(q)}]^{-1}h) \|_{[L_2(\Omega_q)]^6} &\leq C_1^* \| [\mathscr{H}^{(q)}]^{-1} h \|_{[H^{-3/2}(S_q)]^6} \\ &\leq C_2^* \|h\|_{[H^{-1/2}(S_q)]^6} \leq C_3^* \|h\|_{[L_2(S_q)]^6}\end{aligned}$$

with some positive constants C_1^*, C_2^* and C_3^* independent of h. So, finally we obtain that

$$\langle \mathscr{A}^{(q)} h, h \rangle_{S_q} \geq c_1 C_1^2 \|h\|^2_{[H^{1/2}(S_q)]^6} - c_2 (C_3^*)^2 \|h\|^2_{[L_2(S_q)]^6},$$

which proves the item (c).

Now the item (e) follows from (c) and the nonnegativity property of the operator $\mathscr{A}^{(q)}$, while (d) follows from (e). The lemma is proved. □

Further, we introduce the vector function space $\mathbb{H}^l$ and the convex set of vector functions $\mathbb{K}_{\varphi_0}$:

$$\begin{aligned}&\mathbb{H}^l = \{ g = (g^{(1)}, g^{(2)})^\top : g^{(q)} = (\varphi^{(q)}, \psi^{(q)})^\top \in [H^l(S_q)]^6, q = 1, 2 \}, \quad l \in \mathbb{R}, \\ &\mathbb{K}_{\varphi_0} = \{ g \in \mathbb{H}^{1/2} : r_{S_q^D} g^{(q)} = 0, r_{S_c} \{ \varphi^{(1)} \cdot n^{(1)} + \varphi^{(2)} \cdot n^{(2)} \}^+ \leq \varphi_0 \},\end{aligned}$$

where φ_0 is defined by formula (2.11). It is clear that the set $\mathbb{K}_{\varphi_0}$ is convex and closed in the space $\mathbb{H}^{1/2}$. On the convex closed set $\mathbb{K}_{\varphi_0}$ we consider the following variational inequality: Find $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathbb{K}_{\varphi_0}$, such that the boundary variational inequality

$$\sum_{q=1}^{2} \langle \mathscr{A}^{(q)} h_0^{(q)}, h^{(q)} - h_0^{(q)} \rangle_{S_q} \geq \sum_{q=1}^{2} \langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} (h^{(q)} - h_0^{(q)}) \rangle_{S_q^N} \tag{2.20}$$

is fulfilled for all $h = (h^{(1)}, h^{(2)})^\top \in \mathbb{K}_{\varphi_0}$.

3. Equivalence of problem (A_0) and variational inequality (2.20)

In this section we prove the equivalence of the boundary variational inequality (2.20) and contact problem (A_0).

THEOREM 3.1. *If* $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathbb{K}_{\varphi_0}$ *is a solution of the boundary variational inequality* (2.20)*, then* $U = (U^{(1)}, U^{(2)})^\top$ *with* $U^{(q)} = G^{(q)} h_0^{(q)} \in [H^1(\Omega_q)]^6$ *is a solution of Problem* (A_0)*, and vice versa, if* $U = (U^{(1)}, U^{(2)})^\top$ *is a solution of Problem* (A_0)*, then* $h_0 = (h_0^{(1)}, h_0^{(2)})^\top$ *with* $h_0^{(q)} = \{U^{(q)}\}^+$ *is a solution of the boundary variational inequality* (2.20).

PROOF. Let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathbb{K}_{\varphi_0}$ with $h_0^{(q)} = (\vartheta_0^{(q)}, \psi_0^{(q)})^\top$ be a solution of the variational inequality (2.20). We can show that then the vector-function $U = (U^{(1)}, U^{(2)})^\top$, where $U^{(q)} = G^{(q)} h_0^{(q)}$, is a solution of Problem (A_0). Here the Green operator $G^{(q)}$ is defined by formula (2.16). Taking into account the definition of the operator $G^{(q)}$ it is not difficult to show that the conditions (2.3), (2.4), and (2.6) are fulfilled. Let

$$f = (f^{(1)}, f^{(2)})^\top \in [\tilde{H}^{1/2}(S_1^N)]^6 \times [\tilde{H}^{1/2}(S_2^N)]^6$$

be an arbitrary vector and substitute $h_0 \pm f$ instead of h in (2.20) to obtain:

$$\sum_{q=1}^{2} \langle r_{S_q^N} \mathscr{A}^{(q)} h_0^{(q)}, r_{S_q^N} f^{(q)} \rangle_{S_q^N} = \sum_{q=1}^{2} \langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} f^{(q)} \rangle_{S_q^N}.$$

