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Algebraic Geometry — *Quotients of Fano surfaces*, by XAVIER ROULLEAU, communicated on 9 March 2012 by Carlo Sbordone.

ABSTRACT. — Fano surfaces parametrize the lines of smooth cubic threefolds. In this paper, we study their quotients by some of their automorphism sub-groups. We obtain in that way some interesting surfaces of general type.

KEY WORDS: Surfaces of general type, Fano surface of a cubic threefold, Quotient singularities, Intersection theory on normal surfaces.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 14J29 (primary); 14J17, 14J50, 14C17 (secondary).

INTRODUCTION

It is classical to study quotients of surfaces by automorphism groups in order to obtain new surfaces. For example, Godeaux obtained one of the first surfaces of general type with vanishing geometric genus by taking the quotient of a quintic hypersurface in $\mathbb{P}^3$ by an order 5 fixpoint free action.

In this paper, we study quotients of Fano surfaces. These surfaces are by definition modular varieties: they parametrize the lines on smooth cubic threefolds. This modular property allows to understand them very well. In fact, we can handle the Fano surface S of a cubic threefold $F \hookrightarrow \mathbb{P}^4$ almost like a hypersurface in $\mathbb{P}^3$: we can think of F as giving the equation of S from which we can read of the properties of the irregular surface S. In particular, we can obtain the classification of the automorphism groups of these surfaces. In the present paper we study the minimal desingularisation of the quotients of these surfaces by some subgroups of automorphisms. We compute their Chern numbers c_1^2, c_2, irregularity q and geometric genus p_g, their minimality and their Kodaira dimension κ.

Using the classification of cyclic groups of prime order acting on cubic threefolds done in [11], we give in the following table the classification of the minimal desingularisation of the quotients of Fano surfaces by groups of prime order, and we give examples of quotients by some automorphisms of order 4 and 15:

O	Type	c_1^2	c_2	q	p_g	χ	g	Singularities	Min	κ
2	I	18	54	1	6	6	3	$27A_1$	yes	2
2	II	12	12	3	4	2		A_1	yes	2
3	III(1)	15	9	3	4	2			yes	2
3	III(2)	15	33	1	4	4	4	$9A_2$	yes	2
3	III(3)	6	54	0	4	5		$27A_{3,1}$	yes	2
3	III(4)	-3	3	2	1	0			no	0
4	IV(1)	6	18	1	2	2	4	$6A_1 + A_3$	yes	2
4	IV(2)	0	36	1	3	3	1	$12A_1 + 3A_3$	yes	1
5	V	9	15	1	2	2	4	$2A_4$	yes	2
11	XI	-5	17	0	0	1		$5A_{11,3}$	no	$-\infty$
15	XV	-4	16	0	0	1		$5A_{3,1} + 2A_{15,4}$	no	$-\infty$

The first and second column give the order and type of the automorphism, the column g is the genus of the fibration onto the Albanese variety when it is an elliptic curve, the column Singularities gives the number and type of singularities on the quotient surface, Min indicates if the minimal desingularisation surface is minimal. For the surfaces which are quotient by the following groups G, we obtain:

G	c_1^2	c_2	q	p_g	χ	g	Singularities	Min	κ
$(\mathbb{Z}/2\mathbb{Z})^2$ (type I)	5	43	0	3	4		$24A_1$	yes	2
S_3 (type I)	3	45	0	3	4		$27A_1$	yes	2
$(\mathbb{Z}/3\mathbb{Z})^2$	5	19	1	2	2	2	$6A_2$	yes	2
$\mathbb{D}_2$ (type II)	-3	3	2	1	0			no	0
$\mathbb{D}_3$ (type II)	0	12	1	1	1	1	$A_1 + 3A_2$	yes	1
$\mathbb{D}_5$ (type II)	-2	2	1	0	0	0	A_1	no	$-\infty$
$S_3 \times \mathbb{Z}/3\mathbb{Z}$	1	23	0	1	2		$9A_1 + 3A_2$	yes	2

Where $\mathbb{D}_n$ is the dihedral group of order $2n$. In each of the cyclic and non-cyclic cases, we obtain surfaces of all Kodaira dimensions: rational, abelian, minimal elliptic and of general type.

The rather exceptional fact that Fano surfaces are modular varieties enables us know exactly which singularities are on the quotient surface. Moreover, the situation is so good that we can determine the four invariants c_1^2, c_2, q, p_g separately and then double-check our computations by using the Noether formula.

We use intersection theory on singular normal surfaces as defined by Mumford in [14]. In particular this intersection theory is applied in Propositions 20 and 22 in order to find the Kodaira dimension of some surfaces, and we think that this has independent interest.

Although there are a lot of papers on the subject, the fine classification of surfaces of general type with small birational invariants in not achieved, in particular for the irregular ones. Let us discuss the place of the surfaces we obtain in the geography of surfaces of general type.

The surfaces of type I are discussed in [20]. Some examples of irregular surfaces with $p_g = 4$ and birational canonical map are discussed in [6]. Our type II, III(1), III(2), III(3) surfaces have $p_g = 4$ too. The surfaces of type II is discussed in [6], the surface III(3) is described in [13], but our examples III(1) and III(2) are, to our knowledge, new. We think also that the surfaces of type IV(1) and V are new.

The surface coming from the group $G = S_3$ is a Horikawa surface [12]. The moduli space of surfaces coming from the group $G = (\mathbb{Z}/3\mathbb{Z})^2$ has recently be work out in [9].

Our last example is a surface with $K^2 = p_g = 1$. In [4] and [5], Catanese study the moduli of such surfaces, obtaining counter examples to the global Torelli Theorem.

The paper is divided as follows: in the first section, we remind classical results from intersection theory and computation of invariants of quotient surfaces, in the second we recall the known facts about Fano surfaces and in the third and fourth, we compute the invariants of the resolutions of the quotient surfaces.

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1. GENERALITIES ON QUOTIENTS AND INTERSECTION THEORY

Let us recall, mainly without proof, some well-known Lemmas for computing the invariants of the minimal resolution of the quotient of a surface S by an automorphism group G.

We will use intersection theory of $\mathbb{Q}$-Cartier divisors on compact normal surfaces as defined by Mumford [14], a good reference on that topic is Fulton's book [8]. Let Y be a normal surface and let $g : Z \to Y$ be a resolution of the singularities of Y. We denote by C_i, $i \in I$ the irreducible reduced components of the exceptional curves of g. The intersection matrix $(C_i C_j)_{i,j}$ is negative definite. For a divisor C on Y let $\overline{C}$ be the strict transform on Z of C. Let g^*C and a_i, $i \in I$ be the $\mathbb{Q}$-divisor and the positive rational numbers uniquely defined by:

$$\overline{C} = g^*C - \sum a_i C_i$$

and the relations $C_i g^* C = 0$ for all i. The intersection number $CC' \in \mathbb{Q}$ of C and C' is defined by $g^* C g^* C'$. It is bilinear and independent of g. Let K_Y be the canonical $\mathbb{Q}$-divisor on Y, then $K_Z = \overline{K}_Y$ and:

LEMMA 1. *Let $K_Z = g^* K_Y - \sum a_i C_i$ be the canonical divisor of Z. Then:*

$$K_Z^2 = K_Y^2 + \left(\sum a_i C_i\right)^2.$$

If all the components C_i of an exceptional divisor of the resolution $Z \to Y$ are (-2)-curves, then $a_i = 0$ for all of those C_i.

Let us suppose that there exists a smooth surface S and a finite automorphism group G such that Y is the quotient of S by G and let $\pi : S \to S/G = Y$ be the quotient map. For each reduced divisor R on the surface S, let H_R be the isotropy group of R:

$$H_R = \{g \in G / g_{|R} = Id_R\}.$$

Let $|E|$ denote the order of a set E; $|H_R|$ is the ramification index of the quotient map $\pi : S \to S/G$ over R. For a curve C on S/G, we have $\pi^* C = \sum_{R \subset \pi^{-1} C} |H_R| R$ and for another divisor C', we have:

$$CC' = \frac{1}{|G|} \pi^* C \pi^* C'.$$

We say that the divisor C on $Y = S/G$ is nef if $CC' \geq 0$ for all curves C'. We have:

LEMMA 2. *Let $K_{S/G}$ be the canonical $\mathbb{Q}$-divisor on S/G. Then:*

$$K_S = \pi^* K_{S/G} + \sum_R (|H_R| - 1) R$$

in particular:

$$K_{S/G}^2 = \frac{1}{|G|} \left(K_S - \sum_R (|H_R| - 1) R\right)^2.$$

If $K_S - \sum_R (|H_R| - 1) R$ is nef, then $K_{S/G}$ is nef. If $K_{S/G}$ is nef and $K_Z = g^* K_{S/G}$, then K_Z is nef.

