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Algebraic Geometry — *Points of order two on theta divisors*, by Valeria Ornella Marcucci and Gian Pietro Pirola, communicated on 9 March 2012.

Abstract. — We give a bound on the number of points of order two on the theta divisor of a principally polarized abelian variety A. When A is the Jacobian of a curve C the result can be applied in estimating the number of effective square roots of a fixed line bundle on C.

Key words: Abelian variety, theta divisor, torsion points.

Mathematics Subject Classification: 14K25 (primary); 14H40 (secondary).

Introduction

In this paper we give an upper bound on the number of 2-torsion points lying on a theta divisor of a principally polarized abelian variety. Given any principally polarized abelian variety A of dimension g and symmetric theta divisor $\Theta \subset A$, Θ contains at least $2^{g-1}(2^g - 1)$ points of order two, the odd theta characteristics. Moreover, in [Mum66] and [Igu72, Chapter IV, Section 5] it is proved that Θ cannot contain all points of order two on A.

In this work we use the projective representation of the theta group to prove the following:

Given a principally polarized abelian variety A, any translated $t_a^*\Theta$ of a theta divisor $\Theta \subset A$ contains at most $2^{2g} - 2^g$ points of order 2 ($2^{2g} - (g+1)2^g$ if $t_a^*\Theta$ is irreducible and not symmetric).

Our bound is far from being sharp. In [SM94] Salvati Manni proved that there are at least $g(2g+1)$ points of order two outside an irreducible theta divisor. This gives a better bound for $g < 7$. We conjecture that the right estimate should be $2^{2g} - 3^g$ as in the case of a product of elliptic curves.

When A is the Jacobian of a curve C the result can be applied in estimating the number of effective square roots of a fixed line bundle on C (cf. Section 2).

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1. Main result

In this section we prove our main result.

THEOREM 1.1. *Let A be a principally polarized abelian variety of dimension g and let Θ be a symmetric theta divisor.*

1. *For each $a \in A$ there are at most $2^{2g} - 2^g$ points of order two lying on $t_a^*\Theta$.*
2. *Let $a \in A$ and assume that Θ is irreducible and $t_a^*\Theta$ is not symmetric with respect to the origin. Then there are at most $2^{2g} - (g+1)2^g$ points of order two lying on $t_a^*\Theta$.*

PROOF. Denote by $(K, \langle \cdot, \cdot \rangle)$ the group of 2-torsion points on A with the perfect pairing induced by the polarization. Let

$$\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$$

be a basis of K over the field of order two such that

$$\langle a_i, b_j \rangle = \delta_{ij}, \quad \langle a_i, a_j \rangle = 0, \quad \langle b_i, b_j \rangle = 0,$$

and let

$$H := \langle a_1, \ldots, a_g \rangle \tag{1}$$

be the subgroup of K generated by the elements $a_1, \ldots, a_g$. Consider the projective morphism $\varphi : A \to \mathbb{P}^{2^g - 1}$ associated to the divisor 2Θ. By the construction of the projective representation of the theta group $K(2\Theta)$ (see [Mum66], [Kem91, Chapter 4] and [Kem89]), we know that the elements of $\varphi(H)$ are a basis of the projective space. In the same way, the images of the elements of a coset H_b of H in K generate the projective space $\mathbb{P}^{2^g-1}$.

Suppose by contradiction that there exists a subset $S \subset K$ such that all points of S lie on $t_a^*\Theta$ and $|S| > 2^{2g} - 2^g$. By the previous argument, since $H_b \subset S$ for some b, the points of $\varphi(S)$ generate the entire projective space $\mathbb{P}^{2^g-1}$. On the other hand, by the Theorem of the Square ([Mum08, Chapter II, Section 6, Corollary 4]),

$$t_a^*\Theta + t_{-a}^*\Theta \equiv 2\Theta.$$

It follows that the points of $\varphi(S)$ lie on an hyperplane of $\mathbb{P}^{2^g-1}$. This proves (1).

Now we prove the second part. Suppose by contradiction that there exists a subset $S \subset K$ such that all points of S lie on $t_a^*\Theta$ and $|S| > 2^{2g} - (g+1)2^g$. We claim that

$$\boxed{\text{the points in } \varphi(S) \text{ lie on a } 2^g - g - 2\text{-plane in } \mathbb{P}^{2^g-1}.} \tag{$*$}$$

Given a point $\varepsilon \in S$, it holds also $\varepsilon \in t_{-a}^*\Theta$. Thus $S \subset t_a^*\Theta \cap t_{-a}^*\Theta$. If $t_a^*\Theta$ is not symmetric and irreducible, $t_a^*\Theta \cap t_{-a}^*\Theta$ has codimension 2 in A and we can consider the natural exact sequence

$$0 \to \mathcal{O}_A(-2\Theta) \to \mathcal{O}_A(-t_{-a}^*\Theta) \oplus \mathcal{O}_A(-t_a^*\Theta) \to I_{t_a^*\Theta \cap t_{-a}^*\Theta} \to 0;$$

by tensoring it with $\mathcal{O}_A(2\Theta)$ we get

$$0 \to \mathcal{O}_A \to \mathcal{O}_A(t_a^*\Theta) \oplus \mathcal{O}_A(t_{-a}^*\Theta) \to I_{t_a^*\Theta \cap t_{-a}^*\Theta} \otimes \mathcal{O}_A(2\Theta) \to 0.$$