Whence the equallity

$$r_{S_q^N} \mathscr{A}^{(q)} h_0^{(q)} = r_{S_q^N} \{T^{(q)}(\partial, n^{(q)})(G^{(q)} h_0^{(q)})\}^+ = r_{S_q^N} \Psi^{(q)}$$

almost everywhere on S_q^N follows, i.e., conditions (2.5) are fulfilled. Therefore we can rewrite (2.20) as follows:

$$(3.1) \qquad \sum_{q=1}^{2} \langle r_{S_c} \mathscr{A}^{(q)} h_0^{(q)}, r_{S_c}(h^{(q)} - h_0^{(q)}) \rangle_{S_c}$$

$$= \sum_{q=1}^{2} [\langle r_{S_c} \{\mathscr{T}^{(q)} U^{(q)}\}^+, r_{S_c}(\vartheta^{(q)} - \vartheta_0^{(q)}) \rangle_{S_c}$$

$$+ \langle r_{S_c} \{\mathscr{M}^{(q)} U^{(q)}\}^+, r_{S_c}(\psi^{(q)} - \psi_0^{(q)}) \rangle_{S_c}] \geq 0$$

$$\forall h = (h^{(1)}, h^{(2)})^\top \in \mathbb{K}_{\varphi_0}, \quad h^{(q)} = (\vartheta^{(q)}, \psi^{(q)})^\top \in [H^{1/2}(S_q)]^6.$$

Let $h = (h^{(1)}, h^{(2)})^\top$, $h^{(1)} = (\vartheta_0^{(1)}, \psi_0^{(1)} \pm \varphi)^\top$, $h^{(2)} = (\vartheta_0^{(2)}, \psi_0^{(2)})^\top$, where $\varphi \in [\tilde{H}^{1/2}(S_c)]^3$ is an arbitrary vector-function. Clearly, $h \in \mathbb{K}_{\varphi_0}$, and from (3.1) we find

$$\langle r_{S_c} \{\mathscr{M}^{(1)} U^{(1)}\}^+, \varphi \rangle_{S_c} = 0 \quad \forall \varphi \in [\tilde{H}^{1/2}(S_c)]^3,$$

i.e., $r_{S_c} \{\mathscr{M}^{(1)} U^{(1)}\}^+ = 0$. Analogously, we find that the equality

$$r_{S_c} \{\mathscr{M}^{(2)} U^{(2)}\}^+ = 0$$

is fulfilled which together with the latter one results in (2.10). Taking into account the above equalities, we can rewrite (3.1) as

$$\sum_{q=1}^{2} \langle r_{S_c}\{\mathcal{T}^{(q)} U^{(q)}\}^+, r_{S_c}(\vartheta^{(q)} - \vartheta_0^{(q)})\rangle_{S_c} \geq 0 \tag{3.2}$$
$$\forall h = (h^{(1)}, h^{(2)})^\top \in \mathbb{K}_{\varphi_0}.$$

Let now $h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}^{1/2}$ be such that $h^{(q)} = (\vartheta^{(q)}, \psi^{(q)})^\top$,

$$\psi^{(q)} = \psi_0^{(q)}, \quad q = 1, 2, \quad \vartheta^{(2)} = \vartheta_0^{(2)}, \quad \vartheta^{(1)} \cdot n^{(1)} = \vartheta_0^{(1)} \cdot n^{(1)}$$

and $\vartheta_t^{(1)} = \vartheta_{0t}^{(1)} \pm \varphi$, where $\varphi \in [\tilde{H}^{1/2}(S_c)]^3$ is again an arbitrary vector function. Clearly, $h \in \mathbb{K}_{\varphi_0}$, and (3.2) yields

$$r_{S_c}\{(\mathcal{T}^{(1)} U^{(1)})_t\}^+ = 0.$$

Quite analogously, we find that the equality

$$r_{S_c}\{(\mathcal{T}^{(2)} U^{(2)})_t\}^+ = 0$$

is fulfilled, and hence the validity of condition (2.9) is proved.