For any integer $n \geq 1$, we define the stratum on S:

$$S_n = \{s / |Stab_G(s)| = n\},$$

where $Stab_G(s)$ is the stabilizer of the point s in S. Using the inclusion-exclusion principle and the multiplicativity property of étale maps for the Euler number e, we obtain:

LEMMA 3. *The Euler number of S/G is given by the formula:*

$$e(S/G) = \sum_{n\geq 1} \frac{n}{|G|} e(S_n) = \frac{1}{|G|}\Big(e(S) + \sum_{n\geq 2}(n-1)e(S_n)\Big).$$

The Euler number of the minimal resolution $Z \to S/G$ is the sum of $e(S/G)$ and the number of irreducible components of the exceptional curves of $Z \to S/G$.

Let Ω_X^i be the bundle of holomorphic i-forms on a smooth variety X and let $\omega_X = \bigwedge^{\dim X} \Omega_X$. We denote the irregularity by q_X and the geometric genus by $p_g(X)$.

LEMMA 4. *Let Z be the minimal resolution of the surface S/G. We have:*

$$H^0(Z, \Omega_Z^i) \simeq H^0(S, \Omega_S^i)^G.$$

In particular: $p_g(Z) = \dim H^0(S, \omega_S)^G$ and $q_Z = \dim H^0(S, \Omega_S)^G$.

PROOF. In [10], pp. 349–354, Griffiths gives a definition of differential forms for singular varieties. This notion coincide with the usual one when the variety X is smooth and we have $H^0(Z, \Omega_Z^i) \simeq H^0(X, \Omega_X^i)$ for any resolution of singularities Z of X. Moreover, by [10] formula (2.8), we have $H^0(S/G, \Omega_{S/G}^i) = H^0(S, \Omega_S^i)^G$, therefore: $H^0(Z, \Omega_Z^i) = H^0(S, \Omega_S^i)^G$. □

2. GENERALITIES ON FANO SURFACES

Here we recall the known facts about Fano surfaces. We use mainly the results of Clemens-Griffiths [7], Tyurin [21], [22], Bombieri Swinnerton-Dyer [2] and also [16], [17], [18] and [19].

Let $S \hookrightarrow G(2,5)$ be the Fano surface parametrizing the lines on a smooth cubic threefold $F \hookrightarrow \mathbb{P}^4$. The Chern numbers of S are $c_1^2 = 45$ and $c_2 = 27$. For a point s in S, we denote by $L_s \hookrightarrow F$ the corresponding line on F. There are 6 lines through a generic point of F. The closure C_s of the incidence:

$$\{t \,|\, s \neq t,\ L_t \textit{ cuts } L_s\}$$

is an ample connected divisor of genus 11 on S, with at most nodal singularities. It has the property that if a plane cuts F into three lines $L_s + L_t + L_u$, then $C_t + C_s + C_u$ is a canonical divisor K_S. In particular $3C_s$ is numerically equivalent to K_S and $C_s^2 = 5$.

The 5 dimensional space $H^0(\Omega_S)^*$ is the tangent space of the Albanese variety $Alb(S)$ of S. As the Albanese map of S is an embedding, we therefore consider the tangent space $T_{S,s}$ (for s in S) as a subspace of $H^0(\Omega_S)^*$. We have:

THEOREM 5 ([7], Tangent Bundle Theorem 13.37). *There exists an isomorphism of vector spaces:*

$$\phi : H^0(\Omega_S)^* \to H^0(F, \mathcal{O}(1))$$

such that for all s in S we have: $H^0(L_s, \mathcal{O}(1)) = \phi(T_{S,s})$.

In words: the tangent space to the point s, translated to the point 0 of $Alb(S)$, is identified by the linear map ϕ to the plane subjacent to the line $L_s \hookrightarrow \mathbb{P}^4$. This powerful Theorem has many important consequences:

THEOREM 6 ([18], [19]). *Let σ be an automorphism of S, let $d\sigma$ denotes its action on $H^0(\Omega_S)^*$. The element $d\sigma$ acts naturally on the cubic F and a point s of S is a fixed point of σ if and only if the line L_s is stable under the action of $d\sigma$.*

If a point s is a fixed point of σ, the action of $d\sigma_s : T_{S,s} \to T_{S,s}$ is given by the restriction of $d\sigma$ to the 2 dimensional vector space $H^0(L_s, \mathcal{O}(1))$.

The map $\sigma \to d\sigma$ is an isomorphism between the automorphism groups of S and F.

The following Lemma enables us to compute the geometric genus and irregularity of the quotient surfaces:

LEMMA 7 ([7], (10.14)). *The natural map $\bigwedge^2 H^0(\Omega_S) \to H^0(S, \omega_S)$ is an isomorphism.*

Let us fix some notations that we will use thereafter:

DEFINITION 8. Let σ be an automorphism of S and let $a_1, \ldots, a_k$, $k \leq 5$ the eigenvalues of $d\sigma$. We denote by V_{a_i} the eigenspace in $H^0(F, \mathcal{O}(1)) = H^0(\Omega_S)^*$ with eigenvalue a_i, and by $\mathbb{P}(V_{a_i}) \hookrightarrow \mathbb{P}^4$ its projectivisation.

Now we can read on the cubic F which points are fixed by σ: they are the stable lines L in F under the action of $d\sigma$. Let n be the order of the automorphism σ. In general, we know the action of $d\sigma$ on $F \hookrightarrow \mathbb{P}^4$, thus we know its action on $H^0(\Omega_S)^*$ only up to a n^{th} root of unity. But in our previous papers we computed these actions. Let us give an example. There is an obvious order two automorphism σ acting on the Fano surface of the cubic:

$$F = \{x_1^2 x_2 + G(x_2, x_3, x_4, x_5) = 0\}.$$

The lines which are stable by σ are:

i) the lines in the cone intersection of $\{x_2 = 0\}$ and F, parametrized by a smooth plane cubic curve $E \hookrightarrow S$,
ii) the 27 lines on the smooth intersection of F by $\{x_1 = 0\}$.

The automorphism $d\sigma$ acting on $H^0(\Omega_S)^*$ is:

$$f : x \to (x_1, -x_2, -x_3, -x_4, -x_5)$$

or $-f$. Let s be one of the 27 isolated points of σ. We have $T_{S,s} \subset \{x_1 = 0\}$ and as s is an isolated point of σ, $d\sigma_s$ acts by $x \to -x$ on $T_{S,s}$, therefore $d\sigma = f$. Such kind of order 2 automorphisms are called of type I; their trace on $H^0(\Omega_S)$ is -3. Let us recall:

PROPOSITION 9 ([18], Thm. 13). *There is a natural bijection between the set of elliptic curves $E \hookrightarrow S$ on S and the set of involutions σ_E of type* I. *The intersection number of the curves E, E' is given by the formula:*

$$EE' = \begin{cases} -3 & \text{if} \quad E = E' \\ 0 & \text{if} \quad o(\sigma_E \sigma_{E'}) = 3 \\ 1 & \text{if} \quad o(\sigma_E \sigma_{E'}) = 2 \end{cases}$$

where $o(g)$ denotes the order of an automorphism g.

If s is a point on E, then $C_s = E + F_s$ where F_s is the fiber over s of a fibration $\gamma_E : S \to E$ invariant by σ_E, and such that the lines L_t, $L_{\sigma_E t}$, $L_{\gamma_E t}$ in F are coplanar for all t in S.

There is another class of involutions acting on Fano surfaces, called of type II. The trace of their action on $H^0(\Omega_S)$ equals 1. An involution that is the product of two involutions of type I has type II.

PROPOSITION 10 ([17], Thm. 3). *The fixed point set of an involution of type* II *is the union of an isolated point t and a smooth genus* 4 *curve R_t. There exists a genus* 2 *curve D_t on S such that*

$$C_t = D_t + R_t$$

The curve D_t is smooth or sum of two elliptic curves which intersect in t. Let $\sigma, \sigma', \ldots$ be involutions of type II *generating a group such that all involutions have type* II. *The intersection number of the curves $R_t, R_{t'} \ldots$ is given by the formula:*

$$R_t R_{t'} = \begin{cases} -3 & \text{if} \quad \sigma = \sigma' \\ 1 & \text{if} \quad o(\sigma\sigma') = 2 \text{ or } 6 \\ 3 & \text{if} \quad o(\sigma\sigma') = 3 \\ 2 & \text{if} \quad o(\sigma\sigma') = 5 \end{cases}$$

where $o(f)$ is the order of the element f. For the intersection $D_t D_{t'}$, we have: $D_t D_{t'} = R_t R_{t'} - 1$.

We denote by $x_1, \ldots, x_5$ a basis of the space $H^0(\Omega_S)$ of global sections of the cotangent sheaf and by $e_1, \ldots, e_5$ the dual basis.

