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Algebraic Geometry — *On the existence of curves with a triple point on a K3 surface*, by Concettina Galati, presented on 13 January 2012 by Ciro Ciliberto.

Abstract. — Let (S, H) be a general primitively polarized $K3$ surface of genus p and let $p_a(nH)$ be the arithmetic genus of nH. We prove the existence in $|\mathcal{O}_S(nH)|$ of curves with a triple point and A_k-singularities. In particular, we show the existence of curves of geometric genus g in $|\mathcal{O}_S(nH)|$ with a triple point and nodes as singularities and corresponding to regular points of their equisingular deformation locus, for every $1 \leq g \leq p_a(nH) - 3$ and $(\mathrm{p}, \mathrm{n}) \neq (4, 1)$. Our result is obtained by studying the versal deformation space of a non-planar quadruple point.

Key words: Severi varieties, $K3$ surfaces, versal deformations, space curve singularities, triple point.

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1. Introduction

Let S be a complex smooth projective $K3$ surface and let H be a very ample line bundle on S of sectional genus $\mathrm{p} = p_a(H) \geq 2$. The pair (S, H) is called a *primitively polarized* $K3$ *surface of genus* p. It is well-known that a (very) general such pair satisfies $\operatorname{Pic} S \cong \mathbb{Z}[H]$. We denote by $\mathcal{V}^S_{nH,\delta} \subset |\mathcal{O}_S(nH)| = |nH|$ the *Zariski closure* of the Severi variety of δ-nodal curves, defined as the locus of *irreducible* and reduced curves with exactly δ nodes as singularities. The non-emptiness of these varieties for every $\delta \leq \dim(|nH|) = p_a(nH)$, where $p_a(nH)$ is the arithmetic genus of nH, has been established in [5]. Like the Severi variety of δ-nodal plane curves, $\mathcal{V}^S_{nH,\delta}$ has several good properties. By [20], we know that it is smooth of expected dimension at every point $[C]$ corresponding to a δ-nodal curve. This implies that every irreducible component V of $\mathcal{V}^S_{nH,\delta}$ has codimension δ in $|nH|$ and it is contained in at least one irreducible component of the Severi variety $\mathcal{V}^S_{nH,\delta-1}$. Moreover, it is also known that $\mathcal{V}^S_{nH,\delta}$ coincides with the variety $\mathcal{V}^S_{nH,g} \subset |nH|$, defined as the Zariski closure of the locus of reduced and irreducible curves of geometric genus g (cf. [5, Lemma 3.1] and [12, Remark 2.6]). Unlike the Severi variety of δ-nodal plane curves, nothing is known about the irreducibility of $\mathcal{V}^S_{nH,\delta}$. Classically, the irreducibility problem of Severi varieties is related to the problem of the description of their boundary (cf. [17]).

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Problem 1. *Let $V \subset \mathscr{V}^S_{nH,\delta}$ be an irreducible component and let $V^o \subset V$ be the locus of δ-nodal curves. What is inside the boundary $V \backslash V^o$? Does V contain divisors V_n, V_{tac}, V_c and V_{tr} whose general element corresponds to a curve with $\delta + 1$ nodes, a tacnode and $\delta - 2$ nodes, a cusp and $\delta - 1$ nodes and a triple point and $\delta - 3$ nodes, respectively?*

Because of the literature about Severi varieties of plane curves (cf. [7], [8] and [10]), the divisors V_n, V_{tac}, V_c and V_{tr} of $V \subset \mathscr{V}^S_{nH,\delta}$, when non-empty, are expected to play an important role in the description of the Picard group $\mathrm{Pic}(V)$ or, more precisely, of the Picard group of a "good partial compactification" of the locus $V^o \subset V$ of δ-nodal curves. Proving the existence of these four divisors in *every* irreducible component V of $\mathscr{V}^S_{nH,\delta}$ is a very difficult problem. The first progress towards answering the previous question has been made in [12]. In that paper, the authors prove the existence of irreducible curves in $|nH|$ of every allowed genus with a tacnode or a cusp and nodes as further singularities. This article is devoted to the existence of irreducible curves in $|nH|$ of geometric genus $1 \leq g \leq p_a(nH) - 3$ with a triple point and nodes as further singularities. Before introducing our result, we make some observations concerning Problem 1, describing the type of singularity of V along V_n, V_{tac}, V_c and V_{tr}, whenever these loci are non-empty. Let us denote by $\mathscr{V}^S_{nH,g,tac}$, $\mathscr{V}^S_{nH,g,c}$ and $\mathscr{V}^S_{nH,g,tr}$ the Zariski closure of the locus in $|nH|$ of reduced and irreducible curves of geometric genus g with a tacnode, a cusp and a triple point, respectively, and nodes as further singularities. Let W be an irreducible component of any of these varieties or of $\mathscr{V}^S_{nH,g-1}$. Then, by [5, Lemma 3.1], we have that $\dim(W) = g - 1$. Thus W is a divisor in at least one irreducible component V of $\mathscr{V}^S_{nH,\delta = p_a(nH)-g} = \mathscr{V}^S_{nH,g}$ and, by using the same arguments as in [6, Section 1], we also know what V looks like in a neighborhood of the general point of W. If W is an irreducible component of $\mathscr{V}^S_{nH,g-1}$, then $\mathscr{V}^S_{nH,g}$ has an ordinary multiple point of order $\delta + 1$ at the general point of W. In particular, at least in principle, there may not be a unique irreducible component V of $\mathscr{V}^S_{nH,g}$ containing W. On the contrary, if W is any irreducible component of $\mathscr{V}^S_{nH,g,tac}$, $\mathscr{V}^S_{nH,g,c}$ or $\mathscr{V}^S_{nH,g,tr}$, then W is contained in only one irreducible component V of $\mathscr{V}^S_{nH,g}$. In particular, $\mathscr{V}^S_{nH,g}$ is smooth at the general point of every irreducible component of $\mathscr{V}^S_{nH,g,tac}$ and $\mathscr{V}^S_{nH,g,tr}$ and it looks like the product of a cuspidal curve and a smooth $(g-1)$-dimensional variety in a neighborhood of the general point of an irreducible component of $\mathscr{V}^S_{nH,g,c}$. We finally observe that, unlikely the case of Severi varieties of plane curves, if $V \subset \mathscr{V}^S_{nH,g}$ is an irreducible component, a priori, there may exist divisors in V parametrizing curves with worse singularities than an ordinary triple point, a tacnode or a cusp and nodes. We now describe the results of this paper. Our theorem about non-emptiness of $\mathscr{V}^S_{nH,g,tr}$ is based on the following local problem.

Problem 2. *Let $\mathscr{X} \to \mathbb{A}^1$ be a smooth family of surfaces with smooth general fiber $\mathscr{X}_t$ and whose special fiber $\mathscr{X}_0 = A \cup B$ is reducible, having two irreducible components intersecting transversally along a smooth curve $E = A \cap B$. What kind of curve singularities on $\mathscr{X}_0$ at a point $p \in E$ may be deformed to a triple point on $\mathscr{X}_t$?*

In the first part of Section 3, we prove that, if a triple point on $\mathscr{X}_t$ specializes to a general point $p \in E$ along a smooth bisection γ of $\mathscr{X} \to \mathbb{A}^1$, then the limit curve singularity on $\mathscr{X}_0$ is a space quadruple point, union of two nodes having one branch tangent to E. The result is obtained by a very simple argument of limit linear systems theory, with the same techniques as in [11]. In Lemma 3.2 we find the analytic type of this quadruple point. This allows us to compute the versal deformation space of our limit singularity and to prove that, under suitable hypotheses, it actually deforms to an ordinary triple point singularity on $\mathscr{X}_t$, see Theorem 3.9. In the last section, we consider the case that $\mathscr{X}_t$ is a general primitively polarized $K3$ surface and $A = R_1$ and $B = R_2$ are two rational normal scrolls. In Lemma 4.1, we prove the existence on $\mathscr{X}_0 = R_1 \cup R_2$ of suitable curves with a non-planar quadruple point as above, tacnodes (of suitable multiplicities) and nodes and that may be deformed to curves in $|\mathcal{O}_{\mathscr{X}_t}(nH)|$ with an ordinary triple point and nodes as singularities. In particular, this existence result is obtained as a corollary of the following theorem, which is to be considered the main theorem of this paper.

THEOREM 1.1. *Let (S, H) be a general primitively polarized $K3$ surface of genus $\mathrm{p} = p_a(H)$. Then, for every $(\mathrm{p}, \mathrm{n}) \neq (4, 1)$ and for every $(m-1)$-tuple of non-negative integers $d_2, \dots, d_m$ such that*

$$(1) \qquad \sum_{k=2}^{m} (k-1) d_k = n(\mathrm{p}-2) - 3 = 2\mathrm{nl} - 2\mathrm{n} - 3$$

if $\mathrm{p} = 2\mathrm{l}$ *is even, or*

$$(2) \qquad \sum_{k=2}^{m} (k-1) d_k = n(\mathrm{p}-1) - 4 = 2\mathrm{nl} - 4,$$

if $\mathrm{p} = 2\mathrm{l} + 1$ *is odd, there exist reduced irreducible curves C in the linear system $|nH|$ on S such that:*

- *C has an ordinary triple point, $p_a(nH) - \sum_{k=2}^{m}(k-1)d_k - 4$ nodes and d_k singularities of type A_{k-1}, for every $k = 2, \dots, m$, and no further singularities;*
- *C corresponds to a regular point of the equisingular deformation locus $ES(C)$. Equivalently,* $\dim(T_{[C]}ES(C)) = 0$.

Finally, the singularities of C may be smoothed independently. In particular, under the hypotheses (1) *and* (2), *for every $\delta_k \leq d_k$ and for every $\delta \leq$* $\dim(|nH|) - \sum_{k=2}^{m}(k-1)d_k - 4$, *there exist curves C in the linear system $|nH|$ on S with an ordinary triple point, δ_k singularities of type A_{k-1}, for every $k = 2, \dots, m$, and δ nodes as further singularities and corresponding to regular points of their equisingular deformation locus.*

By Corollary 4.3, the family in $|H|$ of curves with a triple point and δ_k singularities of type A_{k-1} is non-empty if it has expected dimension at least equal to one or it has expected dimension equal to zero and $\delta_2 \geq 1$.