Further, let $\varphi \in \tilde{H}^{1/2}(S_c)$ be an arbitrary scalar function and choose $h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}^{1/2}$ as follows

$$h^{(q)} = (\vartheta^{(q)}, \psi_0^{(q)})^\top, \quad q = 1, 2,$$
$$\vartheta^{(1)} = \vartheta_0^{(1)} + n^{(1)}\varphi, \quad \vartheta^{(2)} = \vartheta_0^{(2)} - n^{(2)}\varphi.$$

Then it can be easily seen that $h \in \mathbb{K}_{\varphi_0}$ and from (3.2) it follows that

$$\langle r_{S_c}\{(\mathcal{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+ - r_{S_c}\{(\mathcal{T}^{(2)} U^{(2)})_{n^{(2)}}\}^+, r_{S_c}\varphi\rangle_{S_c} \geq 0 \quad \forall \varphi \in \tilde{H}^{1/2}(S_c).$$

Whence we can conclude that the first condition in (2.7) is fulfilled.

Let now $\psi^{(q)} = \psi_0^{(q)}$, $q = 1, 2$, $\vartheta^{(2)} = \vartheta_0^{(2)}$ and $\vartheta^{(1)} \cdot n^{(1)} = \vartheta_0^{(1)} \cdot n^{(1)} + \varphi$, where $\varphi \in \tilde{H}^{1/2}(S_c)$ is an arbitrary scalar function satisfying the condition $\varphi \leq 0$. Then $h \in \mathbb{K}_{\varphi_0}$ and

$$\langle r_{S_c}\{(\mathcal{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+, r_{S_c}\varphi\rangle_{S_c} \geq 0 \quad \forall \varphi \in \tilde{H}^{1/2}(S_c), \quad \varphi \leq 0,$$

i.e., conditions (2.7) are fulfilled completely.

It remains to prove the condition (2.8). Taking into account all the above obtained relations, from (3.2) it follows that for all $h \in \mathbb{K}_{\varphi_0}$

$$\langle r_{S_c}\{(\mathcal{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+, r_{S_c}[(\vartheta^{(1)} - \vartheta_0^{(1)}) \cdot n^{(1)} + (\vartheta^{(2)} - \vartheta_0^{(2)}) \cdot n^{(2)}]\rangle_{S_c} \geq 0. \tag{3.3}$$

Let now $h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}^{1/2}$, $h^{(q)} = (\vartheta^{(q)}, \psi^{(q)})^\top \in [\tilde{H}^{1/2}(S_c)]^6$, $q = 1, 2$, $r_{S_c}\vartheta^{(1)} \cdot n^{(1)} = \varphi_0$ and $r_{S_c}\vartheta^{(2)} \cdot n^{(2)} = 0$. Then (3.3) implies that

$$\langle r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+, \varphi_0 - r_{S_c}(\vartheta_0^{(1)} \cdot n^{(1)} + \vartheta_0^{(2)} \cdot n^{(2)})\rangle_{S_c} \geq 0. \tag{3.4}$$

Analogously, if $h^{(q)} = (\vartheta^{(q)}, \psi^{(q)})^\top \in [\tilde{H}^{1/2}(S_c)]^6$, $q = 1, 2$, and

$$r_{S_c}\vartheta^{(1)} \cdot n^{(1)} = 2r_{S_c}\vartheta_0^{(1)} \cdot n^{(1)} - \varphi_0, \quad r_{S_c}\vartheta^{(2)} \cdot n^{(2)} = 2r_{S_c}\vartheta_0^{(2)} \cdot n^{(2)},$$

then (3.3) yields

$$\langle r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+, r_{S_c}(\vartheta_0^{(1)} \cdot n^{(1)} + \vartheta_0^{(2)} \cdot n^{(2)}) - \varphi_0\rangle_{S_c} \geq 0. \tag{3.5}$$

The inequalities (3.4) and (3.5) are equivalent to the condition (2.8). Consequently, the first part of Theorem 3.1 is proved.

Assume now that $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top \in [H^1(\Omega_q)]^6$, $q = 1, 2$, and $U = (U^{(1)}, U^{(2)})^\top$ is a solution of Problem (A_0). We set

$$h_0^{(q)} = (\vartheta_0^{(q)}, \psi_0^{(q)})^\top := \{U^{(q)}\}^+, \quad q = 1, 2,$$

i.e., $\vartheta_0^{(q)} = \{u^{(q)}\}^+$ and $\psi_0^{(q)} = \{\omega^{(q)}\}^+$. Then since $U = (U^{(1)}, U^{(2)})^\top$ is a solution of Problem (A_0), by virtue of conditions (2.4) and (2.6) it is clear that $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathbb{K}_{\varphi_0}$ and due to the definition of the operator $G^{(q)}$ (see (2.16)) the solution $U^{(q)}$ in Ω_q can be represented uniquely by the formula