3. QUOTIENTS BY CYCLIC GROUPS

Let σ be an automorphism of a Fano surface S. We denote by $\pi : S \to S/\sigma$ the quotient map and by $g : Z \to S/\sigma$ the minimal resolution of S/σ.

■ Let $E \hookrightarrow S$ be an elliptic curve and let $\sigma = \sigma_E$ be the corresponding type I involution. The fixed point set of σ is the union of the smooth elliptic curve E and 27 points.

PROPOSITION 11. *The surface S/σ contains $27A_1$ singularities. The resolution Z of S/σ is minimal and has invariants:*

$$c_1^2 = 18, \quad c_2 = 54, \quad q = 1, \quad p_g = 6.$$

The Albanese variety of Z is E and the natural fibration $Z \to E$ has genus 3 fibers.

PROOF. There is a natural fibration $\gamma : S \to E$ invariant under σ_E such that for all s in E, we have $C_s = E + F_s$ with F_s the fiber of γ at s. The divisor $\pi^* K_{S/\sigma} = K_S - E = C_s + C_{\sigma s} + F_{\gamma s}$ is ample, therefore $K_Z = g^* K_{S/\sigma}$ is nef and

$$K_Z^2 = K_{S/\sigma}^2 = \frac{1}{2}(K_S - E)^2 = 18.$$

The invariant sub-spaces of $H^0(\Omega_S)$ and $H^0(S, \omega_S)$ by σ_E have dimension 1 and 6, that implies that $c_2 = 54$.

A fiber F_s of γ has genus 7; as $F_s E = 4$, the quotient fiber F_s/σ_E has genus 3. □

The fibers of the Albanese fibration of Z are genus 3 curves. In [20], Takahashi prove that surfaces with $q = 1$, $K^2 = 3p_g \geq 12$ and Albanese fibers of genus 3 are canonical i.e. their canonical map is birational.

■ Let σ an involution of S of type II. The fixed point set of σ is the union of a point t and a smooth genus 4 curve R_t.

PROPOSITION 12. *The minimal resolution Z of the quotient surface S/σ is minimal and has invariants:*

$$c_1^2 = 12, \quad c_2 = 12, \quad q = 3, \quad p_g = 4, \quad h^{1,1} = 14.$$

PROOF. The image of t on the surface Z/σ is a node. We have:

$$e(Z) - 1 = \frac{1}{2}(e(S) + 1 + e(R_t)).$$

As $e(R_t) = -6$, we get $e(Z) = 12$. Moreover, we have:

$$K_Z^2 = K_{S/\sigma}^2 = \frac{1}{2}(K_S - R_t)^2 = \frac{1}{2}(45 - 2 \cdot 9 - 3) = 12.$$

The other invariants are easily computed. Let D_t be the residual divisor such that $C_t = D_t + R_t$. Let $\equiv$ denotes the numerical equivalence. As $K_S - R_t \equiv 2C_t + D_t$ is nef, $K_{S/\sigma}$ is nef and $K_Z \equiv g^* K_{S/\sigma}$ is nef, therefore Z is minimal. □

A smooth polarisation Θ of type $(1,1,2)$ on an Abelian Threefold has the same invariants as the surface Z, see [6].

■ Let α be a primitive third root of unity. Let σ be an order 3 automorphism of S such that the eigenvalues of $d\sigma$ acting on $H^0(\Omega_S)$ are α^2, α, 1, 1, 1 (automorphism of type III(1)).

PROPOSITION 13. *The automorphism σ has no fixpoints. The quotient surface $S/\sigma = Z$ is smooth, minimal, and has invariants:*

$$c_1^2 = 15, \quad c_2 = 9, \quad q = 3, \quad p_g = 4.$$

PROOF. Up to a change of coordinates, the cubic can be written as:

$$F = \{x_1^3 + x_2^3 + ax_1x_2x_3 + C(x_3, x_4, x_5) = 0\}$$

with C a cubic form. As F is smooth, there are no lines into the intersection of F and the plane $\mathbb{P}(V_1)$. Moreover, there are no lines in F going through the points $\mathbb{P}(V_\alpha)$ and $\mathbb{P}(V_{\alpha^2})$, therefore the automorphism σ has no fixed points and the surface $S/\sigma = Z$ is smooth. We have moreover: $\pi^* K_Z = K_S$, thus K_Z is ample. As π is étale $K_Z^2 = \frac{1}{3}K_S^2$ and $c_2(Z) = \frac{1}{3}c_2(S)$. Since the action of σ on $H^0(\Omega_S)$ is known, we can compute the other invariants. □

In view of [6], where Catanese and Schreyer discuss about irregular surfaces with $p_g = 4$, we collect further informations on the surface Z.

Let $w_1, w_2 \in H^0(\Omega_S)$ be two linearly independent 1-forms on S. Recall that by the Tangent Bundle Theorem, the canonical divisor associated to the form $w_1 \wedge w_2$ parametrizes the lines on $F \hookrightarrow \mathbb{P}^4$ that cut the plane $\{w_1 = w_2 = 0\} \hookrightarrow \mathbb{P}^4$.

A basis of the σ-invariant canonical forms on S is $x_1 \wedge x_2$, $x_3 \wedge x_4$, $x_3 \wedge x_5$, $x_4 \wedge x_5$. Thus, a point s in S is a base point of the corresponding 3 dimensional linear system if the line L_s cuts the 4 planes: $x_1 = x_2 = 0$, $x_3 = x_4 = 0$, $x_3 = x_5 = 0$, $x_4 = x_5 = 0$. But this is impossible, therefore the system is base point free and the canonical system of Z too.

The Albanese map of Z is not a fibration (by [7], there is no fibration of a Fano surface onto a curve of genus > 1). It would be interesting to study deeper Z in the spirit of [6], in particular we can ask if the canonical map is birational.

■ Let σ be an order 3 automorphism of S such that the eigenvalues of $d\sigma$ acting on $H^0(\Omega_S)$ are $(\alpha^2, \alpha^2, \alpha, \alpha, 1)$ (automorphism of type III(2)).

PROPOSITION 14. *The 9 singularities of the quotient S/σ are cusps A_2. The minimal resolution Z of this surface has invariants:*

$$c_1^2 = 15, \quad c_2 = 33, \quad q = 1, \quad p_g = 4, \quad h^{1,1} = 27$$

and is minimal. The fibers of the fibration onto the Albanese variety have genus 4.

PROOF. Up to a change of coordinates, σ acts on the cubic:

$$F = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + \ell_1(x_1, x_2)\ell_2(x_3, x_4)x_5 = 0\}.$$

The lines $\mathbb{P}(V_{\alpha^2})$ and $\mathbb{P}(V_\alpha)$ and the point $\mathbb{P}(V_1)$ are not contained on F. The stable lines are the 9 lines on F that cut the disjoint lines $\mathbb{P}(V_\alpha)$ and $\mathbb{P}(V_{\alpha^2})$. Let s be one of the fixed points of σ. The eigenvalues of $d\sigma$ acting on $T_{S,s}$ are α, α^2, therefore the image of s on the quotient surface S/σ is a A_2 singularity, resolved by a chain of 2 (-2)-curves. We have:

$$K_Z^2 = K_{S/\sigma}^2 = \frac{45}{3} = 15.$$

moreover $e(Z) - 9 \cdot 2 = \frac{1}{3}(27 + 2 \cdot 9)$ and $e(Z) = 33$.

Let us compute the genus of the fibers. By [16] Cor. 26, there is a fibration of S onto an elliptic curve E, invariant by σ and with fibers F of genus 10. Therefore the quotient surface S/σ has a fibration $S/\sigma \to E$ by fibers of genus 4. □

By using the same method as for surfaces of type III(1), we see that the canonical system of Z has no has point. Again, it would be interesting to study deeper Z in the spirit of [6].

■ Let σ be an order 3 automorphism of S such that the eigenvalues of $d\sigma$ acting on $H^0(\Omega_S)$ are $(\alpha^2, \alpha, \alpha, \alpha, \alpha)$ (automorphism of type III(3)). The space $\mathbb{P}(V_\alpha)$ is a hyperplane, $\mathbb{P}(V_{\alpha^2})$ is one point outside F. The hyperplane $\mathbb{P}(V_\alpha)$ cuts F into a smooth cubic surface Y, therefore σ fixes 27 isolated points.

PROPOSITION 15. *The quotient Y/σ has $27A_{3,1}$ singularities. Its minimal resolution Z has invariants*

$$c_1^2 = 6, \quad c_2 = 54, \quad q = 0, \quad p_g = 4, \quad h^{1,1} = 44.$$

Up to the change of coordinates, the cubic F has equation

$$F = \{x_1^3 + G(x_2, \ldots, x_5) = 0\},$$

with G a cubic form. The surface Z is studied by Ikeda [13]; it is the resolution of the double cover of the smooth cubic surface $Y = \{G = 0\}$, ramified along the intersection of Y with its Hessian. By [13], the surface Z is a minimal surface, its canonical system is base point free and the image of the canonical map is Y.