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2. Notation and terminology

Throughout the paper an irreducible curve C will be a reduced and irreducible projective curve, even if not specified. We will be concerned here only with reduced and locally complete intersection curves C in a linear system $|D|$ on a (possibly singular) surface X, having, as singularities, space singularities, plane ordinary triple points or plane singularities of type A_k. We recall that an ordinary triple point has analytic equation $y^3 = x^3$ while an A_k-singularity has analytic equation $y^2 - x^{k+1}$. Singularities of type A_1 are nodes, A_2-singularities are cusps and an A_{2m-1}-singularity is a so called m-tacnode. Whenever not specified, a tacnode will be a 2-tacnode, also named a simple tacnode. For curves with ordinary plane triple points and plane singularities of type A_k, equisingular deformations from the analytic point of view coincide with equisingular deformations from the Zariski point of view (cf. [6, Definition (3.13)]). Because of this, given a curve as above, we define the equisingular deformation locus $ES(C) \subset |D|$ as the Zariski closure of the locus of analytically equisingular deformations of C, without discrepancy with classical terminology. Finally, we recall that, if X is a smooth surface and C has as singularities an ordinary triple point and d_k singularities of type A_k, then the dimension of the tangent space to $ES(C)$ at the point $[C]$ corresponding to the curve C is given by $\dim(T_{[C]}ES(C)) \geq \dim|D| - 4 - \sum_k d_k k$ (cf. [6]). If equality holds, we say that $ES(C)$ *is regular at* $[C]$. The regularity of $ES(C)$ is a sufficient and necessary condition for the surjectivity of the standard morphism $H^0(C, \mathcal{N}_{C|X}) \to T^1_C$. In this case, we say that *the singularities of C may be smoothed independently*, with obvious meaning because of the versality properties of T^1_C.

3. How to obtain curves with a triple point on a smooth surface

Let $\mathcal{X} \to \mathbb{A}^1$ be a smooth family of projective complex surfaces with smooth general fiber $\mathcal{X}_t$ and whose special fiber $\mathcal{X}_0 = A \cup B$ has two irreducible components, intersecting transversally along a smooth curve $E = A \cap B$. Let $D \subset \mathcal{X}$ be a Cartier divisor and let us set $D_t = D \cap \mathcal{X}_t$. We want to find sufficient conditions for the existence of curves with a triple point in $|D_t| = |\mathcal{O}_{\mathcal{X}_t}(D_t)|$. We first ask the following question. Assume that, for general t, there exists a reduced and irreducible divisor $C_t \in |D_t|$ with a triple point. Assume, moreover, that the curve C_t degenerates to a curve $C_0 \in |D_0| = |\mathcal{O}_{\mathcal{X}_0}(D_0)|$ in such a way that the triple point of C_t comes to a general point $p \in E = A \cap B$. We ask for the type of singularity of C_0 at p. Actually we do not want to find all possible curve singularities at p that are limit of a triple point on $\mathcal{X}_t$. We only want to find a suitable limit singularity. We first observe that, since p belongs to the singular locus of $\mathcal{X}_0$ and $\mathcal{X}$ is smooth at p, there are no sections of $\mathcal{X} \to \mathbb{A}^1$ passing through p. Thus the triple point of the curve C_t, as $t \to 0$, must move along a multisection γ' of $\mathcal{X} \to \mathbb{A}^1$. In order to deal with a divisor S in a smooth family of surfaces $\mathcal{Y} \to \mathbb{A}^1$ with a triple point at the general point of a section of $\mathcal{Y} \to \mathbb{A}^1$, we make a base change of $\mathcal{X} \to \mathbb{A}^1$.

Let $\mathcal{Y} \to \mathbb{A}^1$ be the smooth family of surfaces

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[dr] & \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \mathbb{A}^1 \arrow[r, "\nu_2"] & \mathbb{A}^1 \end{tikzcd}$$

obtained from $\mathcal{X} \to \mathbb{A}^1$ after a base change of order two totally ramified at 0 and by normalizing the achieved family. Now the family $\mathcal{Y}$ has general fibre $\mathcal{Y}_t \simeq \mathcal{X}_t$ and special fibre $\mathcal{Y}_0 = A \cup \mathcal{E} \cup B$, where, by abusing notation, A and B are the proper transforms of A and B in $\mathcal{X}$ and $\mathcal{E}$ is a $\mathbb{P}^1$-bundle on E. In particular, $A \cap \mathcal{E}$ and $B \cap \mathcal{E}$ are two sections of $\mathcal{E}$ isomorphic to E. Denote by F the fibre $\mathcal{E}$ corresponding to the point $p \in E \subset \mathcal{X}_0$ and let γ be a section of $\mathcal{Y}$ intersecting F at a general point q. Assume there exists a divisor $S \subset \mathcal{Y}$ having a triple point at the general point of γ.

Step 1. Let $\pi_1 : \mathcal{Y}^1 \to \mathcal{Y}$ be the blow-up of $\mathcal{Y}$ along γ with new exceptional divisor Γ and special fibre $\mathcal{Y}^1_0 = A \cup \mathcal{E}' \cup B$, where $\mathcal{E}'$ is the blow-up of $\mathcal{E}$ at q. Still denoting by F the proper transform of $F \subset \mathcal{Y}$ in $\mathcal{Y}^1$, we have that F has self-intersection $(F)^2_{\mathcal{E}'} = -1$ on $\mathcal{E}'$. Moreover, if S^1 is the proper transform of S in $\mathcal{Y}^1$, we have that

$$S^1 \sim \pi_1^*(S) - 3\Gamma. \tag{3}$$

We deduce that $S^1 F = -3$ and hence $F \subset S^1$.

Step 2. Let now $\pi_2 : \mathcal{Y}^2 \to \mathcal{Y}^1$ be the blow-up of $\mathcal{Y}^1$ along F with new exceptional divisor $\Theta \simeq \mathbb{F}_1$ and new special fibre $\mathcal{Y}^2_0 = A' \cup \mathcal{E}' \cup \Theta \cup B'$, where A' and B' are the blow-ups of A and B at $F \cap A$ and $B \cap F$ respectively. Denoting again by F the proper transform of $F \subset \mathcal{Y}^1$ in $\mathcal{Y}^2$, we have that F is the (-1)-curve of Θ. Moreover, if S^2 is the proper transform of S^1 in $\mathcal{Y}^2$, by (3), we deduce that

$$S^2|_\Theta \sim \pi_2^*(S^1)|_\Theta - m_F\Theta|_\Theta \sim -3f_\Theta + m_F(F + 2f_\Theta) \sim (2m_F - 3)f_\Theta + m_F F, \tag{4}$$

where f_Θ is the fibre of Θ and m_F is the multiplicity of S^1 along F. Furthermore, since $S^2|_\Theta$ is an effective divisor, we have that $m_F \geq 2$. In particular, if $m_F = 2$ then $|S^2|_\Theta| = |f_\Theta + 2F|$ contains F in the base locus with multiplicity 1. Hence $S^2|_\Theta = F + L$, with $L \sim f_\Theta + F$. Using again that S^2 is a Cartier divisor, we find that $S^2|_{A'}$ (resp. $S^2|_{B'}$) has two smooth branches intersecting $\Theta \cap A'$ transversally at $F \cap A'$ and $L \cap A'$ (resp. $F \cap B'$ and $L \cap B'$), as in Figure 1.

Now let $\mathcal{S} \subset \mathcal{X}$ be the image of S. If t is a general point of $\mathbb{A}^1$ and $\{t_1, t_2\} = \nu_2^{-1}(t)$, then the fibre of $\mathcal{S}$ over t is $\mathcal{S}_t = S_{t_1} \cup S_{t_2}$, where S_{t_i} is the fibre of S over t_i; while the special fibre $\mathcal{S}_0 = 2(S|_A \cup S|_B)$ of $\mathcal{S}$ is the image curve,

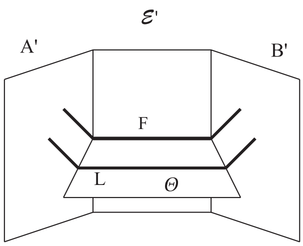

Figure 1

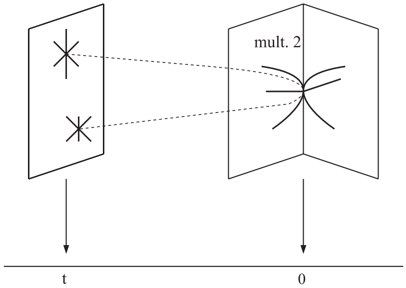

Figure 2

counted with multiplicity 2, of the special fibre S_0 of S under the contraction of $\mathscr{E}$ (see Figure 2).

Remark 3.1. *The curve $S|_A \cup S|_B \subset \mathscr{X}_0$ belongs to the subvariety $Q_p \subset |D_0|$, defined as the Zariski closure of the locus of curves $C = C_A \cup C_B$, where $C_A \subset A$ and $C_B \subset B$ have a node at $p \in E = A \cap B$ with one branch tangent to E. Notice that every such curve C has a non-planar quadruple point at p and that Q_p is a linear subspace of $|D_0|$ of codimension at most* 5.

Lemma 3.2. *Let $[C] \in Q_p$ be a point corresponding to a curve $C = C_A \cup C_B$ with a non-planar quadruple point at p as in the previous remark. Then the analytic equations of C at p are given by*

$$\begin{cases} (y + x - z^2)z = 0 \\ xy = t \\ t = 0. \end{cases} \tag{5}$$

Remark 3.3. *We want to observe that, as for plane curve singularities, the classification of simple complete intersection space singularities is known,* (*cf.* [14]). *The singularity defined by* (5) *is not a simple singularity.*

Before proving Lemma 3.2, we need to recall a basic result about space curve singularities. We refer to [14] and use the same notation and terminology.

Definition 3.4. *Let $\mathbb{Q}_+$ be the set of positive rational numbers. A polynomial $p(\underline{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be quasi-homogeneous of type $(d; a_1, \ldots, a_n) \in \mathbb{Q}_+ \times \mathbb{Q}_+^n$ if $p(\underline{x})$ is a linear combination of monomials $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ such that:*

$$\sum_{i=1}^{n} a_i \alpha_i = d.$$

The n-tuple $\underline{a} = (a_1, \ldots, a_n)$ is called a system of weights. We shall also say that $p(\underline{x})$ has degree d if every variable x_i has weight a_i, for every $i \leq n$.

For every fixed system of weights $\underline{a} = (a_1, \dots, a_n)$, we will denote by $\Gamma^{\underline{a}}$ the induced graduation

$$\mathbb{C}[x_1, \dots, x_n] = \bigoplus_{d \geq 0} G_d^{\underline{a}}$$

on $\mathbb{C}[x_1, \dots, x_n]$, where $G_d^{\underline{a}}$ is the set of quasi-homogeneous polynomials of type $(d; a_1, \dots, a_n)$.