$$U^{(q)} = G^{(q)}h_0^{(q)}, \quad q = 1, 2.$$

Taking into account the boundary conditions of Problem (A_0) and the definition of the Steklov-Poincaré operator, for every $h = (h^{(1)}, h^{(2)})^\top \in \mathbb{K}_{\varphi_0}$ with $h^{(q)} = (\vartheta^{(q)}, \psi^{(q)})^\top$, $q = 1, 2$, we have

$$\begin{aligned}
&\sum_{q=1}^{2} \langle \mathscr{A}^{(q)}h_0^{(q)}, h^{(q)} - h_0^{(q)}\rangle_{S_q} \\
&\quad = \sum_{q=1}^{2} \langle \{T^{(q)}(\partial, n^{(q)})G^{(q)}h_0^{(q)}\}^+, h^{(q)} - h_0^{(q)}\rangle_{S_q} \\
&\quad = \sum_{q=1}^{2} \langle r_{S_q^N}\Psi^{(q)}, r_{S_q^N}(h^{(q)} - h_0^{(q)})\rangle_{S_q^N} \\
&\qquad + \sum_{q=1}^{2} \langle r_{S_c}\{T^{(q)}(\partial, n^{(q)})G^{(q)}h_0^{(q)}\}^+, r_{S_c}(h^{(q)} - h_0^{(q)})\rangle_{S_c}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q=1}^{2} \langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N}(h^{(q)} - h_0^{(q)}) \rangle_{S_q^N} \\
&\quad + \sum_{q=1}^{2} \langle r_{S_c}\{(\mathscr{T}^{(q)} U^{(q)})_{n^{(q)}}\}^+, r_{S_c}(\vartheta^{(q)} \cdot n^{(q)} - \vartheta_0^{(q)} \cdot n^{(q)}) \rangle_{S_c} \\
&= \sum_{q=1}^{2} \langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N}(h^{(q)} - h_0^{(q)}) \rangle_{S_q^N} \\
&\quad + \sum_{q=1}^{2} \langle r_{S_c}\{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+, r_{S_c}(\vartheta^{(1)} \cdot n^{(1)} - \vartheta_0^{(1)} \cdot n^{(1)}) \rangle_{S_c} \\
&\quad + \sum_{q=1}^{2} \langle r_{S_c}\{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+, r_{S_c}(\vartheta^{(2)} \cdot n^{(2)} - \vartheta_0^{(2)} \cdot n^{(2)}) \rangle_{S_c} \\
&= \sum_{q=1}^{2} \langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N}(h^{(q)} - h_0^{(q)}) \rangle_{S_q^N} \\
&\quad + \langle r_{S_c}\{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+, r_{S_c}(\vartheta^{(1)} \cdot n^{(1)} + \vartheta^{(2)} \cdot n^{(2)}) - \varphi_0 \rangle_{S_c} \\
&\quad - \langle r_{S_c}\{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+, r_{S_c}(\vartheta_0^{(1)} \cdot n^{(1)} + \vartheta_0^{(2)} \cdot n^{(2)}) - \varphi_0 \rangle_{S_c} \\
&= \sum_{q=1}^{2} \langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N}(h^{(q)} - h_0^{(q)}) \rangle_{S_q^N} \\
&\quad + \langle r_{S_c}\{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+, r_{S_c}(\vartheta^{(1)} \cdot n^{(1)} + \vartheta^{(2)} \cdot n^{(2)}) - \varphi_0 \rangle_{S_c}.
\end{aligned}
$$

Since $h = (h^{(1)}, h^{(2)})^\top \in \mathbb{K}_{\varphi_0}$ and the condition (2.7) is fulfilled, the second term of the last equality is nonnegative, and hence

$$
\sum_{q=1}^{2} \langle \mathscr{A}^{(q)} h_0^{(q)}, h^{(q)} - h_0^{(q)} \rangle_{S_q} \geq \sum_{q=1}^{2} \langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N}(h^{(q)} - h_0^{(q)}) \rangle_{S_q^N}.
$$