PROOF. The automorphism $d\sigma$ acts on the hyperplane V_α by multiplication by α, therefore it acts on the tangent space $T_{S,s}$ of an isolated fixed point s by multiplication by α and the resulting singularity on S/σ is a $A_{3,1}$. We have:

$$K_{S/\sigma}^2 = \frac{1}{3}K_S^2 = 15$$

and $K_Z \equiv g^* K_{S/\sigma} - \frac{1}{3}\sum_{i=1}^{i=27} E_i$ with E_i the (-3)-curves over the $A_{3,1}$ singularities. Therefore

$$K_Z^2 = 15 + \frac{1}{9}27 \cdot (-3) = 6.$$

The computation of $q = 0$ and $p_g = 4$ is immediate. □

■ Let σ be an order 3 automorphism of S such that the eigenvalues of $d\sigma$ are $(\alpha, \alpha, \alpha, 1, 1)$ (automorphism of type III(4)). Then $\mathbb{P}(V_\alpha)$ is a plane and $\mathbb{P}(V_1)$ is a line. The family of lines going through the plane $\mathbb{P}(V_\alpha)$ and the line $\mathbb{P}(V_1)$ is the union of 3 disjoint elliptic curves.

PROPOSITION 16. *The quotient map $\pi : S \to S/\sigma$ is a triple cover branched over 3 elliptic curves of the blow-up in three points of an abelian surface.*

PROOF. See [16]. □

■ Let σ be an order 4 automorphism of S such that the eigenvalues of $d\sigma$ acting on $H^0(\Omega_S)^*$ are -1, -1, 1, i, $-i$.

PROPOSITION 17. *The quotient surface contains 6 nodes and one A_3 singularity. The minimal resolution Z of S/σ is minimal and has invariants:*

$$c_1^2 = 6, \quad c_2 = 18, \quad q = 1, \quad p_g = 2, \quad h^{1,1} = 16.$$

The fibers of the natural fibration of Z onto its Albanese variety have genus 4.

PROOF. Up to a change of coordinates, the automorphism

$$x \to (-x_4, -x_1, -x_2, -x_3, -x_5)$$

acts on the cubic:

$$F = \{x_5^3 + ax_5^2\sigma_1 + x_5(b\sigma_1^2 + c\sigma_2) + P(\sigma_1, \sigma_2, \sigma_3) = 0\},$$

where $\sigma_i = x_1^i + \cdots + x_4^i$ and P is a polynomial such that $P(\sigma_1, \sigma_2, \sigma_3)$ is homogenous of degree 3 in the variables x_j. The basis of V_{-1}, V_{-i}, V_i and V_1 are respectively: $e_1 + e_2 + e_3 + e_4$, e_5, $v_{-i} = e_1 - ie_2 - e_3 + ie_4$, $v_i = e_1 + ie_2 - e_3 - ie_4$ and $v_1 = e_1 - e_2 + e_3 - e_4$. The line through $\mathbb{C}v_{-i}$ and $\mathbb{C}v_i$ is on F, this is also the unique isolated stable line L_t of σ^2, involution of type II. There are three lines on F are going trough the line $\mathbb{P}(V_{-1})$ and the point v_i and three other lines through the line $\mathbb{P}(V_{-1})$ and the point v_{-i}. These 6 lines correspond to the intersection points of D_t and R_t, where $C_t = R_t + D_t$ are as in Proposition 10. Their images on S/σ are nodes.

As the eigenvalues of $d\sigma$ acting on $\mathbb{C}v_{-i} + \mathbb{C}v_i$ are $(-i, i)$, the image of t on S/σ is an $A_3 = A_{4,3}$ singularity, resolved by 3 (-2) curves. Let us compute the Euler number:

$$e(Z) - 6 - 3 = \frac{1}{4}(e(S) + (e(R_t) - 6) + 3 \cdot 7)$$

thus $e(Z) = 18$. The quotient map π is ramified with index 2 over R_t, thus $K_S = \pi^* K_{S/v} + R_t$. The divisor $K_S - R_t = 2C_t + D_t$ is nef, therefore $K_{S/\sigma}$ is nef and $K_Z = g^* K_{S/\sigma}$ is nef, thus Z is minimal, moreover:

$$K_Z^2 = K_{S/\sigma}^2 = \frac{1}{4}(K_S - R_t)^2 = 6.$$

It is immediate to check that $q = 1$ and $p_g = 2$.

By [16], Theorem 18, there exists a σ-invariant fibration $S \to E$ onto an elliptic curve with generic fibers D of genus 13 and R_t is contained in a fiber, therefore $D \to D/\sigma$ is étale and the fiber D/σ has genus 4. □

In [20], Takahashi constructed all canonical surfaces with $q = 1$ and $K^2 = 3p_g$ with $p_g \geq 4$. The fibers of the Albanese fibration of such surfaces are genus 3 curves. As far as the author knows, the above surface Z seems new on the line of surfaces with $q = 1$ and $K^2 = 3p_g$ (the fibers of the Albanese fibration of Z have genus 4).

■ Let σ be an order 4 automorphism of S such that the automorphism $\sigma : x \to (ix_1, ix_2, ix_3, -ix_4, x_5)$ acts on the cubic threefold F.

PROPOSITION 18. *The quotient surface S/σ contains* 12 *nodes and* 3 *singularities A_3. The minimal resolution Z of S/σ is a minimal properly elliptic surface with invariants:*

$$c_1^2 = 0, \quad c_2 = 36, \quad q = 1, \quad p_g = 3.$$

PROOF. Up to a change of coordinates, the cubic is:

$$F = \{x_5^2 x_4 + x_4^2 x_1 + C(x_1, x_2, x_3) = 0\}.$$

The point $\mathbb{C}e_5$ is the vertex of a cone in F whose basis is an elliptic curve $E \hookrightarrow S$. The automorphism σ^2 is a type I involution, fixing E and 27 isolated points. A point s on E correspond to a line

$$L_s = \{(\lambda x_1 : \lambda x_2 : \lambda x_3 : 0 : \mu x_5)/(\lambda : \mu) \in \mathbb{P}^1, C(x_1, x_2, x_3) = 0\},$$

and such a line is stable under σ, therefore E is fixed by σ. The space $\mathbb{C}e_5$ is the tangent space to E in the Albanese variety of S, therefore, as σ fixes E, the automorphism $d\sigma$ is equal to $x \to (ix_1, ix_2, ix_3, -ix_4, x_5)$ (the eigenvalues of $d\sigma$ acting on $H^0(\Omega_S)^*$ are i, i, i, $-i$, 1). The line through $\mathbb{P}(V_1)$ and $\mathbb{P}(V_{-i})$ is not on F and there are 3 lines going through $\mathbb{P}(V_{-i})$ and that cut the plane $\mathbb{P}(V_i)$, these three lines are among the 27 isolated fixed lines of σ^2 and give $3A_3$ singularities. The images on S/σ of the remaining 24 isolated fixed points of σ^2 are 12 nodes. We have

$$e(Z) - (12 + 3 \cdot 3) = \frac{1}{4}(27 + 24 + 3e(E) + 3 \cdot 3) = 15$$

and $e(Z) = 36$. Let be $\gamma_E : S \to E$ be the σ^2-invariant fibration, associated to E. It is also σ-invariant. Let F_s be the fiber over s. Then $\pi^* K_{S/\sigma} = K_S - 3E$ is numerically equivalent to $3F_s$, therefore $K_S^2 = K_{S/\sigma}^2 = \frac{1}{4}(3F_s)^2 = 0$. Moreover, as F_s is nef, K_Z is nef and then Z is minimal. The invariants $q = 1$, $p_g = 3$ are readily computed. □

■ Let σ be an order 5 automorphism of S such that the eigenvalues of $d\sigma$ acting on $H^0(\Omega_S)^*$ are $(1, \xi, \xi^2, \xi^3, \xi^4)$, with ξ a primitive 5^{th} root of unity.

PROPOSITION 19. *The quotient surface S/σ has $2A_4$ singularities. The minimal resolution Z of S/σ is minimal and its invariants are:*

$$c_1^2 = 9, \quad c_2 = 15, \quad q = 1, \quad p_g = 2, \quad h^{1,1} = 13.$$

The general fiber of the Albanese map of Z has genus 4.