Definition 3.5. *An element $f = (f_1, \dots, f_p) \in (C[x_1, \dots, x_n])^p$ is quasi-homogeneous of type $(\underline{d}, \underline{a}) = (d_1, \dots, d_p; a_1, \dots, a_n)$ if, for every $i = 1, \dots, p$, the component f_i of f is of type $(d_i; a_1, \dots, a_n)$. The $(p+n)$-tuple $(\underline{d}, \underline{a}) = (d_1, \dots, d_p; a_1, \dots, a_n)$ is called a system of degrees and weights.*

For every fixed system of degrees and weights $(\underline{d}, \underline{a})$, one defines a graduation $\Gamma^{(\underline{d}, \underline{a})}$ on $(C[x_1, \dots, x_n])^p$ by setting

$$(C[x_1, \dots, x_n])^p = \bigoplus_{v \in \mathbb{Z}} G_v^{(\underline{d}, \underline{a})},$$

where $G_v^{(\underline{d}, \underline{a})} = \{(g_1, \dots, g_p) \in (C[x_1, \dots, x_n])^p \mid g_i \in G_{d_i+v}^{\underline{a}}\}$. Moreover, we denote by $v_{\underline{a}}$ the valuation naturally associated to $\Gamma^{\underline{a}}$ and by $v_{\underline{d}, \underline{a}}$ the valuation associated to $\Gamma^{\underline{a}, \underline{d}}$ and defined by

$$v_{\underline{d}, \underline{a}}(h) = \inf_i [v_{\underline{a}}(h_i) - d_i],$$

for every $h \in (C[x_1, \dots, x_n])^p$. Finally, we recall that, if $I_{n,p}$ denotes the set of germs of applications $f = (f_1, \dots, f_p) : (C^n, \underline{0}) \to (C^p, \underline{0})$ such $(X, \underline{0}) = (f^{-1}(\underline{0}), \underline{0})$ is a germ of a complete intersection analytic variety with an isolated singularity at the origin, then we have the versal deformation space $T^1(f) = T^1_{X, \underline{0}}$ (cf. [15] or [19]) and, by [18], a graduation is naturally induced on

$$T^1(f) = \bigoplus_{v \in \mathbb{Z}} T^1(f)_v, \quad \text{where } T^1(f)_v \subset G_v^{(\underline{d}, \underline{a})} \text{ for every } v \in \mathbb{Z}.$$

Proposition 3.6 (Merle, [14, Proposition 1]). *Let $f \in I_{n,p}$ be a quasi-homogeneous element of type $(\underline{d}, \underline{a})$. Let $g \in (\mathbb{C}[x_1, \dots, x_n])^p$ be any element such that:*

$$v_{(\underline{d}, \underline{a})}(g) > \sup(0, \alpha),$$

where $\alpha = \mathrm{Sup}_{v \in \mathbb{Z}} \{v \mid T^1(f)_v \neq 0\}$. Then the germs of singularities $((f+g)^{-1}(\underline{0}), \underline{0})$ and $(f^{-1}(\underline{0}), \underline{0})$ are analytically equivalent.

Proof of Lemma 3.2. Let $[C'] \in Q_p$ be a point associated to a curve $C' = C'_A \cup C'_B$, where $C'_A \subset A$ and $C'_B \subset B$ have a node at $p \in E$ with one branch tangent to $E = A \cap B$. Let (x, y, z, t) be analytic coordinates of $\mathscr{X}$ at $p = \underline{0}$ in such a way that $A : x = t = 0$ and $B : y = t = 0$. Then the analytic equations of $C' \subset \mathcal{X}$ at p are given by

$$\begin{cases} p(x, y, z) = 0, \\ xy = 0, \\ t = 0, \end{cases} \tag{6}$$

where $p(x, y, z) = 0$ is the equation of an analytic surface S in $\mathbb{A}^3$ having a singularity of multiplicity 2 at $(0, 0, 0)$. Moreover, the tangent cone of S at p must contain the line $x = y = 0$. Thus, up to a linear transformation, we may assume that the analytic equations of $C' \subset \mathcal{X}$ at p are given by

$$p(x, y, z) = xz + yz + ax^2 + by^2 + z^3 + p_3(x, y, z) = 0, \quad xy = 0, \quad t = 0,$$

where $p_3(x, y, z)$ is a polynomial of degree at least 3 with no z^3-term. We want to prove that the complete intersection curve singularity given by (6) is analytically equivalent to the curve singularity given by the equations (5). We first observe that the element $f = (xz + yz + z^3, xy) \in I_{3,2}$ is quasi-homogeneous of type $(3, 4; 2, 2, 1)$. Moreover, every term of the polynomial $p(x, y, z) - xz - yz - z^3 = ax^2 + by^2 + p_3(x, y, z)$ has degree strictly greater than 3, if the variables (x, y, z) have weights $(2, 2, 1)$. In order to apply Proposition 3.6, we need to compute $T^1(f)$. Let C be a reduced Cartier divisor on $\mathcal{X}_0 = A \cup B$ having a singularity of analytical equations given by (5) at p. Then, since C is a local complete intersection subvariety of $\mathcal{X}$, we have the following standard exact sequence of sheaves on C

$$0 \to \Theta_C \to \Theta_{\mathcal{X}}|_C \xrightarrow{\alpha} \mathcal{N}_{C|\mathcal{X}} \to T^1_C \to 0, \tag{7}$$

where $\Theta_C \simeq Hom(\Omega^1_C, \mathcal{O}_C)$ is the tangent sheaf of C, defined as the dual of the sheaf of differentials of C, $\Theta_{\mathcal{X}}|_C$ is the tangent sheaf of $\mathcal{X}$ restricted to C, $\mathcal{N}_{C|\mathcal{X}}$ is the normal bundle of C in $\mathcal{X}$ and T^1_C is the first cotangent sheaf of C, which is supported at the singular locus $\mathrm{Sing}(C)$ of C and whose stalk $T^1_{C,q}$ at every singular point q of C is the versal deformation space of the singularity. We also recall that the global sections of the image sheaf $\mathcal{N}'_{C|\mathcal{X}}$ of α are, by the versality properties of T^1_C, the infinitesimal deformations of C in $\mathcal{X}$ preserving singularities of C and their analytic type. For this reason, $\mathcal{N}'_{C|\mathcal{X}}$ is called the *equisingular deformation sheaf of C in $\mathcal{X}$*. Now observe that $T^1(f) = T^1_{C,p}$. In order to compute $T^1_{C,p}$ we use the following standard identifications:

- of the local ring $\mathcal{O}_{C,p} = \mathcal{O}_{\mathcal{X},p}/\mathcal{I}_{C|\mathcal{X},p}$ of C at p with $\mathbb{C}[x, y, z]/(f_1, f_2)$, where $f_1(x, y, z) = xz + yz + z^3$ and $f_2(x, y, z) = xy$,
- of the $\mathcal{O}_{C,p}$-module $N_{C|\mathcal{X},p}$ with the free $\mathcal{O}_{\mathcal{X},p}$-module $Hom_{\mathcal{O}_{\mathcal{X},p}}(\mathcal{I}_{C|\mathcal{X},p}, \mathcal{O}_{C,p})$, generated by morphisms f_1^* and f_2^*, defined by

$$f_i^*(s_1(x, y, z)f_1(x, y, z) + s_2(x, y, z)f_2(x, y, z)) = s_i(x, y, z), \quad \text{for } i = 1, 2$$

and, finally,

- of the $\mathcal{O}_{C,p}$-module

$$
\begin{aligned}
(\Theta_{\mathscr{X}}|_C)_p &\simeq \Theta_{\mathscr{X},p}/(I_{C,p} \otimes \Theta_{\mathscr{X},p}) \\
&\simeq \langle \partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial t \rangle_{\mathcal{O}_{C,p}} / \langle \partial/\partial t - x\partial/\partial y - y\partial/\partial x \rangle
\end{aligned}
$$

  with the free $\mathcal{O}_{\mathscr{X},p}$-module generated by the derivatives $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$.

With these identifications, the localization map $\alpha_p : (\Theta_{\mathscr{X}}|_C)_p \to \mathscr{N}_{C|\mathscr{X},p}$ at p of the sheaf map α in (7) is defined by

$$
\alpha_p(\partial/\partial x) = (s = s_1 f_1 + s_2 f_2 \mapsto \partial s/\partial x =_{\mathcal{O}_{C,p}} s_1 \partial f_1/\partial x + s_2 \partial f_2/\partial x)
$$

and, similarly, for $\alpha_p(\partial/\partial y)$ and $\alpha_p(\partial/\partial z)$. In particular, we have that

$$
\begin{aligned}
&\alpha_p(\partial/\partial x) = z f_1^*(s) + y f_2^*(s), \\
&\alpha_p(\partial/\partial y) = z f_1^*(s) + x f_2^*(s) \quad \text{and} \\
&\alpha_p(\partial/\partial z) = (x + y + 3z^2) f_1^*(s).
\end{aligned} \tag{8}
$$

It follows that the versal deformation space of the non planar quadruple point of C at p is the affine space

$$
T^1_{C,p} \simeq \mathcal{O}_{C,p} f_1^* \oplus \mathcal{O}_{C,p} f_2^* / _{\langle z f_1^* + y f_2^*,\, z f_1^* + x f_2^*,\, (3z^2 + x + y) f_2^* \rangle}. \tag{9}
$$

In particular, $T^1_{C,p} = T^1(f) \simeq \mathbb{C}^7$ (in accordance with [13, Proposition on p. 165]) and, if we fix affine coordinates $(b_1, b_2, b_3, a_1, a_2, a_3, a_4)$ on $T^1_{C,p}$, then the versal deformation family $\mathscr{C}_p \to T^1_{C,p} = T^1(f)$ has equations

$$
\begin{cases}
(y + x - z^2)z + a_1 + a_2 x + a_3 y + a_4 z = F(x, y, z) = 0, \\
xy + b_1 + b_2 z + b_3 z^2 = G(x, y, z) = 0.
\end{cases} \tag{10}
$$

Furthermore, by the equality

$$
\begin{aligned}
\begin{pmatrix} F(x, y, z) \\ G(x, y, z) \end{pmatrix} &= \begin{pmatrix} 0 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_2 z \end{pmatrix} + \begin{pmatrix} a_4 z \\ b_3 z^2 \end{pmatrix} \\
&\quad + \begin{pmatrix} a_3 y + a_2 x \\ 0 \end{pmatrix} + \begin{pmatrix} (y + x - z^2)z \\ xy \end{pmatrix},
\end{aligned}
$$

we deduce that the graduation on $T^1(f)$ induced by $\Gamma^{(3,4;2,2,1)}$ is given by

$$
T^1(f) = \bigoplus_{v=-4}^{0} T^1(f)_v.
$$

In particular, we obtain that $\alpha = \sup_{v \in \mathbb{Z}} \{ v \mid T^1(f)_v \neq 0 \} = 0$. By Proposition 3.6, using that $v_{3,4;2,2,1}(p(x, y, z) - xz - yz - z^3, xy) > 3 > 0 = \alpha$ we get that the singularity defined by (6) is analytically equivalent to the singularity of equations (5), as desired. □