Passing to the corresponding sequence on the global sections, we have

$$\begin{aligned} (2) \qquad & 0 \to H^0(A, \mathcal{O}_A) \to H^0(A, \mathcal{O}_A(t_a^*\Theta)) \oplus H^0(A, \mathcal{O}_A(t_{-a}^*\Theta)) \\ & \to H^0(I_{t_a^*\Theta \cap t_{-a}^*\Theta} \otimes \mathcal{O}_A(2\Theta)) \to H^1(A, \mathcal{O}_A) \to 0, \end{aligned}$$

since, by the Kodaira vanishing theorem (see e.g. [GH94, Chapter 1, Section 2]),

$$H^1(A, \mathcal{O}_A(t_a^*\Theta)) = H^1(A, \mathcal{O}_A(t_{-a}^*\Theta)) = 0.$$

It follows that

$$\dim H^0(I_{t_a^*\Theta \cap t_{-a}^*\Theta} \otimes \mathcal{O}_A(2\Theta)) = g + 1.$$

Thus the points in $\varphi(t_a^*\Theta \cap t_{-a}^*\Theta)$ lie on a $2^g - g - 2$-plane of $\mathbb{P}^{2^g - 1}$ and the claim $(*)$ is proved.

To conclude the proof of (2) we notice that if $|S| > 2^{2g} - (g+1)2^g$ then $|S \cap H_b| > 2^g - (g+1)$ for some coset H_b of H (see (1)). Then it follows that $\varphi(S)$ contains at least $2^g - g$ independent points and we get a contradiction. □

Remark 1.2. One might expect the right bound to be $2^{2g} - 3^g$ and that this is realized only in the case of a product of elliptic curves.

Remark 1.3. The argument of Theorem 1.1 can be also used to obtain a bound on the number of n-torsion points (with $n > 2$) lying on a theta divisor.

2. Applications

In this section we apply Theorem 1.1 to the case of Jacobians. This gives a generalization of [MP, Proposition 2.5].

Proposition 2.1. *Let C be a curve of genus g and M be a line bundle of degree $d \le g - 1$. Given an integer $k \le g - 1 - d$, for each $L \in \mathrm{Pic}^{2k}(C)$ there are at least 2^g line bundles $\eta \in \mathrm{Pic}^k(C)$ such that $\eta^2 \simeq L$ and $h^0(\eta \otimes M) = 0$.*

Proof. We prove the statement for $M \simeq \mathcal{O}_C$ and $k = g - 1$. The general case follows from this by replacing L with $M^2 \otimes L \otimes \mathcal{O}_C(p)^{2n}$, where p is an arbitrary point of C and $n := g - 1 - k - d$. Denote by Θ the divisor of effective line bundles of degree $g-1$ in $\mathrm{Pic}^{g-1}(C)$. Given the morphism

$$m_2 : \mathrm{Pic}^{g-1}(C) \to \mathrm{Pic}^{2g-2}(C)$$
$$\eta \mapsto \eta^2,$$

we want to prove that $|m_2^{-1}(L) \cap \Theta| \le 2^{2g} - 2^g$. Let $\alpha \in m_2^{-1}(L)$, we have

$$m_2^{-1}(L) = \{\alpha \otimes \sigma \text{ s.t. } \sigma^2 = \mathcal{O}_C\}.$$

If $|m_2^{-1}(L) \cap \Theta| > 2^{2g} - 2^g$, then there are more than $2^{2g} - 2^g$ points of order two lying on a translated of a symmetric theta divisor of $J(C)$ and, by (1) of Theorem 1.1, we get a contradiction. □

Remark 2.2. If we apply Proposition 2.1 to $M = \mathcal{O}_C$, $L = \omega_C$, we get that on a curve of genus g there are at most $2^{2g} - 2^g$ effective theta characteristics. We notice that when $g = 2$ they are the 6 line bundles of type $\mathcal{O}_C(p)$ where p is a Weierstrass point. When $g = 3$ and C is not hyperelliptic, they correspond to the 28 bi-tangent lines to the canonical curve.

Corollary 2.3. *Let C be a curve of genus g and $M_1, \dots M_N$ be a finite number of line bundles of degree $d \le g - 1$. Given an integer $k \le g - 1 - d$, if η is a generic line bundle of degree k such that $h^0(\eta^2) > 0$, then*

$$h^0(\eta \otimes M_i) = 0 \quad \forall i = 1, \dots, N.$$

Proof. Let

$$\Lambda := \{\eta \in \mathrm{Pic}^k(C) : h^0(\eta^2) > 0\},$$

and, for each $i = 1, \dots N$, consider its closed subset

$$\Lambda_i := \{\eta \in \Lambda : h^0(M_i \otimes \eta) > 0\}.$$

We remark that Λ is a connected 2^{2g}-étale covering of the image of the $2k$-th symmetric product of C in $\mathrm{Pic}^{2k}(C)$. By Proposition 2.1, for each effective $L \in \mathrm{Pic}^{2k}(C)$ there exists $\eta \in \Lambda \backslash \Lambda_i$ such that $\eta^2 \simeq L$. It follows that Λ_i is a proper subset of Λ. Since Λ is irreducible, also the set

$$\bigcup_{i=1}^{N} \Lambda_i = \{\eta \in \mathrm{Pic}^k(C) : h^0(M_i \otimes \eta) > 0 \text{ for some } i\}$$

is a proper closed subset of Λ. □

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