This completes the proof of the theorem. □

4. Existence results

Here we show the existence of a solution to the boundary variational inequality (2.20). To this end, on the convex closed set $\mathbb{K}_{\varphi_0}$ we consider the functional

$$
\begin{aligned}
J(h) &= \frac{1}{2} \sum_{q=1}^{2} \langle \mathscr{A}^{(q)} h^{(q)}, h^{(q)} \rangle_{S_q} - \sum_{q=1}^{2} \langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} h^{(q)} \rangle_{S_q^N} \\
&\forall h = (h^{(1)}, h^{(2)})^\top \in \mathbb{K}_{\varphi_0}.
\end{aligned} \tag{4.1}
$$

It can be easily shown that in view of the symmetry property of the operator $\mathscr{A}^{(q)}$ (see Lemma 2.1(a)), the existence of solutions of the variational inequality (2.20) is equivalent to the existence of the minimizing element in the set $\mathbb{K}_{\varphi_0}$ of the functional (4.1), i.e., the variational inequality (2.20) is equivalent to the following minimizing problem: Find $h_0 \in \mathbb{K}_{\varphi_0}$ such that

$$J(h_0) = \inf_{h \in \mathbb{K}_{\varphi_0}} J(h). \tag{4.2}$$

The functional (4.1) is continuous and convex. Let us prove that the functional J is coercive on the set $\mathbb{K}_{\varphi_0}$, i.e., $J(h) \to +\infty$, when $h \in \mathbb{K}_{\varphi_0}$ and

$$\|h\|^2_{\mathbb{H}^{1/2}} := \sum_{q=1}^{2} \|h^{(q)}\|^2_{[H^{1/2}(S_q)]^6} \to +\infty.$$

Since the operator $\mathscr{A}^{(q)}$ is coercive on the set $\mathbb{K}_{\varphi_0}$ (see Lemma 2.1(d)), the coerciveness of the functional J in the above mentioned sense follows from the following obvious estimate

$$J(h) \geq C \sum_{q=1}^{2} \|h^{(q)}\|^2_{[H^{1/2}(S_q)]^6} - C_1 \sum_{q=1}^{2} \|h^{(q)}\|_{[H^{1/2}(S_q)]^6}$$
$$\forall h = (h^{(1)}, h^{(2)})^\top \in \mathbb{K}_{\varphi_0},$$

where the positive constants C and C_1 do not depend on h.

The general theory of variational inequalities (see [GLT1], [Fi1]) makes it possible to conclude that the problem (4.2) is uniquely solvable. Finally, we arrive at the following assertion.

Theorem 4.1. *Let* $\Psi^{(q)} \in [\tilde{H}^{-1/2}(S_q^N)]^6$, $q = 1, 2$. *Then the variational inequality* (2.20) *has a unique solution* $h_0^{(q)} \in [H^{1/2}(S_q)]^6$, *and* $U^{(q)} = G^{(q)} h_0^{(q)}$ *will be a unique solution of Problem* (A_0).

Remark 4.1. Let $\Psi^{(q)} \in [\tilde{H}^{-1/2}(S_q^N)]^6$ and $\mathscr{G}^{(q)} \in [L_2(\Omega_q)]^6$, $q = 1, 2$. Then Problem (A) has a unique solution representable in the form $(U^{(1)} + U_0^{(1)}, U^{(2)} + U_0^{(2)})^\top$, where a pair $(U^{(1)}, U^{(2)})^\top$ is a unique solution of Problem (A_0) and $U_0^{(q)}$, $q = 1, 2$, are solutions of the auxiliary problems (2.2).

5. The semicoercive case

5.1. *Formulation of the problem*

Let $S_q^D = \emptyset$, $\overline{S_q^N} \cup \overline{S_c} = S_q$, $\mathscr{G}^{(q)} \in [L_2(\Omega_q)]^6$ and $\Psi^{(q)} \in [\tilde{H}^{-1/2}(S_q^N)]^6$, $q = 1, 2$. Consider the so-called semicoercive contact problem.