Remark 3.7 (Equisingular infinitesimal deformations). *Let $C \subset \mathscr{X}_0$ be a curve as in Lemma* 3.2*. By the proof of the lemma, we may deduce the equations at p of an equisingular infinitesimal deformation of C in $\mathscr{X}$. Indeed, by the equalities* (8)*, we have that, if $s \in \mathscr{N}'_{C|\mathscr{X},p}$, then the equations of s at p are given by*

$$\begin{cases} z(x+y+z^2) + \varepsilon(zu_x + zu_y + (3z^2 + x + y)u_z) = 0, \\ xy + \varepsilon(xu_y + yu_x) = 0, \end{cases} \tag{11}$$

where

$$u = u_x(x,y,z)\partial/\partial x + u_y(x,y,z)\partial/\partial y + u_z(x,y,z)\partial/\partial z \in (\Theta_{\mathscr{X}}|_C)_p.$$

Now $xy + \varepsilon(xu_y + yu_x) = 0$ is the equation of an infinitesimal deformation $\mathscr{X}_0$ vanishing along the singular locus E. By [2, Section 2]*, we know that these infinitesimal deformations are the infinitesimal deformations of $\mathscr{X}_0$ preserving the singular locus. Since $\mathscr{X}_0$ is the only singular fibre of $\mathscr{X}$, we have that $xu_y + yu_x = 0$ in* (11)*. Since this does not depend on the type of singularity of C at p, we deduce that $H^0(C, \mathscr{N}'_{C|\mathscr{X}}) = H^0(C, \mathscr{N}'_{C|\mathscr{X}_0})$. Moreover, we obtain that u_x and u_y are polynomials with no z^r-terms, for every r, and such that $u_y(\underline{0}) = u_x(\underline{0}) = 0$ and no x^r-terms (resp. y^r-terms) appear in u_y (resp. u_x). Thus the equations of $s \in \mathscr{N}'_{C|\mathscr{X},p}$ at p are given by*

$$\begin{cases} z(x+y+z^2) + \varepsilon(zxu'_x + zyu'_y + (3z^2 + x + y)(c + u'_z)) = 0, \\ xy = 0, \end{cases} \tag{12}$$

where we set $u_x(x,z) = xu'_x(x,z)$, $u_y(y,z) = yu'_y(y,z)$ and $u_z(x,y,z) = c + u'_z(x,y,z)$, with $u'_z(\underline{0}) = 0$. Finally, $H^0(C, \mathscr{N}'_{C|\mathscr{X}}) = H^0(C, \mathscr{N}'_{C|\mathscr{X}_0})$ is a linear space of codimension ≤ 4 in $|D_0|$, contained in the linear system T_p of curves $C'_A \cup C'_B \subset \mathscr{X}_0$, with $C'_A \subset A$, $C'_B \subset B$ and C'_A and C'_B tangent to E at p.

We may now provide sufficient conditions for the existence of curves with a triple point and possibly further singularities in the linear system $|D_t|$.

Definition 3.8. *Let $\mathscr{X}$ be a family of surfaces as above and let $D \subset \mathscr{X}$ be a Cartier divisor. Let $\mathscr{W}^{\mathscr{X}|\mathbb{A}^1}_{D,g,tr} \subset |\mathscr{O}_{\mathscr{X}}(D)| \times (\mathbb{A}^1 \backslash \{0\})$ be the locally closed subset defined as follows*

$$\mathscr{W}^{\mathscr{X}|\mathbb{A}^1}_{D,g,tr} = \{([D'],t) \mid \mathscr{X}_t \text{ is smooth, } D' \cap \mathscr{X}_t := D'_t \in |\mathscr{O}_{\mathscr{X}_t}(D_t)| \text{ is irreducible of genus } g\text{, with a triple point and nodes as singularities.}\}$$

There is a naturally defined rational map $\pi : \mathscr{W}^{\mathscr{X}|\mathbb{A}^1}_{D,g,tr} \to \mathscr{H}^{\mathscr{X}|\mathbb{A}^1}$, where $\mathscr{H}^{\mathscr{X}|\mathbb{A}^1}$ is the relative Hilbert scheme of $\mathscr{X} \to \mathbb{A}^1$. We will denote by $\mathscr{V}^{\mathscr{X}|\mathbb{A}^1}_{D,g,tr}$ the Zariski closure of the image of π and we will name it the universal Severi variety of curves of genus g in $|D|$ with a triple point and nodes. The restriction of this variety $\mathscr{V}^{\mathscr{X}|\mathbb{A}^1}_{D,g,tr} \cap |D_t| = \mathscr{V}^{\mathscr{X}_t}_{D_t,g,tr}$, where $t \in \mathbb{A}^1$ is general, is the Severi variety of genus g curves in $|D_t|$ with a triple point and nodes as singularities.

We observe that, if $V \subset \mathscr{V}^{\mathscr{X}_t}_{D_t,g,tr}$ is an irreducible component, then V coincides with the equisingular deformation locus $ES(C) \subset |D_t|$ of the curve C corresponding to the general point $[C] \in V$, defined in Section 2.

THEOREM 3.9. *With the notation above, let $C = C_A \cup C_B \subset \mathscr{X}_0$ be a reduced divisor in the linear system $|D_0|$ such that $C_A \subset A$ and $C_B \subset B$ have a node at a general point p of $E = A \cap B$ with one branch tangent to E. Suppose that C_A and C_B are smooth and they intersect E transversally outside p. Assume, moreover, that:*

1) $h^1(A, \mathscr{O}_A) = h^1(B, \mathscr{O}_B) = h^1(\mathscr{X}_t, \mathscr{O}_{\mathscr{X}_t}) = 0$, *for every t;*
2) $\dim(|D_t|) = \dim(|D_0|)$, *for a general t;*
3) $h^0(C, \mathscr{N}'_{C|\mathscr{X}}) = \dim(|D_0|) - 4$.

Then the universal Severi variety $\mathscr{V}^{\mathscr{X}|\mathbb{A}^1}_{D,p_a(D)-3,tr} \subset \mathscr{H}^{\mathscr{X}|\mathbb{A}^1}$ is non-empty. More precisely, the point $[C] \in \mathscr{H}^{\mathscr{X}|\mathbb{A}^1}$ corresponding to the curve C belongs to an irreducible component $\mathscr{Q}$ of the special fibre $\mathscr{V}_0$ of $\mathscr{V}^{\mathscr{X}|\mathbb{A}^1}_{D,p_a(D)-3,tr} \to \mathbb{A}^1$, contained in $\mathscr{V}_0$ with multiplicity 2. Finally, $\mathscr{V}^{\mathscr{X}|\mathbb{A}^1}_{D,p_a(D)-3,tr}$ is smooth at $[C]$ and the irreducible component $V_t^{\mathscr{Q}}$ of the general fibre $\mathscr{V}_t$ of $\mathscr{V}$, specializing to $\mathscr{Q}$, has expected dimension in $|D_t|$.

PROOF. We want to obtain curves in the linear system $|\mathscr{O}_{\mathscr{X}_t}(D_t)|$ with a triple point as deformations of $C \subset \mathscr{X}_0$. The scheme parametrizing deformations of C in $\mathscr{X}$ is an irreducible component $\mathscr{H}$ of the relative Hilbert scheme $\mathscr{H}^{\mathscr{X}|\mathbb{A}^1}$ of the family $\mathscr{X}$. In particular, by [19, Proposition 4.4.7] and the hypotheses 1) and 2), we have that $\mathscr{H}$ is smooth at the point $[C] \in \mathscr{H}$ corresponding to C. Now, by hypothesis, C has a non-planar quadruple point at p, nodes on $E \setminus \{p\}$ and no further singularities. Moreover, it is well-known that, no matter how we deform C to a curve on $\mathscr{X}_t$, the nodes of C on E are smoothed (see, for example, [11, Section 2]). We want to prove that C may be deformed to a curve on $\mathscr{X}_t$ in such a way that the non-planar quadruple point of C at p is deformed to a triple point. This will follows from a local analysis. First recall that, by Lemma 3.2, we may choose analytic coordinates (x, y, z, t) of $\mathscr{X}$ at p in such a way that the equations of C at p are given by (5) and the versal deformation family $\mathscr{C}_p \to T^1_{C,p}$ has equations given by (10), where $(b_1, b_2, b_3, a_1, a_2, a_3, a_4)$ are the affine coordinates on $T^1_{C,p} \simeq \mathbb{C}^7$. By versality, denoting by $\mathscr{D} \to \mathscr{H}$ the universal family parametrized by $\mathscr{H}$, there exist étale neighborhoods U_p of $[C]$ in $\mathscr{H}$, U'_p of p in $\mathscr{D}$ and V_p of $\underline{0}$ in $T^1_{C,q}$ and a map $\phi_p : U_p \to V_p$ so that the family $\mathscr{D}|_{U_p} \cap U'_p$ is isomorphic to the pull-back of $\mathscr{C}_p|_{V_p}$, with respect to ϕ_p,

$$
(13)\qquad
\begin{tikzcd}
\mathscr{C}_p \arrow[d] & \mathscr{C}_p|_{V_p} \arrow[l] \arrow[d] & U_p \times_{V_p} \mathscr{C}_p|_{V_p} \arrow[l] \arrow[r, "\simeq"] \arrow[dr] & \mathscr{D}|_{U_p} \cap U'_p \arrow[r] \arrow[d] & \mathscr{D} \arrow[d] \\
T^1_{C,p} & V_p \arrow[l] & & U_p \arrow[ll, "\phi_p"'] \arrow[r] & \mathscr{H}.
\end{tikzcd}
$$