PROOF. Up to a change of coordinates, the cubic is given by:

$$F = \{x_1^2x_3 + x_3^2x_4 + x_4^2x_2 + x_2^2x_1 + x_5(ax_1x_4 + bx_2x_3) + x_5^3 = 0\},$$

and σ acts by:

$$x \to (\xi x_1, \xi x_2^2, \xi^3 x_3, \xi^4 x_4, x_5).$$

The eigenspace V_{ξ^k} is generated by e_k. The lines through two points $\mathbb{C}e_k$, $\mathbb{C}e_{k'}$ and contained in F are the line L_w through $\mathbb{C}e_1$ and $\mathbb{C}e_4$ and the line $L_{w'}$ through $\mathbb{C}e_2$ and $\mathbb{C}e_3$. The eigenvalues of $d\sigma_w : T_{S,w} \to T_{S,w}$ are ξ^4 and ξ, the eigenvalues of $d\sigma_{w'} : T_{S,w'} \to T_{S,w'}$ are ξ^3 and ξ^2, therefore the images of w, w' on S/σ are $2A_4$ singularities. We have:

$$e(Z) - 2 \cdot 4 = \frac{1}{5}(e(S) + 4 \cdot 2)$$

thus $e(Z) = 15$. As $K_{S/\sigma}$ is ample, K_Z is nef. Moreover $K_Z^2 = \frac{1}{5}K_S^2 = 9$. By [17], Thm. 3 D), there exists a fibration $S \to \mathbb{E}$ onto an elliptic curve $\mathbb{E}$ that is invariant by σ and with fibers of genus 16. We deduce that the general fiber of the Albanese map of Z has genus 4. □

■ Let Z be the resolution of the quotient of the Fano surface S of the Klein cubic:

$$F = \{x_1^2x_2 + x_2^2x_4 + x_4^2x_3 + x_3^2x_5 + x_5^2x_1 = 0\}$$

by the order 11 automorphism σ acting on F by:

$$x \to (\xi x_1, \xi^9 x_2, \xi^3 x_3, \xi^4 x_4, \xi^5 x_5)$$

where ξ is a 11^{th} primitive root of unity (S is unique to have an order 11 automorphism, see [19]).

PROPOSITION 20. *The invariants of the surface Z are:*

$$c_1^2 = -5, \quad c_2 = 17, \quad q = p_g = 0.$$

The surface S/σ contains 5 *singularities $A_{11,3}$.*

PROOF. Let us denote by L_{ij} the line $x_s = x_t = x_u = 0$ where $\{i, j, s, t, u\} = \{1, 2, 3, 4, 5\}$. The lines on F that are stable by σ are L_{13}, L_{23}, L_{25}, L_{45}, L_{14}. The

5 corresponding fix points s_{ij} on S give 5 singularities $A_{11,3}$ on S/σ resolved by curves A_{ij}, B_{ij} with $(A_{ij})^2 = -3$, $(B_{ij})^2 = -4$ and $A_{ij}B_{ij} = 1$. We have:

$$K_Z = g^*K_X - \frac{1}{11}\sum(6A_{ij} + 7B_{ij})$$

with $g^*K_X^2 = K_X^2 = \frac{45}{11}$, thus $K_Z^2 = -5$. The Euler number is:

$$e = \frac{1}{11}(27 + 10 \cdot 5) + 10 = 17.$$

Moreover, we check immediately that the invariants subspaces of $H^0(S, \Omega_S)$ and $H^0(S, \omega_S)$ by σ are trivial, therefore the quotient surface has $q = p_g = 0$. □

PROPOSITION 21. *The surface Z is rational.*

PROOF. Let us prove the existence of a smooth rational curve C such that $C^2 = 0$ on a blow-down of Z.

Let us denote by C_{ij} the incidence divisor for the stable line L_{ij} corresponding to the fixed point s_{ij}. The automorphism σ acts on C_{ij}. Using the equation of F, we see that among the 5 fixed points of σ, the curve C_{13} contains s_{14} and s_{23} and moreover, as the line L_{13} is double (there is a plane X such that $XF = 2L_{13} + L_{14}$), the point s_{13} is on C_{13}.

The permutation $\tau = (1, 2, 4, 3, 5)$ acts on the Klein cubic threefold and with the order 11 automorphism σ, it generates an order 55 group such that the group generated by σ is distinguished. By these order 5 symmetries, we therefore know which fixed points of σ are on the curve C_{ij} (τ acts on the indices of the C_{ij}, s_{ij} etc ...). Any incidence divisor C is a double cover of a plane quintic Γ that can be explicitly computed using [2], equation (6). For the divisor C_{13}, the corresponding quintic Γ has equation:

$$4x_2^3x_4x_5 - x_2x_4^4 - x_5^5 = 0$$

in the plane with coordinates x_2, x_4, x_5. The curve Γ has only one nodal singularity, and therefore by [2] Lemma 2, the curve C_{13} has only one nodal singularity. By using the order 5 symmetry τ, the same property holds for the all the C_{ij}. Let D_{ij} be the reduced image by the quotient map π of C_{ij} and let $\overline{D}_{ij}$ the strict transform of D_{ij} in Z by the minimal resolution $g : Z \to S/\sigma$. We can write:

$$\overline{D}_{13} = g^*D_{13} - \frac{1}{11}(a_{13}A_{13} + b_{13}B_{13} + a_{23}A_{23} + b_{23}B_{23} + a_{14}A_{14} + b_{14}B_{14})$$

for a_{ij}, b_{ij} rational. Let be $M = \begin{pmatrix} -3 & 1 \\ 1 & -4 \end{pmatrix}$, $M^{-1} = -\frac{1}{11}\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$. We have $\overline{D}_{13}A_{ij}, \overline{D}_{13}B_{ij} \in \mathbb{Z}^+$, therefore $(a_{ij}, b_{ij})M \in (11\mathbb{Z}^-, 11\mathbb{Z}^-)$, thus a_{ij}, b_{ij} are positive integers. Using the order 5 symmetry, we get:

$$\overline{D}_{25} = g^*D_{25} - \frac{1}{11}(a_{13}A_{25} + b_{13}B_{25} + a_{23}A_{45} + b_{23}B_{45} + a_{14}A_{23} + b_{14}B_{23}).$$

Moreover $\overline{D}_{13}\overline{D}_{25} \in \mathbb{Z}^+$, thus

$$\overline{D}_{13}\overline{D}_{25} = \frac{1}{11^2}\left(55 + (a_{23}, b_{23})M\begin{pmatrix} a_{14} \\ b_{14}\end{pmatrix}\right) \in \mathbb{Z}^+.$$

Let us define $(a_{23}, b_{23})M = (-11u_1, -11u_2)$ with u_1, u_2 positive integers. We have:

$$\frac{1}{11^2}(55 - 11(a_{14}u_1 + b_{14}u_2)) \in \mathbb{Z}^+$$

as $a_{14}, b_{14} \in \mathbb{Z}^+$, we get $\overline{D}_{12}\overline{D}_{25} = 0$ and $a_{14}u_1 + b_{14}u_2 = 5$. Taking care of $\overline{D}_{13}A_{13}, \overline{D}_{13}B_{13} \in \mathbb{Z}^+$, we get the following 8 possibilities for $(a_{14}, b_{14}, u_1, u_2)$:

$$(4,1,1,1), (1,3,2,1), (5,4,1,0), (5,15,1,0),$$
$$(1,3,5,0), (4,1,0,5), (9,5,0,1), (20,5,0,1).$$

Thus $(a_{14}, b_{14}, a_{23}, b_{23})$ is one of the following:

$$(4,1,5,4), (1,3,9,5), (5,4,4,1), (5,15,4,1),$$
$$(1,3,20,5), (4,1,5,15), (9,5,1,3), (20,5,1,3).$$

Using the order 5 symmetry $\tau = (1,2,4,3,5)$, we get:

$$\overline{D}_{14} = g^*D_{14} - \frac{1}{11}(a_{13}A_{14} + b_{13}B_{14} + a_{23}A_{13} + b_{23}B_{13} + a_{14}A_{45} + b_{14}B_{45}).$$

We have $\overline{D}_{13}\overline{D}_{14} \in \mathbb{Z}^+$, therefore:

$$\frac{1}{11^2}\left(55 + (a_{13}, b_{13})M\begin{pmatrix} a_{14} + a_{23} \\ b_{14} + b_{23}\end{pmatrix}\right) \in \mathbb{Z}^+.$$

Moreover:

$$\begin{pmatrix} a_{14} + a_{23} \\ b_{14} + b_{23}\end{pmatrix} = \begin{pmatrix} 9 \\ 5\end{pmatrix}, \begin{pmatrix} 10 \\ 8\end{pmatrix}, \begin{pmatrix} 21 \\ 8\end{pmatrix} \text{ or } \begin{pmatrix} 9 \\ 16\end{pmatrix}.$$