Problem (B): Find a pair of vector-functions $U = (U^{(1)}, U^{(2)})^{\top} \in \mathbb{H}^1$ with $U^{(q)}$ being a weak solution of the equation

$$L^{(q)}(\partial)U^{(q)} + \mathscr{G}^{(q)} = 0, \quad q = 1, 2, \tag{5.1}$$

in the domain Ω_q, and satisfying the boundary conditions on S_q^N

$$r_{S_q^N}\{T^{(q)}(\partial, n^{(q)})U^{(q)}\}^+ = r_{S_q^N}\Psi^{(q)}, \quad q = 1, 2,$$

and the contact conditions on S_c:

$$\begin{aligned}
&r_{S_c}\{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+ \leq 0,\\
&r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+ = r_{S_c}\{(\mathscr{T}^{(2)}U^{(2)})_{n^{(2)}}\}^+ \leq 0,\\
&\langle r_{S_c}\{(\mathscr{T}^{(1)}U^{(1)})_{n^{(1)}}\}^+, r_{S_c}\{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+\rangle_{S_c} = 0,\\
&r_{S_c}\{(\mathscr{T}^{(q)}U^{(q)})_t\}^+ = 0, \quad q = 1, 2,\\
&r_{S_c}\{\mathscr{M}^{(q)}U^{(q)}\}^+ = 0, \quad q = 1, 2.
\end{aligned}$$

To reduce this problem to a boundary variational inequality, we have to reduce the inhomogeneous equation (5.1) to the homogeneous one. To this end, let us consider the auxiliary problem: Find a pair of vector-functions $U_0 = (U_0^{(1)}, U_0^{(2)})^{\top} \in \mathbb{H}^1$ with $U_0^{(q)} = (u_0^{(q)}, \omega_0^{(q)})^{\top}$ being a weak solution of equation (5.1) and satisfying the boundary conditions:

$$\begin{aligned}
&r_{S_q^N}\{T^{(q)}(\partial, n^{(q)})U_0^{(q)}\}^+ = 0, \quad r_{S_c}\{u_0^{(q)} \cdot n^{(q)}\}^+ = 0,\\
&r_{S_c}\{(\mathscr{T}^{(q)}U_0^{(q)})_t\}^+ = 0, \quad r_{S_c}\{\mathscr{M}^{(q)}U_0^{(q)}\}^+ = 0, \quad q = 1, 2.
\end{aligned} \tag{5.2}$$

It is known that this problem has a unique solution when S_c, a part of the boundary S_q, is neither rotational nor ruled surface (see [GGN1]). Then, if the pair $W = (W^{(1)}, W^{(2)})^{\top}$ with $W^{(q)} \in [H^1(\Omega_q)]^6$, $q = 1, 2$, is a solution of Problem (B) and $U_0^{(q)}$ is a solution of the auxiliary problem (5.2), then the difference $U^{(q)} = W^{(q)} - U_0^{(q)}$ will be a solution of the following problem.

Problem (B_0): Find vector-functions $U^{(q)} \in [H^1(\Omega_q)]^6$, $q = 1, 2$, which are weak solutions of the homogeneous equations

$$L^{(q)}(\partial)U^{(q)} = 0 \quad \text{in } \Omega_q, \qquad q = 1, 2,$$

and satisfy the boundary conditions on S_q^N

$$r_{S_q^N}\{T^{(q)}(\partial, n^{(q)})U^{(q)}\}^+ = r_{S_q^N}\Psi^{(q)}, \quad q = 1, 2,$$

and the contact conditions on S_c

$$
\begin{aligned}
& r_{S_c}\{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+ \le 0, \\
& r_{S_c}\{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+ + \varphi_0^{(1)} = r_{S_c}\{(\mathscr{T}^{(2)} U^{(2)})_{n^{(2)}}\}^+ + \varphi_0^{(2)} \le 0, \\
& \langle r_{S_c}\{(\mathscr{T}^{(1)} U^{(1)})_{n^{(1)}}\}^+ + \varphi_0^{(1)}, r_{S_c}\{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+ \rangle_{S_c} = 0, \\
& r_{S_c}\{(\mathscr{T}^{(q)} U^{(q)})_t\}^+ = 0, \quad q = 1, 2, \\
& r_{S_c}\{\mathscr{M}^{(q)} U^{(q)}\}^+ = 0, \quad q = 1, 2,
\end{aligned}
$$

where

$$
\varphi_0^{(q)} = r_{S_c}\{(\mathscr{T}^{(q)} U_0^{(q)})_{n^{(q)}}\}^+, \quad q = 1, 2.
$$

Consider a convex closed set

$$
\begin{aligned}
\mathscr{K} = \{h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}^{1/2} : h^{(q)} = (\vartheta^{(q)}, \psi^{(q)})^\top, \\
r_{S_c}\{\vartheta^{(1)} \cdot n^{(1)} + \vartheta^{(2)} \cdot n^{(2)}\} \le 0\}
\end{aligned}
$$

and formulate the following boundary variational inequality:

Find $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathscr{K}$ such that the inequality

$$
\begin{aligned}
(5.3) \quad \sum_{q=1}^{2} \langle \mathscr{A}^{(q)} h_0^{(q)}, h^{(q)} - h_0^{(q)} \rangle_{S_q} \ge \sum_{q=1}^{2} [\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N}(h^{(q)} - h_0^{(q)}) \rangle_{S_q^N} \\
- \langle \varphi_0^{(q)}, r_{S_c}\{\vartheta^{(q)} \cdot n^{(q)} - \vartheta_0^{(q)} \cdot n^{(q)}\} \rangle_{S_c}]
\end{aligned}
$$

is fulfilled for all $h = (h^{(1)}, h^{(2)})^\top \in \mathscr{K}$. As in the coercive case, one can show that inequality (5.3) is equivalent to Problem (B_0).

Theorem 5.1. *Let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathscr{K}$ be a solution of the variational inequality* (5.3) *and $G^{(q)}$ denote the Green operator for the Dirichlet problem in Ω_q. Then $U = (U^{(1)}, U^{(2)})^\top$ with $U^{(q)} = G^{(q)} h_0^{(q)}$ solves Problem (B_0), and vice versa, if $U = (U^{(1)}, U^{(2)})^\top$ is a solution of Problem (B_0), then $h_0 = (h_0^{(1)}, h_0^{(2)})^\top$ with $h_0^{(q)} = \{U^{(q)}\}_{S_q}^+$ solves the variational inequality* (5.3).

The proof of this theorem is quite similar to that of Theorem 3.1 and we omit it here. To prove the existence of a solution to the variational inequality (5.3) we proceed as follows. Let

$$
\Lambda := \{\chi = (\chi^{(1)}, \chi^{(2)})^\top : \chi^{(q)} \in \Lambda^{(q)}(S_q) \ (q = 1, 2)\},
$$

where $\Lambda^{(q)}(S_q)$ is defined by (2.19), and let

$$
\begin{aligned}
\mathscr{R} &:= \mathscr{K} \cap \Lambda \\
&= \{\chi \in \Lambda : ([a^{(1)} \times x] + b^{(1)}) \cdot n^{(1)} + ([a^{(2)} \times x] + b^{(2)}) \cdot n^{(2)} \le 0 \text{ for } x \in S_c\}.
\end{aligned}
$$

Since $n^{(1)} = -n^{(2)}$ at the points of S_c, we have

$$\mathscr{R} = \{\chi = (\chi^{(1)}, \chi^{(2)})^\top \in \Lambda : ([(a^{(1)} - a^{(2)}) \times x] + b^{(1)} - b^{(2)}) \cdot n^{(1)} \leq 0, x \in S_c\}.$$

Let now $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathscr{K}$ be a solution of inequality (5.3). Clearly, $h_0 + \chi \in \mathscr{K}$ for $\chi \in \mathscr{R}$. Substitute $h_0 + \chi$ into (5.3) instead of h and take into account that $\ker \mathscr{A}^{(q)} = \Lambda^{(q)}(S_q)$. We get the inequality

$$(5.4) \quad \sum_{q=1}^{2} [\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} \chi^{(q)} \rangle_{S_q^N} - \langle \varphi_0^{(q)}, r_{S_c} \{[a^{(q)} \times x] + b^{(q)}\} \cdot n^{(q)} \rangle_{S_c}] \leq 0$$

which should hold for all $\chi \in \mathscr{R}$. Thus we have obtained that (5.4) is the necessary condition for the existence of a solution of the variational inequality (5.3). Further, let

$$\mathscr{R}^* := \{\chi \in \Lambda : ([(a^{(1)} - a^{(2)}) \times x] + b^{(1)} - b^{(2)}) \cdot n^{(1)} = 0, x \in S_c\}$$

and assume that inequality (5.4) is fulfilled in a strong sense, i.e., equality (5.4) holds if and only if $\chi \in \mathscr{R}^*$. Due to the general theory of variational inequalities (see [Fi1], [GLT1]), the condition (5.4) becomes sufficient for the solvability of inequality (5.3). Consider the uniqueness question of a solution of (5.3). Suppose that $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathscr{K}$ and $\tilde{h}_0 = (\tilde{h}_0^{(1)}, \tilde{h}_0^{(2)})^\top \in \mathscr{K}$ are two solutions of the variational inequality (5.3). Then