We need to describe the image $\phi_p(U_p) \subset V_p$. First we want to prove that

$$\phi_p(U_p \cap |D_0|) = \Gamma \cap V_p, \tag{14}$$

where

$$\Gamma : b_1 = b_2 = b_3 = 0.$$

Obviously, $\phi_p(U_p \cap |D_0|) \subset \Gamma \cap V_p$. To prove that the equality holds, it is enough to show that the differential $d\phi_{p[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}_0}) \to T_{\underline{0}}\Gamma$ is surjective. By standard deformation theory, by identifying the versal deformation space of a singularity with its tangent space at $\underline{0}$, the differential of ϕ_p at $[C]$ can be identified with the map

$$d\phi_{p[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \to H^0(C, T^1_C) \to T^1_{C,p},$$

induced by the exact sequence (7). In particular, using that, by Remark 3.7, $H^0(C, \mathcal{N}'_{C|\mathcal{X}}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}_0})$ and that the nodes of C on E are necessarily preserved when we deform C on $\mathcal{X}_0$, we have that

$$\ker d\phi_{p[C]} = H^0(C, \mathcal{N}'_{C|\mathcal{X}}).$$

Now, again by Remark 3.7, we know that $h^0(C, \mathcal{N}'_{C|\mathcal{X}}) = h^0(C, \mathcal{N}'_{C|\mathcal{X}_0}) \geq \dim(|D_0|) - 4 = \dim(|D_0|) - \dim(\Gamma)$. Actually, by the hypothesis 3), we have that $h^0(C, \mathcal{N}'_{C|\mathcal{X}_0})$ has the expected dimension and hence the equality (14) is verified. Using that $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ is a hyperplane in $H^0(C, \mathcal{N}_{C|\mathcal{X}})$ and that $\phi_p(U_p \cap |D_0|) \subset \phi_p(U_p)$, this implies, in particular, that $\phi_p(U_p) \subset V_p \subset T^1_{C,p}$ is a subvariety of dimension $\dim(\phi_p(U_p \cap |D_0|)) + 1 = 5$, smooth at $\underline{0}$. We want to determine the equations of tangent space $T_{\underline{0}}\phi_p(U_p) = d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}}))$. For this purpose, it is enough to find the image by $d\phi_p$ of the infinitesimal deformation $\sigma \in H^0(C, \mathcal{N}_{C|\mathcal{X}}) \backslash H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ having equations

$$\begin{cases} z(x + y + z^2) = 0 \\ xy = \varepsilon. \end{cases} \tag{15}$$

The image of σ is trivially the vector $(1, 0, \ldots, 0)$. We deduce that

$$T_{\underline{0}}\phi_p(U_p) = d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}})) : b_2 = b_3 = 0.$$

Now we want to prove that the locus in $\phi_p(U_p) \backslash \phi_p(U_p \cap |D_0|)$ of points corresponding to curves with an ordinary triple point as singularity is not empty and its Zariski closure is a smooth curve T tangent to $\Gamma = \phi_p(U_p \cap |D_0|)$ at $\underline{0}$. This will imply the theorem by versality and by a straightforward dimension count. Because of smoothness of $\phi_p(U_p)$ at $\underline{0}$, it is enough to prove that the locus of points $(b_1, 0, 0, a_1, \ldots, a_4) \in T_{\underline{0}}\phi_p(U_p)$ with $b_1 \neq 0$ and corresponding to a curve with an ordinary triple point is not empty and its Zariski closure is a smooth curve T tangent to $\Gamma = \phi_p(U_p \cap |D_0|)$ at $\underline{0}$.

Let $(b_1, 0, 0, a_1, \dots, a_4) \in T^1_{C,p}$ be a point with $b_1 \neq 0$. Then the fibre $\mathscr{C}_{(b_1,0,0,a_1,\dots,a_4)}$ of the versal family $\mathscr{C}_p \to T^1_{C,p}$ has equation

$$\mathscr{C}_{(b_1,0,0,a_1,\dots,a_4)} : z\left(z^2 + \frac{b_1}{y} + y\right) + a_1 + a_2\frac{b_1}{y} + a_3 y + a_4 z = 0, \quad y \neq 0$$

or, equivalently,

$$\mathscr{C}_{(b_1,0,0,a_1,\dots,a_4)} : F(y,z) = z^3 y + zy^2 + b_1 z + a_1 y + a_2 b_1 + a_3 y^2 + a_4 zy, \quad y \neq 0.$$

Now a point $(y_0, z_0) \in \mathscr{C}_{(b_1,0,0,a_1,\dots,a_4)}$ is a singular point of multiplicity at least three if and only if

$$\frac{\partial F(y,z)}{\partial y} = z_0^3 + 2y_0 z_0 + a_1 + 2a_3 y_0 + a_4 z_0 = 0, \tag{16}$$

$$\frac{\partial F(y,z)}{\partial z} = 3z_0^2 y_0 + y_0^2 + b_1 + a_4 y_0 = 0, \tag{17}$$

$$\frac{\partial F(y,z)}{\partial z^2} = 6z_0 y_0 = 0, \tag{18}$$

$$\frac{\partial F(y,z)}{\partial y^2} = 2z_0 + 2a_3 = 0, \tag{19}$$

$$\frac{\partial F(y,z)}{\partial y \partial z} = 3z_0^2 + 2y_0 + a_4 = 0. \tag{20}$$

By the hypothesis $b_1 \neq 0$ and the equalities (18) and (19), we find that $z_0 = 0$. By substituting $z_0 = 0$ in the equalities (16), (17), (19), (20) and $F(y_0, 0) = 0$, we find that

$$\begin{aligned} a_1 = a_2 = a_3 &= 0, \\ y_0^2 + b_1 + a_4 y_0 &= 0, \\ z_0 &= 0, \\ 2y_0 + a_4 &= 0. \end{aligned}$$

Conversely, for every point $(b_1, 0, 0, a_1, \dots, a_4) \in T^1_{C,p}$ such that $b_1 \neq 0$, $a_1 = a_2 = a_3 = 0$ and $a_4^2 = 4b_1$, we have that the corresponding curve has a triple point at $\left(-\frac{a_4}{2}, 0\right)$, with tangent cone of equation $z\left(\left(y + \frac{a_4}{2}\right)^2 + z^2\right)$ and no further singularities. The curve

$$T : a_1 = a_2 = a_3 = b_2 = b_3 = 0, \quad a_4^2 = 4b_1$$

is smooth and tangent to $\Gamma : b_1 = b_2 = b_3 = 0$ at $\underline{0}$. □

The following corollary is a straight consequence of the proof of Theorem 3.9.

COROLLARY 3.10. *Let $\mathscr{X}$ be a family of regular surfaces and $D \subset \mathscr{X}$ a Cartier divisor as in the statement of Theorem 3.9. Let $C' \subset |D_0|$ be any reduced curve with a space quadruple point of equations* (5) *at a point $p \in E$ and possibly further singularities. Then, using the same notation as in the proof of the previous theorem, the image H'_p of the morphism*

$$H^0(C', \mathscr{N}_{C'|\mathscr{X}}) \to T^1_{C',p}$$

is contained in the 5-dimensional plane of equations $H_p : b_2 = b_3 = 0$. If $H'_p = H_p$ then there exist deformations $C_t \in |D_t|$ of C on $\mathscr{X}_t$ with a triple point, obtained as deformation of the singularity of C at p.

4. Curves with a triple point and nodes on general $K3$ surfaces

This section is devoted to the proof of Theorem 1.1. We will prove the theorem by using the very classical degeneration technique introduced in [3]. Let (S, H) be a general primitively polarized $K3$ surface of genus $\mathrm{p} = p_a(H)$ in $\mathbb{P}^{\mathrm{p}}$. We will degenerate S to a union of two rational normal scrolls $R = R_1 \cup R_2$. On R we will prove the existence of suitable curves $C \in |\mathcal{O}_R(nH)|$ with a space quadruple point given by equations (5), tacnodes and nodes. Finally, we will show that the curves C deform to curves on S with the desired singularities.

We first explain the degeneration argument, introducing notation. Fix an integer $\mathrm{p} \geq 3$ and set $l := \lfloor \frac{\mathrm{p}}{2} \rfloor$. Let $E \subset \mathbb{P}^{\mathrm{p}}$ be a smooth elliptic normal curve of degree $\mathrm{p}+1$. Consider two general line bundles $L_1, L_2 \in \mathrm{Pic}^2(E)$. We denote by R_1 and R_2 the (unique) rational normal scrolls of degree $\mathrm{p}-1$ in $\mathbb{P}^{\mathrm{p}}$ defined by L_1 and L_2, respectively. Notice that $R_1 \cong R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{F}_0$ if p is odd whereas $R_1 \cong R_2 \cong \mathbb{F}_1$ if p is even. Moreover, R_1 and R_2 intersect transversally along the curve E which is anticanonical in each R_i (cf. [3, Lemma 1]). More precisely, for odd p, where $R_1 \cong R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, we let $\sigma_i = \mathbb{P}^1 \times \{pt\}$ and $F_i = \{pt\} \times \mathbb{P}^1$ on R_i be the generators of $\mathrm{Pic}\, R_i$, with $i = 1, 2$. For even p, where $R_1 \cong R_2 \cong \mathbb{F}_1$, we let σ_i be the section of negative self-intersection and F_i be the class of a fiber. Then the embedding of R_i into $\mathbb{P}^{\mathrm{p}}$ is given by the line bundle $\sigma_i + lF_i$, for $i = 1, 2$. Let now $R := R_1 \cup R_2$ and let $\mathscr{U}_{\mathrm{p}}$ be the component of the Hilbert scheme of $\mathbb{P}^{\mathrm{p}}$ containing R. Then we have that $\dim(\mathscr{U}_{\mathrm{p}}) = \mathrm{p}^2 + 2\mathrm{p} + 19$ and, by [3, Theorems 1 and 2], the general point $[S] \in \mathscr{U}_{\mathrm{p}}$ represents a smooth, projective $K3$ surface S of degree $2\mathrm{p}-2$ in $\mathbb{P}^{\mathrm{p}}$ such that $\mathrm{Pic}\, S \cong \mathbb{Z}[\mathcal{O}_S(1)] = \mathbb{Z}[H]$. We denote by $\mathscr{S} \to T$ a general deformation of $\mathscr{S}_0 = R$ over a one-dimensional disc T contained in $\mathscr{U}_{\mathrm{p}}$. In particular, the general fiber is a smooth projective $K3$ surface $\mathscr{S}_t$ in $\mathbb{P}^p$ with $\mathrm{Pic}\, \mathscr{S}_t \cong \mathbb{Z}[\mathcal{O}_{\mathscr{S}_t}(1)]$. Now $\mathscr{S}$ is smooth except for 16 rational double points $\{\xi_1, \ldots, \xi_{16}\}$ lying on E. In particular, $\{\xi_1, \ldots, \xi_{16}\}$ are the zeroes of the section of the first cotangent bundle T^1_R of R, determined by the first order embedded deformation associated to $\mathscr{S} \to T$, [3, pp. 644–647]. By blowing-up $\mathscr{S}$ at these points and by contracting the corresponding exceptional components (all isomorphic to $\mathbb{F}_0$) on R_2, we get a smooth family of surfaces $\mathscr{X} \to T$, such that $\mathscr{X}_t \simeq \mathscr{S}_t$ and $\mathscr{X}_0 = R_1 \cup \tilde{R}_2$, where $\tilde{R}_2$ is the blowing-up of R_2 at the points $\{\xi_1, \ldots, \xi_{16}\}$, with new exceptional curves $E_1, \dots, E_{16}$. We will name $\{\xi_1, \dots, \xi_{16}\}$ *the special points of* E.