As above, let us define $(a_{14} + a_{23}, b_{14} + b_{23})M = (-11w_1, -11w_2)$ with w_1, w_2 positive integers. We obtain:

$$(w_1, w_2) = (2,1); (2,2); (5,1) \text{ or } (1,5).$$

As $a_{13}w_1 + b_{13}w_2 = 5$, we get the following possibilities with respect to the 4 above pairs (w_1, w_2):

$$\begin{pmatrix} a_{13} \\ b_{13}\end{pmatrix} = \begin{pmatrix} 2 \\ 1\end{pmatrix}, \begin{pmatrix} 1 \\ 3\end{pmatrix}, \begin{pmatrix} 0 \\ 5\end{pmatrix}; \emptyset; \begin{pmatrix} 1 \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ 5\end{pmatrix}; \begin{pmatrix} 0 \\ 1\end{pmatrix}, \begin{pmatrix} 5 \\ 0\end{pmatrix}$$

but as $\overline{D}_{13}A_{13} \geq 0$ is an integer, the only solution is $(a_{13}, b_{13}) = (1, 3)$, $(w_1, w_2) = (2, 1)$ and $(a_{14}, b_{14}, a_{23}, b_{23})$ equals $(4, 1, 5, 4)$ or $(5, 4, 4, 1)$. We obtain that: $\overline{D}_{13}^2 = -1$ and $K_Z\overline{D}_{13} = -1$ and by symmetry, the curves $\overline{D}_{ij}$ are 5 disjoint (-1)-curves. Let us suppose that $(a_{14}, b_{14}, a_{23}, b_{23})$ is $(5, 4, 4, 1)$, then:

$$\overline{D}_{13} = g^*D_{13} - \frac{1}{11}(A_{13} + 3B_{13} + 4A_{23} + B_{23} + 5A_{14} + 4B_{14})$$

$$\overline{D}_{25} = g^*D_{25} - \frac{1}{11}(A_{25} + 3B_{25} + 4A_{45} + B_{45} + 5A_{23} + 4B_{23})$$

$$\overline{D}_{14} = g^*D_{14} - \frac{1}{11}(A_{14} + 3B_{14} + 4A_{13} + B_{13} + 5A_{45} + 4B_{45})$$

$$\overline{D}_{23} = g^*D_{23} - \frac{1}{11}(A_{23} + 3B_{23} + 4A_{25} + B_{25} + 5A_{13} + 4B_{13})$$

$$\overline{D}_{45} = g^*D_{45} - \frac{1}{11}(A_{45} + 3B_{45} + 4A_{14} + B_{14} + 5A_{25} + 4B_{25}).$$

We have $A_{13}\overline{D}_{ij} = 0, 0, 1, 1, 0$ and $B_{13}\overline{D}_{ij} = 1, 0, 0, 1, 0$, moreover:

$$\overline{D}_{14}A_{13} = \overline{D}_{14}A_{45} = \overline{D}_{23}A_{13} = \overline{D}_{23}A_{25} = 1.$$

The images of A_{13} and A_{45} by the blow-down map of the five $\overline{D}_{ij}$ are two (-1)-curves A'_{13} and A'_{45} such that $A'_{13}A'_{45} = 1$ therefore, as Z is regular, it is a rational surface. In the same way, if we suppose that $(a_{14}, b_{14}, a_{23}, b_{23})$ is $(4, 1, 5, 4)$, we obtain that the surface Z is rational. □

■ Let S be the Fano surface of the cubic:

$$x_1^2x_3 + x_3^2x_4 + x_4^2x_2 + x_2^2x_1 + x_5^3 = 0.$$

The order 15 automorphism:

$$\sigma : x \to (\mu x_1, \mu^7 x_2, \mu^{13} x_3, \mu^4 x_4, \mu^5 x_5)$$

$(\mu^{15} = 1)$ acts on S.

PROPOSITION 22. *The surface S/σ contains $5A_{3,1} + 2A_{15,4}$ singularities. The minimal resolution Z of S/σ has invariants:*

$$c_1^2 = -4, \quad c_2 = 16, \quad q = p_g = 0, \quad h^{1,1} = 14.$$

PROOF. The automorphism σ fixes 2 isolated points s_{14}, s_{23} (corresponding to the lines $\mathbb{C}e_1 + \mathbb{C}e_4$ and $\mathbb{C}e_2 + \mathbb{C}e_3$) and acts on their tangent spaces by the diagonal matrix with diagonal elements (μ^4, μ) giving $2A_{15,4}$ singularities denoted by a and b. The singularity a is resolved by two (-4)-curves T_a, U_a such that $T_aU_a = 1$, the singularity B is resolved by T_b, U_b with the same configuration. The automor-

phism σ^5 fixes 27 isolated points (lines in the hyperplane $x_5 = 0$) and acts on the tangent space at these points by multiplication by μ^5. The points s_{14}, s_{23} are among theses 27 points. The other 25 fixed points of σ^5 gives $5A_{3,1}$ singularities on S/G resolved by 5 (-3)-curves T_i. We have: $q = p_g = 0$. For the Euler number:

$$e(S/G) = \frac{1}{15}(27 + (3-1)\cdot 25 + (15-1)\cdot 2) = 7$$

and $e(S) = 7 + 5 + 2\cdot 2 = 16$. For the canonical bundle:

$$K_Z = g^* K_{S/G} - \frac{1}{3}\left(2(U_a + T_a) + 2(U_b + T_b) + \sum_{i=1}^{i=5} T_i\right)$$

thus $K_Z^2 = -4$. □

PROPOSITION 23. *The surface Z is rational.*

PROOF. In order to prove the Proposition, we will prove the existence of a smooth rational curve C such that $C^2 = 0$ on a blow-down of Z. The 27 stable lines under the action of σ^5 are on the cubic surface $X = F \cap \{x_5 = 0\}$. Their corresponding points on S are denoted by:

$$e_1, \ldots, e_6, g_1, \ldots, g_6, f_{ij}, \quad 1 \le i < j \le 6$$

and their configuration is as follows:

The two points e_1 and g_1 are fixed by σ and the corresponding lines L_{e_1} and L_{g_1} ($\mathbb{C}e_1 \oplus \mathbb{C}e_4$ and $\mathbb{C}e_2 \oplus \mathbb{C}e_3$) are skew. The images of e_1 and g_1 on S/G are denoted by a and b. The other points e_i and g_i are such that $L_{e_1}, \ldots, L_{e_6}$, $L_{g_1}, \ldots, L_{g_6}$ is a double six. The $\{g_2, \ldots, g_6\}$ and $\{e_2, \ldots, e_6\}$ are orbits of σ whose images on S/G are denoted by f and g.

Each point f_{ij}, $1 \le i < j \le 6$ on S is the isolated fixed point of a type II involution that is the product of two type I involution, therefore each incidence divisor $C_{f_{ij}}$ splits:

$$C_{f_{ij}} = E + E' + R_{ij}$$

for E, E' the two elliptic curves that cut each other in f_{ij} and with R_{ij} the residual divisor. Each of the 10 elliptic curves E as above contains exactly 3 fixed points and these points are among the f_{ij} (intersection of E by 3 other elliptic curves, see [18]).

We denote by A and B the image of C_{e_1}. We can denote by E_{ij}, $1 \le i < j \le 5$ the ten elliptic curves on S. Their configuration is given by $E_{ij}E_{st} = 1$ if $|\{i,j,s,t\}| = 4$, $E_{ij}E_{st} = 0$ if $|\{i,j,s,t\}| = 3$, $E_{ij}^2 = -3$. The divisors $E_1 = E_{12} + E_{23} + E_{34} + E_{45} + E_{15}$ and $E_2 = E_{13} + E_{24} + E_{35} + E_{14} + E_{25}$ are two orbits of σ and we denote by H, L their images on S/G.

We denote by m, n, p the images of the 15 points f_{ij} (3 orbits): $m = \{f_{12}, f_{13}, f_{14}, f_{15}, f_{16}\}$, $n = \{f_{23}, f_{34}, f_{45}, f_{56}, f_{26}\}$, $p = \{f_{24}, f_{35}, f_{46}, f_{25}, f_{36}\}$.