$$(5.5) \quad \sum_{q=1}^{2} \langle \mathscr{A}^{(q)} h_0^{(q)}, \tilde{h}_0^{(q)} - h_0^{(q)} \rangle_{S_q} \geq \sum_{q=1}^{2} [\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} (\tilde{h}_0^{(q)} - h_0^{(q)}) \rangle_{S_q^N} - \langle \varphi_0^{(q)}, r_{S_c} \{\tilde{\vartheta}_0^{(q)} \cdot n^{(q)} - \vartheta_0^{(q)} \cdot n^{(q)}\} \rangle_{S_c}]$$

and

$$(5.6) \quad \sum_{q=1}^{2} \langle \mathscr{A}^{(q)} \tilde{h}_0^{(q)}, h_0^{(q)} - \tilde{h}_0^{(q)} \rangle_{S_q} \geq \sum_{q=1}^{2} [\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} (h_0^{(q)} - \tilde{h}_0^{(q)}) \rangle_{S_q^N} - \langle \varphi_0^{(q)}, r_{S_c} \{\vartheta_0^{(q)} \cdot n^{(q)} - \tilde{\vartheta}_0^{(q)} \cdot n^{(q)}\} \rangle_{S_c}].$$

Adding the above inequalities and taking into account the positive definiteness of the operators $\mathscr{A}^{(q)}$, $q = 1, 2$, we obtain

$$\langle \mathscr{A}^{(q)} (h_0^{(q)} - \tilde{h}_0^{(q)}), h_0^{(q)} - \tilde{h}_0^{(q)} \rangle_{S_q} = 0, \quad q = 1, 2.$$

Whence we can write

$$h_0^{(q)} - \tilde{h}_0^{(q)} = \chi^{(q)}(x) = ([a^{(q)} \times x] + b^{(q)}, a^{(q)})^\top, \quad x \in S_q,$$

since $\ker \mathscr{A}^{(q)} = \Lambda^{(q)}(S_q)$. Consequently, in view of the symmetry property of the operator $\mathscr{A}^{(q)}$ (see Lemma 2.1(a)), we have $\langle \mathscr{A}^{(q)} h_0^{(q)}, \chi^{(q)} \rangle_{S_q} = 0$ and $\langle \mathscr{A}^{(q)} \tilde{h}_0^{(q)}, \chi^{(q)} \rangle_{S_q} = 0$. Therefore from (5.5) and (5.6) we conclude that

$$\sum_{q=1}^{2} [\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} \chi^{(q)} \rangle_{S_q^N} - \langle \varphi_0^{(q)}, r_{S_c}([a^{(q)} \times x] + b^{(q)}, a^{(q)}) \cdot n^{(q)} \rangle_{S_c}] \geq 0$$

and

$$\sum_{q=1}^{2} [\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} \chi^{(q)} \rangle_{S_q^N} - \langle \varphi_0^{(q)}, r_{S_c}([a^{(q)} \times x] + b^{(q)}, a^{(q)}) \cdot n^{(q)} \rangle_{S_c}] \leq 0,$$

i.e.,

$$(5.7) \quad \sum_{q=1}^{2} [\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} \chi^{(q)} \rangle_{S_q^N} - \langle \varphi_0^{(q)}, r_{S_c}([a^{(q)} \times x] + b^{(q)}, a^{(q)}) \cdot n^{(q)} \rangle_{S_c}] = 0.$$

Thus we arrive at the following

THEOREM 5.2. *Let $S_q^D = \emptyset$ and $\Psi^{(q)} \in [\tilde{H}^{-1/2}(S_q^N)]^6$, $q = 1, 2$. Then the inequality* (5.4) *is the necessary condition for the existence of a solution of Problem* (B_0). *If the condition* (5.4) *is fulfilled in a strong sense, i.e., the equality* (5.4) *holds true if and only if $\chi \in \mathscr{R}^*$, then Problem* (B_0) *is solvable. Two solutions of Problem* (B_0) *may differ from each other by a vector of rigid displacement for which the condition* (5.7) *is fulfilled. Solutions of Problem* (B) *with $\mathscr{G}^{(q)} \in [L_2(\Omega_q)]^6$, $q = 1, 2$, are represented then in the form $(U^{(1)} + U_0^{(1)}, U^{(2)} + U_0^{(2)})^\top$, where a pair $(U^{(1)}, U^{(2)})^\top$ is a solution of Problem* (B_0) *and $U_0^{(q)}$, $q = 1, 2$, are solutions of the auxiliary problems* (5.2).

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