LEMMA 4.1. *Let $R = R_1 \cup R_2 \subset \mathbb{P}^{\mathrm{p}}$ as above. Then, for every $n \geq 1$ if $\mathrm{p} \geq 5$ and $n \geq 2$ if $\mathrm{p} = 3, 4$, there exists a one parameter family of curves $C = C_1 \cup C_2 \in |\mathcal{O}_R(nH)|$ such that:*

(1) *$C_i \subset R_i$ and $C_i = C_i^1 \cup \dots \cup C_i^{n-1} \cup D_i \cup L_i$, where C_i^j, D_i and L_i are smooth rational curves with:*
- *$C_i^j \sim \sigma_i$, $D_i \sim F_i$, and $L_i \sim \sigma_i + (nl-1)F_i$ if $\mathrm{p} = 2\mathrm{l}+1$ is odd,*
- *$C_i^j \sim \sigma_i + F_i$, $D_i \sim F_i$ and $L_i \sim \sigma_i + (nl-n)F_i$ if $\mathrm{p} = 2\mathrm{l}$ is even,*

for every $1 \leq j \leq n-1$ and $i = 1, 2$;

(2) *there exist distinct points $p, q, q_1, \dots, q_{2n} \in E$, where p is a general point and $q, q_1, \dots, q_{2n}$ are determined by the following relations:*
- *if $\mathrm{p} = 2\mathrm{l}+1$ is odd, $i = 1, 2$ and $1 \leq j \leq n-1$, then*

$$D_1 \cap E \ (\text{resp. } D_2 \cap E) = \begin{cases} p + q_{2n}, \text{ if } n \text{ is odd}, (\text{resp. if } n \text{ is even}), \\ p + q_{2n-1}, \text{ if } n \text{ is even}, (\text{resp. if } n \text{ is odd}), \end{cases} \tag{21}$$

$$C_1^j \cap E \ (\text{resp. } C_2^j \cap E) = \begin{cases} q_{2j-1} + q_{2j+1}, \text{ if } j \text{ is even (resp. if } j \text{ is odd}), \\ q_{2j} + q_{2j+2}, \text{ if } j \text{ is odd (resp. if } j \text{ is even}) \end{cases} \tag{22}$$

$$\text{and} \quad L_i \cap E = (2nl-3)q + 2p + q_i; \tag{23}$$

- *if $\mathrm{p} = 2\mathrm{l}$ is even, $i = 1, 2$ and $1 \leq j \leq n-1$, then*

$$D_1 \cap E \ (\text{resp. } D_2 \cap E) = p + q_{2n} \ (\text{resp. } p + q_{2n-1}), \tag{24}$$

$$C_1^j \cap E = q_{2j-1} + 2q_{2j}, \quad 1 \leq j \leq n-1, \tag{25}$$

$$C_2^j \cap E = 2q_{2j} + q_{2j+1}, \quad 1 \leq j \leq n-2, \tag{26}$$

$$C_2^{n-1} \cap E = 2q_{2n-2} + q_{2n} \quad \text{and} \tag{27}$$

$$\begin{aligned} L_1 \cap E \ (\text{resp. } L_2 \cap E) = {} & (2nl-2n-2)q + 2p + q_{2n-1} \\ & (\text{resp. } (2nl-2n-2)q + 2p + x, \text{ where} \\ & x = q_2 \text{ if } n = 1 \text{ and } x = q_1 \text{ if } n > 1); \end{aligned} \tag{28}$$

(3) *the singularities of C on $R \backslash E$ are nodes, C has a quadruple point analytically equivalent to (5) at $p \in E$ and tacnodes and nodes on $E \backslash p$. In particular, C has a $(2nl-3)$-tacnode at q and nodes at $q_1, \dots, q_{2n}$, if $\mathrm{p} = 2\mathrm{l}+1$; C has a $(2nl-2n-2)$-tacnode at q, a simple tacnode at q_{2k} and nodes at q_{2n}, q_{2n-1} and q_{2k-1}, for every $k = 1, \dots, n-1$, if $\mathrm{p} = 2\mathrm{l}$.*

PROOF. We first consider the case $\mathrm{p} = 2\mathrm{l}+1$ odd. Recall that, in this case, we have that $R_1 \simeq R_2 \simeq \mathbb{F}_0$, $\mathcal{O}_{R_i}(H) = \mathcal{O}_{R_i}(\sigma_i + lF_i)$, where $|\sigma_i|$ and $|F_i|$ are the two rulings on R_i, and the linear equivalence class of E on R_i is $E \sim_{R_i} 2\sigma_i + 2F_i$, $i = 1, 2$.

Let p and q be two distinct points of $E = R_1 \cap R_2 \subset R$ and let us denote by $W^i_{2,2nl-3}(p,q) \subset |\sigma_i + (nl-1)F_i|$ the family of divisors tangent to E at p and q with multiplicity 2 and $2nl-3$, respectively. Then $W^i_{2,2nl-3}(p,q) \subset |\sigma_i + (nl-1)F_i|$ is a linear system of dimension

$$\dim(W^i_{2,2nl-3}(p,q)) \geq \dim(|\sigma_i + (nl-1)F_i|) - 2nl + 1 = 0.$$

Moreover, it is very easy to see that the general element of $W^i_{2,2nl-3}(p,q)$ corresponds to an irreducible and reduced smooth rational curve. By applying [17, Proposition 2.1], one shows that, in fact, $\dim(W^i_{2,2nl-3}(p,q)) = 0$. We deduce that the variety $W^i_{2,2nl-3} \subset |\sigma_i + (nl-1)F_i|$, parametrizing divisors tangent to E at two distinct points with multiplicity 2 and $2nl-3$ respectively, has dimension 2. Furthermore, looking at the tangent space to $W^i_{2,2nl-3}$ at its general element, it is easy to prove that, if p and q are two general points of E, then the unique divisor $L_i \in |\sigma_i + (nl-1)F_i|$, tangent to E at p with multiplicity 2 and to q with multiplicity $2nl-3$, intersects E transversally at a point $q_i \neq p, q$.

Now let p be a general point of E and let $D_i \subset R_i$ be the fibre $D_i \sim F_i$ passing through p. Because of the generality of p and the fact that $ED_i = 2$, we may assume that D_1 and D_2 intersect E transversally at a further point. Again by the generality of p, we may assume that the fibre $C^{n-1}_1 \sim \sigma_1$ (resp. $C^{n-1}_2 \sim \sigma_2$) passing through the point of $D_2 \cap E$ (resp. $D_1 \cap E$), different from p, intersects transversally E at a further point. Let C^{n-2}_2 (resp. C^{n-2}_1) be the fibre of $|\sigma_2|$ (resp. $|\sigma_1|$) passing through this point. We repeat this argument $n-1$ times, getting $2n$ points $q_1, \ldots, q_{2n}$ of E and fibres $C^j_i \sim \sigma_i$, with $1 \leq j \leq n-1$, verifying relations (21) and (22). From what we proved above, by using that

- the family of divisors in $|\sigma_1 + (nl-1)F_1|$ tangent to E at p, with multiplicity 2, and at a further point, with multiplicity $2nl-3$, has dimension 1;
- the point p is general and the points $q_1, \ldots, q_{2n}$ are determined by p by the argument above;

we find that there exists a point q such that the unique divisor $L_1 \in W^1_{2,2nl-3}(p,q)$ passes through q_1. Now $D_1 + L_1 + C^1_1 + \cdots + C^{n-1}_1 \in |\mathcal{O}_{R_1}(nH)|$ and hence $(D_1 + L_1 + C^1_1 + \cdots + C^{n-1}_1)E = (2nl-3)q + 3p + q_1 + \cdots + q_{2n} \in |\mathcal{O}_E(nH)|$. This implies that there exists on R_2 a divisor

$$L_2 \sim \sigma_2 + (nl-1)F_2 \sim (nH)|_{R_2} - D_2 - C^1_2 - \cdots - C^{n-1}_2$$

cutting on E the divisor $(2nl-3)q + 2p + q_2$. Moreover, L_2 is uniquely determined by the equality $\dim(|O_E(\sigma_2 + (nl-1)F_2)| = \dim(|\mathcal{O}_{R_2}(\sigma_2 + (nl-1)F_2)|) = 2nl-1$. Now, if $C_i = C^1_i \cup \cdots \cup C^{n-1}_i \cup D_i \cup L_i \subset R_i$, then there exist only finitely many curves like $C = C_1 \cup C_2 \subset R$ and passing through a fixed general point $p \in E$. If p varies on E, then the curves C, constructed in this way, move in a one parameter family of curves $\mathscr{W} \subset |\mathcal{O}_R(n)|$. By construction and by Lemma 3.2, the curve C has a space quadruple point of analytic equations (5) at p and nodes or tacnodes at points $q, q_1, \ldots, q_{2n}$, as in the statement. It remains to prove that, if p is general or, equivalently, if $[C] \in \mathscr{W}$ is general, then the singularities of C on $R\backslash E$ are nodes. Since $D_iL_i = D_iC_i^j = 1$, for every i and j, this is equivalent to showing that L_i intersects transversally C_i^j, for every i and j. If $n = 1$ there is nothing to show. Just to fix ideas, we prove the statement for $n = 2$. Our argument trivially extends to the general case. Let $[C] \in \mathscr{W} \subset |\mathcal{O}_R(2)|$ be a general element. First observe that, since $\mathscr{W}$ is contained in the equisingular deformation locus $ES(C) \subset |\mathcal{O}_R(2H)|$ of C, it follows that, if D is the curve corresponding to a point $[D]$ of the tangent space $T_{[C]}\mathscr{W}$, then D has at q a $(2nl - 4)$-tacnode (cf. [12, Proof of Theorem 3.3]). Now assume that L_1 intersects C_1^1 at $r \le 2l - 1$ points $\{x_i\}$. Let $m_i \ge 1$ be the intersection multiplicity of L_1 with C_1^1 at x_i. Then, the analytic equation of C at x_i is $y^2 = x^{2m_i}$ and, by [6, Proposition (5.6)], the localization at x_i of the equisingular deformation ideal of C is (y, x^{2m_i-1}). Thus D must be tangent to C_1^1 at every point x_i with multiplicity $2m_i - 1$. Similarly, D must contain the intersection point of C_1^1 with D_1. It follows that the cardinality of intersection of $D|_{R_1}$ and C_1^1 is given by

$$\sum_i (2m_i - 1) + 1 = 2l - 1 + \sum_i m_i - r + 1 > 2l, \quad \text{if } m_i \ge 2 \text{ for some } i.$$