We have $a, g, m \in A$, $b, f, m \in B$. On S/σ, we have:

$$H^2 = L^2 = \frac{1}{15}E_1^2 = \frac{1}{15}E_2^2 = -\frac{1}{3}$$

and $LH = \frac{1}{15}E_1E_2 = \frac{1}{3}$. The curve H (resp. L) is nodal in n, (resp. p) and H, L cut each other in m transversally. Let T_m, T_n, T_p be the (-3)-curves over m, n, p and let $\overline{H}$, $\overline{L}$ be the proper transform of H, L. Then:

$$\overline{H} = g^*H - \frac{1}{3}(T_m + 2T_n), \quad \overline{L} = g^*L - \frac{1}{3}(T_m + 2T_p)$$

therefore $\overline{H}^2 = \overline{L}^2 = -2$ and $\overline{H}\overline{L} = 0$. As $K_Z\overline{H} = K_Z\overline{L} = 0$, the curves $\overline{H}$ and $\overline{L}$ are two (-2)-curves. Since $a, g, m \in A$ and A is nodal in a, we have:

$$\bar{A} = g^*A - \frac{1}{3}(T_a + U_a) - \frac{1}{3}(T_g + T_m)$$

where T_a and U_a are the 2 (-4)-curves over a. Therefore: $\bar{A}^2 = -1$ and as $K_Z\bar{A} = -1$, $\bar{A}$ is a (-1)-curve. We have:

$$\bar{A}\overline{H} = \left(g^*A - \frac{1}{3}(T_a + U_a) - \frac{1}{3}(T_g + T_m)\right)\left(g^*H - \frac{1}{3}(T_m + 2T_n)\right) = 0.$$

In the same way:

$$\bar{B} = g^*B - \frac{1}{3}(T_b + U_b) - \frac{1}{3}(T_f + T_m)$$

is a (-1)-curve and $\bar{A}\bar{B} = 0$, moreover:

$$\bar{B}\overline{H} = \left(g^*B - \frac{1}{4}(T_b + U_b) - \frac{1}{3}(T_f + T_m)\right)\left(g^*H - \frac{1}{3}(T_m + 2T_n)\right) = 0.$$

Consider the curves $\bar{A}$, $\bar{B}$, T_m, $\overline{H}$, $\overline{L}$. They are smooth rational curves and their intersection matrix is:

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & -3 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

By blowing down four times, we obtain a smooth rational curve C such that $C^2 = 0$. As $q = 0$, the surface Z is rational. □

4. Quotients by non-cyclic groups

Let us now study quotients by non-cyclic groups G. We denote by $\pi : S \to S/G$ the quotient map and by $g : Z \to S/G$ the minimal desingularisation.

■ Let E, E' be 2 genus 1 curves on S such that $EE' = 1$ and let $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$ be the group generated by the involutions of type I σ_E, $\sigma_{E'}$. We have:

Proposition 24. *The quotient S/G has 24 nodes. The minimal resolution Z is minimal and has invariants:*

$$c_1^2 = 5, \quad c_2 = 43, \quad q = 0, \quad p_g = 3, \quad h^{1,1} = 35.$$

Proof. An equation of F is:

$$F = \{x_1^2 x_3 + x_2^2 x_4 + G(x_3, x_4, x_5) = 0\}.$$

The involution $\sigma_E \sigma_{E'}$ has type II and fixes the intersection point t of E and E' and the divisor R_t such that $C_t = E + E' + R_t$. The involution σ_E fixes E and 27 points, 3 of them are on E'; the symmetric situation holds for $\sigma_{E'}$. The images on S/G of these $1 + 2 \cdot 3 = 7$ isolated fixed points of G are smooth points. The 24 singular points of S/G are nodes, and the quotient map is ramified with order 2 over the curve C_t, therefore $\pi^* K_{S/G} = K_S - C_t \equiv 2C_t$ and $K_{S/G}$ is ample, moreover:

$$K_Z^2 = K_{S/G}^2 = \frac{1}{4}(K_S - C_t)^2 = 5.$$

The irregularity is 0 and $p_g = 3$, therefore $c_2 = 43$. □

■ Let σ_1, σ_2 be 2 involutions on type II such that $\sigma_3 = \sigma_1 \sigma_2$ is a third involution of type II. They generate a group G isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2 = \mathbb{D}_2$.

Proposition 25. *The surface $Z = S/G$ is smooth and has invariants:*

$$c_1^2 = -3, \quad c_2 = 3, \quad q = 2, \quad p_g = 1, \quad h^{1,1} = 7.$$

The surface Z is the blow up in three points p_1, p_2, p_3 of an abelian surface such that there exist 3 genus 2 curves R_i' that cuts each other in the three point p_i. The map $\pi : S \to Z$ is a $(\mathbb{Z}/2\mathbb{Z})^2$-cover, branched over the strict transform of the three curves R_i'.

Proof. An equation of F is:

$$F = \{x_1^2 x_4 + x_2^2 x_5 + x_3^2 \ell(x_4, x_5) + a x_1 x_2 x_3 + G(x_4, x_5) = 0\}.$$

Each involution σ_i fixes an isolated point t_i and a smooth genus 4 curve R_i and we have $R_i R_j = 1$, $R_i^2 = -3$, $K_S R_i = 9$.

The action of the group G on $H^0(\Omega_S)$ is generated by diagonal matrices with diagonal elements $(-1, -1, 1, 1, 1)$, $(1, -1, -1, 1, 1)$. Therefore the invariant subspaces $H^0(\Omega_S)^G$ and $H^0(S, \omega_S)^G$ have dimension $q = 2$ and $p_g = 1$.

The lines $x_3 = x_4 = x_5 = 0$, $x_1 = x_4 = x_5 = 0$, $x_2 = x_4 = x_5 = 0$, corresponds to the isolated fix points t_i of the σ_i. For $i \neq j$, the point t_i is a non-isolated fixed point of σ_j because the eigenvalues of $d(\sigma_j)_{t_i}$ acting on T_{S,t_i} are $(-1, 1)$. That implies that the images on S/G of the 3 points t_i are smooth points and that $S/G = Z$ is smooth.

The quotient map $S \to S/G$ is ramified only over the 3 curves R_i. Let D_i be the genus 2 curve such that $C_{t_i} = D_i + R_i$. We have:

$$\pi^* K_Z = K_S - R_1 - R_2 - R_3 = D_1 + D_2 + D_3$$

and:

$$K_Z^2 = \frac{1}{4}(K_S - R_1 - R_2 - R_3)^2 = \frac{1}{4}\left(\sum D_i\right)^2 = -3.$$

We deduce that $c_2 = 3$. As $R_i R_j = 1$, the 3 divisors R_i are the edges of a triangle, with t_i the vertex opposite to the edge R_i. The involution σ_i induces an involution of R_j that fixes only two points t_i, t_k ($\{i, j, k\} = \{1, 2, 3\}$). The quotient $R_i' = R_i/G$ has therefore genus 2.

As $\sigma_i(t_j) = t_j$, the involution σ_i acts on the incident divisor $C_i = D_i + R_i$. As $D_i \sum R_i = 10$, the curve $D_i' = D_i/G$ has genus g such that:

$$2 = 4(2g - 2) + 10$$

and $g = 0$. Moreover $4(D_i')^2 = (\pi^* D_i')^2 = D_i^2 = -4$ and D_i' is a (-1)-curve. As $K_Z = \sum D_i'$ and $q = 2$, we see that the minimal model of Z is an Abelian surface. □

■ Let us study the quotient of S by the group $G \simeq (\mathbb{Z}/3\mathbb{Z})^2$ generated by automorphisms a, b such that the eigenvalues of da and db are respectively $(\alpha^2, \alpha, 1, 1, 1)$ and $(1, 1, \alpha^2, \alpha, 1)$.

PROPOSITION 26. *The surface S/G contains 6 cusps. The minimal resolution Z of the quotient surface S/G is minimal and has invariants:*

$$c_1^2 = 5, \quad c_2 = 19, \quad q = 1, \quad p_g = 2, \quad h^{1,1} = 17.$$

The fibers of the fibration of Z onto its Albanese variety have genus 2.

PROOF. An equation of F is:

$$F = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + u x_1 x_2 x_5 + v x_3 x_4 x_5 = 0\}.$$

The automorphisms a, a^2, b, b^2 have no fixed points. The automorphisms ab, a^2b^2 fix 9 isolated points, and also a^2b, ab^2. That gives 6 cusps singularities on S/G. As these singularities are resolved by (-2)-curves, we have:

$$K_Z^2 = K_{S/d}^2 = \frac{K_S^2}{9} = 5,$$

moreover: $e(Z) - 6 \cdot 3 = \frac{1}{9}(27 - 18)$, therefore $e(Z) = 19$. The invariant subspaces $H^0(\Omega_S)^G$ and $H^0(S, \omega_S)^G$ are easily computed.

By [17], there is a fibration $\gamma : S \to E$ of S onto an elliptic curve that is invariant by G and with fibers F of genus 10. Thus the Albanese fibration of Z has fibers of genus $\frac{1}{2}\frac{1}{9}(F^2 + FK_S) + 1 = 2$.

As K_S is ample, and $K_S = \pi^* K_{S/G}$, we see that $K_{S/G}$ is ample; as $K_Z = g^* K_{S/G}$, the canonical divisor K_Z is nef and Z is minimal. □

Recently the moduli space $\mathscr{M}$ of surfaces with $c_1^2 = 5$, $q = 1$, $p_g = 2$ has been studied by Gentile, Oliviero and Polizzi [9]. They give a stratification of $\mathscr{M}$ and prove that $\mathscr{M}$ has at least 2 irreducible components. It would be interesting to know in which component and strata the surface Z belongs.

■ Let G be the permutation group S_3 generated by two involutions σ_E, $\sigma_{E'}$ such that $EE' = 0$. The order 3 automorphism $\tau = \sigma_E \sigma_{E'}$ has no fixed-points (type III(1)). Let be $E'' = \sigma_E(E') = \sigma_{E'}(E)$. Let $g : Z \to S/G$ be the minimal desingularisation of S/G.