We deduce that $C_1^1 \subset D|_{R_1}$. Using again that D passes through every node of C and that, by Lemma 3.2 and Remark 3.7, the curve $D|_{R_i}$ must be tangent to E at p, for $i = 1, 2$, we obtain that D contains the points $C_1^1 \cap E$, $C_2^1 \cap D_2$ and p of D_2. Thus, since the intersection number $D_2D|_{R_2} = 2$, we have that $D_2 \subset D|_{R_2}$. It follows that the analytic equations of $D \subset \mathscr{X}$ at p are given by (12), where $c = 0$ and $u_z'(x, y, z) = zu_z''(x, y, z)$. In particular, we find that $D|_{R_1}$ has a node at p with one branch tangent to E and the other one tangent to D_1. Again, we find that $D|_{R_1}$ contains at least three points of D_1, counted with multiplicity, and hence $D_1 \subset D|_{R_1}$. This implies, by repeating the same argument, that $C_2^1 \subset D|_{R_2}$. Thus, for every $i = 1, 2$, we have that $D|_{R_i} = D_i' \cup D_i \cup C_i^1$, where $D_i' \sim \sigma_i + (2l - 1)F_i$, D_i' is tangent to E at p with multiplicity 2 and at q with multiplicity $2l - 4$ and, finally, D_i' contains q_i. But there is a unique divisor in $|\sigma_i + (2l - 1)F_i|$ with these properties. We deduce that $D_i' = L_i$, for every $i = 1, 2$, $D = C$ and $T_{[C]}\mathscr{W} = \{[C]\}$, getting a contradiction. This completes the proof in the case $\mathrm{p} = 2\mathrm{l} + 1$.

We now consider the case $\mathrm{p} = 2\mathrm{l}$, where $R_1 \simeq R_2 \simeq \mathbb{F}_1$, $\mathcal{O}_{R_i}(H) = \mathcal{O}_{R_i}(\sigma_i + lF_i)$, with $\sigma_i^2 = -1$ and $F_i^2 = 0$, and $E \sim_{R_i} 2\sigma_i + 3F_i$, for every $i = 1, 2$. The proof of the lemma works as in the previous case, except for the construction of the curves $C_i^j \sim \sigma_i + F_i$. Let $p \in E$ be a general point and let $D_i \sim F_i$ be the fibre passing through p, for $i = 1, 2$. Because of the generality of p, the curve D_i intersects E at a further point, say q_{2n} if $i = 1$ and q_{2n-1} if $i = 2$. Now the curves in $|\sigma_2 + F_2|$ passing through q_{2n} cut out on E, outside q_{2n}, a g_2^1 having, because of the generality of p, four simple ramification points. Let C_2^{n-1} be one of the four curves in $|\sigma_2 + F_2|$ passing through q_{2n} and simply tangent to E at a further point $q_{2n-2} \ne q_{2n-1}, q_{2n}$. Then, denote by $C_1^{n-2} \sim \sigma_1 + F_1$ the unique curve tangent to E at q_{2n-2} and let q_{2n-3} be the further intersection point of C_1^{n-2} with E. Now repeat the same argument until you get curves C_i^j, with $i = 1, 2$ and $j = 1, \dots, n - 1$, and points $q_1, \dots, q_{2n}$ satisfying relations (24)–(27). Again by the generality of p, there exists a point q such that the unique (smooth and irreducible) divisor $L_2 \in |\sigma_2 + (nl - n)F_2|$, tangent to E at p with multiplicity 2 and at q with multiplicity $2nl - 2n - 2$, passes through q_1. It follows that there exists a unique (smooth and irreducible) divisor $L_1 \in |\sigma_1 + (nl - n)F_1|$ passing through q_{2n-1} and tangent to E at p and q with multiplicity 2 an $2nl - 2n - 2$ respectively. The curve we constructed has tacnodes, nodes and a space quadruple point of analytic equations (5) on E, as desired. To see that the singularities of this curve outside E are nodes, argue as in the case $\mathrm{p} = 2\mathrm{l} + 1$. $\square$

We may now prove Theorem 1.1.

Proof of Theorem 1.1. Let T^1_R be the first cotangent bundle of R, defined by the standard exact sequence

$$0 \to \Theta_R \to \Theta_{\mathbb{P}^{\mathrm{p}}}|_R \to \mathscr{N}_{R|\mathbb{P}^{\mathrm{p}}} \to T^1_R \to 0. \tag{29}$$

Since R is a variety with normal crossings, by [10, Section 2], we know that T^1_R is locally free of rank one. In particular, $T^1_R \simeq \mathscr{N}_{E|R_1} \otimes \mathscr{N}_{E|R_2}$ is a degree 16 line bundle on E. Fix a general divisor $\xi_1 + \cdots + \xi_{16} \in |T^1_R|$. Since the family of curves constructed in the previous lemma is a one parameter family, we may always assume that there exists a curve $C = C_1 \cup C_2 \in |\mathscr{O}_R(nH)|$, with $C_i = C^1_i \cup \cdots \cup C^{n-1}_i \cup D_i \cup L_i$, as in the statement of Lemma 4.1, such that $q_{2n} = \xi_1$ and $q_j \neq \xi_k$, for every $j \leq 2n - 1$ and $k \leq 16$. Now, by [3, Corollary 1], we have that the induced map $H^0(R, \mathscr{N}_{R|\mathbb{P}^{\mathrm{p}}}) \to H^0(R, T^1_R)$ is surjective. By [3, Theorems 1 and 2] and related references (precisely, [10, Remark 2.6] and [16, Section 2]) and because of the generality of $\xi_1 + \cdots + \xi_{16} \in |T^1_R|$, it follows that there exists a deformation $\mathscr{S} \to T$ of $\mathscr{S}_0 = R$ whose general fiber is a smooth projective $K3$ surface $\mathscr{S}_t$ in $\mathbb{P}^{\mathrm{p}}$ with $\mathrm{Pic}(\mathscr{S}_t) \cong \mathbb{Z}[\mathscr{O}_{\mathscr{S}_t}(1)] \cong \mathbb{Z}[H]$ and such that $\mathscr{S}$ is singular exactly at the points $\xi_1, \dots, \xi_{16} \in E$. Let $\mathscr{X} \to T$ be the smooth family of surfaces obtained by blowing-up $\xi_1, \dots, \xi_{16}$ and by contracting the corresponding exceptional components on R_2, in such a way that $\mathscr{X}_0 = R_1 \cup \tilde{R}_2$, where $\tilde{R}_2$ is the blowing-up of R_2 at the points $\{\xi_1, \dots, \xi_{16}\}$, with new exceptional curves $E_1, \dots, E_{16}$. Let us denote by $\tilde{C}$ the proper transform of C and by $\pi^*(C) = \tilde{C} \cup E_1$ the pull-back of C with respect to $\pi : \mathscr{X} \to \mathscr{S}$. Now $\pi^*(C)$ has one more node at the point $x \in E_1 \cap \tilde{C}$ on $\tilde{R}_2 \backslash E$. We want to prove the existence of irreducible curves $C_t \in |\mathscr{O}_{\mathscr{X}_t}(nH)|$ with the desired singularities by deforming of the curve $\pi^*(C)$. The irreducibility of C_t easily follows from the fact that $\mathscr{O}_{\mathscr{X}_t}(H)$ is indivisible.

We first consider the case $\mathrm{p} = 2l + 1$. In this case the singularities of the curve $\pi^*(C)$ are given by

- $2(n-1)(nl-1)$ nodes $y^i_1, \dots, y^i_{(n-1)nl}$, on $R \backslash E$, arising from the intersection of the curves C^j_i, for $1 \leq j \leq n - 1$, with L_i, for every $i = 1, 2$;
- a node at $x \in E_1$;
- $2n - 2$ nodes $z^i_1, \dots, z^i_{n-1}$, $i = 1, 2$, arising from the intersection of D_i with C^j_i, for every $j \leq n - 1$ and $i = 1, 2$;
- $2n$ nodes at $q_1, \dots, q_{2n} \in E$, where now $q_{2n} = E_1 \cap E$;
- a $(2nl - 3)$-tacnode at q and
- a space quadruple point of analytic equations (5) at p.

Now, as we already observed in the proof of the previous lemma, the tangent space $T_{[\pi^*(C)]}ES(\pi^*(C))$ to the equisingular deformation locus of $\pi^*(C)$ in $|\mathcal{O}_{\mathcal{X}_0}(nH)|$ is contained in the linear system of divisors $D = D^1 \cup D^2 \in |\mathcal{O}_{\mathcal{X}_0}(nH)|$ passing through the nodes of $\pi^*(C)$ on $\mathcal{X}_0 \backslash E$ (x included); having a $(2nl-4)$-tacnode at the point q of $\pi^*(C)$ and having local analytic equations given by (12) at p. This implies that, if $D \in T_{[\pi^*(C)]}ES(\pi^*(C))$, then $D|_{R_2}$ contains the irreducible component of $\tilde{C}$ passing through x. It follows that $E_1 \subset D|_{R_2}$ and so on, until, doing the same local analysis of D at p as in the proof of the previous lemma, we find that

$$\dim(T_{[\pi^*(C)]}ES(\pi^*(C))) = \{[\pi^*(C)]\}.$$

Using that the nodes of $\pi^*(C)$ at the points q_i are trivially preserved by every section of $H^0(C, \mathcal{N}_{\pi^*(C)|\mathcal{X}_0})$, we deduce the injectivity of the standard morphism

$$\Phi : H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}) \to T,$$

where

$$T = \bigoplus_{j,i} T^1_{\pi^*(C),y^i_j} \bigoplus_{j,i} T^1_{\pi^*(C),z^i_j} \oplus T^1_{\pi^*(C),x} \oplus T^1_{\pi^*(C),q} \oplus T^1_{\pi^*(C),p}.$$

In particular, we have that Φ has image of dimension

$$\dim(Im(\Phi)) = h^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}) = 2n^2l + 2 = \dim(|\mathcal{O}_{\mathcal{X}_t}(nH)|) + 1.$$

Morever, by Corollary 3.10 and by [12, Corollary 3.6], we know that the image of the morphism Φ must be contained in

$$T' = \bigoplus_{j,i} T^1_{\pi^*(C),y^i_j} \bigoplus_{j,i} T^1_{\pi^*(C),z^i_j} \oplus T^1_{\pi^*(C),x} \oplus H_p \oplus H_q \subset T,$$

where $H_p \subset T^1_{\pi^*(C),p}$ and $H_q \subset T^1_{\pi^*(C),q}$ are linear spaces of dimension 5 and $2nl-3$ respectively. We first study the map Φ in the case that $2nl-3=1$ i.e. for $n=2$ and $l=1$ or $n=1$ and $l=2$. In this case the curve $\pi^*(C)$ has a 1-tacnode, i.e. a node, at $q \in E$ and $H_q = T^1_{\pi^*(C),q}$. Using again that every node of $\pi^*(C)$ on E is trivially preserved by every section of $H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0})$, we obtain that the induced morphism

$$H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}) \to \bigoplus_{j,i} T^1_{\pi^*(C),y^i_j} \bigoplus_{j,i} T^1_{\pi^*(C),z^i_j} \oplus T^1_{\pi^*(C),x} \oplus H_p$$

is injective. In fact this morphism is also surjective by virtue of the equality

$$2(n-1)(nl-1) + 2n - 2 + 1 + 5 = 2n^2l - 2nl - 2n + 2 + 2n + 4 = 2n^2l + 2.$$

By versality and by the proof of Theorem 3.9, it follows that we may deform $\pi^*(C)$ to an irreducible curve $C_t \in |\mathcal{O}_{\mathcal{X}_t}(nH)|$ such that $T_{C_t}ES(C_t) = 0$, by preserving all nodes of $\pi^*(C)$ on $\mathcal{X}_0 \backslash E$ and by deforming the singularity of $\pi^*(C)$ at p to an ordinary triple point. Now we study the morphism Φ under the assumption that $2nl - 3 \geq 2$. In this case Φ is not surjective. More precisely, we have that

$$\dim(T') = 2(n-1)(nl-1) + 2n - 2 + 1 + 5 + 2nl - 3 = 2n^2 l + 3$$

and $Im(\Phi) = \Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}))$ is a hyperplane in T', containing the image $\Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0}))$ of $H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0})$ as a codimension 1 subspace. Now, using on $T^1_{\pi^*(C),p}$ the affine coordinates $(b_1, b_2, b_3, a_1, a_2, a_3, a_4)$ introduced in the proof of Lemma 3.2, by the proof of Theorem 3.9, by [12, Proof of Theorem 3.3] and by a straightforward dimension count, we have that $\Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0}))$ coincides with the $(2n^2 l + 1)$-plane

$$\bigoplus_{j,i} T^1_{\pi^*(C), y^i_j} \bigoplus_{j,i} T^1_{\pi^*(C), z^i_j} \oplus T^1_{\pi^*(C), x} \oplus \Gamma_p \oplus \Gamma_q,$$

where $\Gamma_p \subset H_p \subset T^1_{\pi^*(C),p}$ is the 4-plane of equations $b_1 = b_2 = b_3 = 0$ and $\Gamma_q \subset H_q \subset T^1_{\pi^*(C),q}$ is the $(2nl-4)$-plane parametrizing $(2nl-3)$-nodal curves. In particular, we have that

$$\Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}})) = \bigoplus_{j,i} T^1_{\pi^*(C), y^i_j} \bigoplus_{j,i} T^1_{\pi^*(C), z^i_j} \oplus T^1_{\pi^*(C), x} \oplus \Omega,$$

where Ω is a hyperplane in $H_p \oplus H_q$ such that $\Gamma_p \oplus \Gamma_q \subset \Omega$. It trivially follows that the projection maps $\rho_p : \Omega \to H_p$ and $\rho_q : \Omega \to H_q$ are surjective. Now, by the proof of Theorem 3.9, we know that the locus of curves with a triple point in H_p is the smooth curve T having equations $a_1 = a_2 = a_3 = b_2 = b_3 = 4b_1 - a_4^2 = 0$ and intersecting Γ_p only at $\underline{0}$. Similarly, by [12, proof of Theorem 3.3], for every $(m-1)$-tuple of non-negative integers $d_2, \ldots, d_m$ such that $\sum_{k=2}^m (k-1)d_k = 2nl - 4$, the locus $V_{1^{d_2}, 2^{d_3}, \ldots, m-1^{d_m}} \subset H_q$ of points corresponding to curves with d_k singularities of type A_{k-1}, for every k, is a reduced (possibly reducible) curve intersecting Γ_q only at $\underline{0}$. It follows that, for every $(m-1)$-tuple of non-negative integers $d_2, \ldots, d_m$ such that $\sum_{k=2}^m (k-1)d_k = 2nl - 4$, the locus $(T \times V_{1^{d_2}, 2^{d_3}, \ldots, m-1^{d_m}}) \cap \Omega$ is a reduced (possibly reducible) curve whose parametric equations may be explicitly computed by arguing exactly as in [1, proof of Lemma 4.4, p. 381–382]. This proves, by versality, that we may deform $\pi^*(C)$ to an irreducible curve $C_t \in |\mathcal{O}_{\mathcal{X}_t}(nH)|$, by preserving all nodes of $\pi^*(C)$ at y^i_j, z^i_j and x, by deforming the singularity of $\pi^*(C)$ at p to an ordinary triple point and the $(2nl-3)$-tacnode of $\pi^*(C)$ to $d_k \geq 0$ singularities of type A_{k-1} for every m-tuple of integers d_k such that $\sum_{k=1}^m ((k-1)d_k) = 2nl - 4$. In particular, if $k = 2$ and $d_2 = 2nl - 4$, the corresponding curve C_t is an elliptic curve in $|\mathcal{O}_{\mathcal{X}_t}(nH)|$ with an ordinary triple point and nodes as singularities. Finally, by the injectivity of the morphism Φ, we have that, if $ES(C_t)$ is the equisingular deformation locus of C_t in $|\mathcal{O}_{\mathcal{X}_t}(nH)|$, then $\dim(T_{[C_t]} ES(C_t)) = 0$ and this implies that the singularities of C_t may be smoothed independently, i.e. that the standard morphism $H^0(C_t, \mathcal{N}_{C_t|\mathcal{X}_t}) \to T^1_{C_t}$ is surjective. This proves the theorem for p = 2l + 1.

In the case p = 2l the singularities of the curve $\pi^*(C)$ are given by

- $2(n-1)(nl-n)$ nodes $y_1^i, \ldots, y_{(n-1)nl}^i$, on $R \setminus E$, arising from the intersection of the curves C_i^j, for $1 \le j \le n-1$, with L_i, for every $i = 1, 2$;
- a node at $x \in E_1$;
- $2n-2$ nodes $z_1^i, \ldots, z_{n-1}^i$, $i = 1, 2$, arising from the intersection of D_i with C_i^j, for every $j \le n-1$ and $i = 1, 2$;
- $n-1$ simple tacnodes at $q_2, q_4, \ldots, q_{2n-2} \in E$ and nodes at $q_1, q_3, \ldots, q_{2n-1}, q_{2n}$, where now $q_{2n} = E_1 \cap E$;
- a $(2nl-2n-2)$-tacnode at q;
- a space quadruple point of analytic equations (5) at p.

With the same argument as above, one may prove that the curve $\pi^*(C)$ may be deformed to a curve $C_t \in |\mathcal{O}_{\mathcal{X}_t}(nH)|$, by preserving all nodes of $\pi^*(C)$ at y_j^i, z_j^i and x, by deforming the singularity of $\pi^*(C)$ at p to an ordinary triple point, every simple tacnode of $\pi^*(C)$ on E to a node and the $(2nl-2n-2)$-tacnode of $\pi^*(C)$ to d_k singularity of type A_{k-1} for every m-tuple of non negative integers d_k such that $\sum_{k=1}^m((k-1)d_k) = 2nl-2n-3$. The curve C_t obtained in this way has the desired singularities, is a reduced point for the equisingular deformation locus and its singularities may be smoothed independently. Finally, if we choose $k = 2$ and $d_2 = 2nl-2n-3$, then C_t is an elliptic irreducible curve with a triple point and nodes as singularities. □

COROLLARY 4.2. *Let (S, H) be a general primitively polarized $K3$ surface of genus $\mathrm{p} = p_a(H)$ as above. Then, for every $1 \le g \le p_a(nH) - 3$ and for every $(\mathrm{p}, \mathrm{n}) \neq (4, 1)$, there exist reduced and irreducible curves in $|nH|$ of geometric genus g with an ordinary triple point and nodes as singularities and corresponding to regular points of their equisingular deformation locus.*

In accordance with [4], we do not expect the existence of rational curves in $|\mathcal{O}_S(nH)|$ with a triple point. When $n = 1$ and $\mathrm{p} \ge 5$, Theorem 1.1 implies that the family in $|H|$ of curves with a triple point and δ_k singularities of type A_{k-1} is non-empty whenever it has expected dimension at least equal to one. The precise statement is the following.

COROLLARY 4.3. *Let (S, H) be a general primitively polarized $K3$ surface of genus $\mathrm{p} = p_a(H) \ge 5$. Then, for every $(m-1)$-tuple of non-negative integers $d_2, \ldots, d_m$ such that*

$$\sum_{k=2}^{m}(k-1)d_k = \mathrm{p} - 5 = \dim(|\mathrm{H}|) - 5,$$

there exist reduced irreducible curves C in the linear system $|H|$ on S such that:

- *C has an ordinary triple point, a node and d_k singularities of type A_{k-1}, for every $k = 2, \ldots, m$, and no further singularities;*

- *C corresponds to a regular point of the equisingular deformation locus $ES(C)$. Equivalently,* $\dim(T_{[C]}ES(C)) = 0$.

Finally, the singularities of C may be smoothed independently.

Remark 4.4. *The case* $\mathrm{p} = 4$ *and* $n = 1$ *is the only case where the existence of elliptic curves with a triple point is expected but it is not treated in this paper. In this case, with the notation above, it is easy to show the existence of a unique curve* $C \in |\mathcal{O}_{\mathcal{X}_0}(H)|$ *with a space quadruple point analytically equivalent to* (5). *Because of the unicity, the argument we used in the proof of Theorem* 1.1 *to compute the dimension of the tangent space to the equisingular deformation locus of C does not apply. Actually, we expect that* $\dim(T_{[C]}ES(C)) > 0$. *Nevertheless, it is easy to prove that C deforms to curves* $C_t \in |\mathcal{O}_{X_t}(H)|$ *with a triple point as singularity on the general K*3 *surface* $\mathcal{X}_t$. *But describing the equisingular deformation locus* $ES(C_t)$ *from the scheme-theoretic point of view seems to us to be a very difficult problem. The case* $\mathrm{p} = 4$ *and* $n = 1$ *will be treated in detail in an upcoming article on related topics.*

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