PROPOSITION 27. *The surface Z is the resolution of the 27 nodes on S/G and has invariants:*

$$c_1^2 = 3, \quad c_2 = 45, \quad q = 0, \quad p_g = 3, \quad h^{1,1} = 31,$$

it is a minimal Horikawa surface with $c_2 = 5c_1^2 + 30$.

PROOF. An equation of F is:

$$F = \{x_1^3 + x_2^3 + x_1 x_2 \ell(x_3, x_4, x_5) + C(x_3, x_4, x_5) = 0\}.$$

Each involution of type I fixes 27 isolated points and these points are not fixed by the 2 other involutions, therefore the surface S/G contains 27 nodes. Let F, F', F'' be fibers of γ_E, $\gamma_{E'}$, $\gamma_{E''}$, then:

$$K' := F + F' + F'' = K_S - (E + E' + E'')$$

is nef; as $K' = \pi^* K_{S/G}$, $K_{S/G}$ is nef thus $K_Z = g^* K_{S/G}$ is nef. Moreover:

$$K_Z^2 = K_{S/G}^2 = \frac{1}{6}(K')^2 = 3.$$

It is easy to check that $q = 0$ and $p_g = 3$. Let us compute the Euler number:

$$e(S/G) = \frac{1}{6}(e(S) + e(E + E' + E'') + 3 \cdot 27) = 18$$

As there are 27 nodes on S/G, $e(Z) = 18 + 27 = 45$. □

■ Let S be a Fano surface and let σ_1, σ_2 be 2 involutions of type II acting on S and generating a group G isomorphic to the dihedral group $\mathbb{D}_3$, with the involution $\sigma_1 \sigma_2 \sigma_1$ of type II.

PROPOSITION 28. *The minimal resolution Z of the quotient surface S/G has invariants:*

$$c_1^2 = 0, \quad c_2 = 12, \quad q = p_g = 1, \quad h^{1,1} = 12.$$

It is a minimal properly elliptic surface. The surface S/G contain 3 *cusps and one node.*

PROOF. The representation of $\mathbb{D}_3$ on $H^0(\Omega_S)$ splits into the sum of twice the unique 2 dimensional irreducible representation $V_{\frac{1}{3}}$ and the trivial representation T (see [17], Section 3.3 for an equation), therefore $q = 1$. The representation of $\mathbb{D}_3$ on $H^0(S, \omega_S)$ is $T + 3D + 3V_{\frac{1}{3}}$, where D is the determinantal representation, thus $p_g = 1$.

The element $\sigma = \sigma_1\sigma_2$ is a type III(2) automorphism that fixes 9 points s_i. There are 3 involutions of type II in G, each of them fixes a curve R_i and an isolated point t_i. The image of the t_i is a A_1 singularity on S/σ. Let D_i be the divisor on S such that $C_{t_i} = D_i + R_i$. We have $R_iR_j = 3$ for $i \neq j$. This gives 3 fixed points, say s_1, s_2, s_3, for the whole group G and the images of the points $s_4, \ldots, s_9$ are two cups on S/σ. The representation of the group G on the tangent space of points s_1, s_2, s_3 is isomorphic to $V_{\frac{1}{3}}$, their images are smooth points on the surface S/G. As S/G has only nodal singularities or cusps, we have $K_Z^2 = K_{S/\sigma}^2$. By [17], we have $D_iD_j = 2$, $R_i^2 = -4$, $K_SR_i = 6$, and $F = \sum_{i=1}^{i=3} D_i$ is a fiber of a fibration $\gamma : S \to E$ onto an elliptic curve E. As

$$K_S - \sum_{i=1}^{i=3} R_i = \sum_{i=1}^{i=3} D_i = F,$$

we obtain that $K_Z^2 = \frac{1}{6}F^2 = 0$ and we deduce that $c_2 = 12$.

The fibration γ is moreover invariant by $\mathbb{D}_3$, thus for a generic fiber F_s of γ, the curve $F_s/\mathbb{D}_3$ is a fiber of the Albanese map of Z. The quotient $F_s \to F_s/\mathbb{D}_3$ is ramified over $F_s \sum R_i = F_s(K_S - F_s) = 18$ points; as $K_S \sum D_i = 18$, the genus of $F/\mathbb{D}_3$ is equal to 1.

As there is a fibration by elliptic curves on S, it has Kodaira dimension less or equal to 1, and since $p_g = 1$, $c_2 = 12$, it is a minimal properly elliptic surface. □

■ Let S be a Fano surface and let $\mathbb{D}_5$ be the dihedral group acting on it, such that the order 2 elements have type II.

PROPOSITION 29. *The minimal resolution Z of the quotient surface $S/\mathbb{D}_5$ has invariants:*

$$c_1^2 = -2, \quad c_2 = 2, \quad q = 1, \quad p_g = 0, \quad h^{1,1} = 4$$

The surface $S/\mathbb{D}_5$ contains a unique nodal singularity. The surface Z is a ruled surface of genus 1.

PROOF. We can take the group generated by the permutations $a = (1,2,3,4,5)$ and $b = (1,3)(4,5)$ acting on the basis vectors $e_1, \ldots, e_5$ of $\mathbb{C}^5$ by permutation of the indices.

The vectors $v_k = \sum_{k=1}^{k=5} \xi^{ki} e_i$, $0 \leq k \leq 4$ are eigenvectors of a, moreover $v_1 \wedge v_4$ and $v_2 \wedge v_3$ are a basis of eigenvectors for the eigenvalue 1 (resp -1) under the action of a (resp. b). The lines corresponding to $s_1 = \mathbb{C}v_1 \wedge v_4$ and $s_2 = \mathbb{C}v_2 \wedge v_3$ are the only ones contained into the cubic F among the points $\mathbb{C}v_i \wedge v_j$ $(1 \leq i < j \leq 5)$ in the grassmannian $G(2,5)$. We deduce that s_1 and s_2 are fixed points for the whole group $\mathbb{D}_5$. On the tangent space of s_1 the action of $\mathbb{D}_5$ is given by $x \to (\xi x_1, \xi^4 x_2)$ and $x \to (x_2, x_1)$. The invariant ring by this action is $\mathbb{C}[x_1^5 + x_2^5, x_1 x_2]$, therefore the image of the s_i are smooth points. As the eigenvalues of db_{s_1} acting on T_{S,s_1} are $(1,-1)$, the fixed curve of b goes through it.

The representation of $\mathbb{D}_5$ on $H^0(\Omega_S)$ splits into the trivial representation and the sum of two 2 dimensional non-isomorphic representations $V_{\frac{1}{5}}$ and $V_{\frac{2}{5}}$, therefore $q = 1$. The representation of $\mathbb{D}_5$ on $H^0(S, \omega_S)$ is $2D + 2V_{\frac{1}{5}} + 2V_{\frac{2}{5}}$, where D is the determinantal representation, thus $p_g = 0$.

The group $\mathbb{D}_5$ contains 5 order 2 elements of type II, each fixes an isolated point t_i and a smooth genus 4 curve R_i, that gives one A_1 singularity of $S/\mathbb{D}_5$. As $R_i R_j = 2$ for $i \neq j$, we deduce that the curves R_i cut each other in s_1 and s_2. Moreover:

$$K_S = \pi^* K_{S/G} + \sum_{i=1}^{5} R_i,$$

therefore:

$$10K_{S/G}^2 = \left(K_S - \sum R_i\right)^2 = -20$$

and as $K_Z^2 = K_{S/G}^2$, we obtain: $K_Z^2 = -2$.

Let us compute the Euler number:

$$e(S/G) = \frac{1}{10}\left(e(S) + e(R_1 + \cdots + R_5 - s_1 - s_2) + 5 + 9 \cdot 2\right) = 1.$$

As we have only one A_1 singularity: $e(S) = 2$.

The divisor $F_t = \sum_{i=1}^{i=5} D_i$ is a connected fiber of genus 16 of a fibration $\gamma : S \to E$ onto an elliptic curve. This fibration is invariant by $\mathbb{D}_5$ (see [17], Theorem 3). The quotient map $F_u \to F_u/\mathbb{D}_5$ for F_u a generic fiber is ramified over $F_u \sum R_i = F_t(5C_t - F_t) = 50$ points, thus the genus of the quotient curve $F_u/\mathbb{D}_5$ is 0. □

■ Let S be a Fano surface with automorphism group containing two involutions of type I $\sigma_E, \sigma_{E'}$ with product of order 3 and commuting with a type III(1) automorphism σ.

PROPOSITION 30. *The minimal resolution Z of the quotient surface S/G is minimal and has invariants:*

$$c_1^2 = 1, \quad c_2 = 23, \quad q = 0, \quad p_g = 1, \quad h^{1,1} = 21.$$

PROOF. Up to a change of coordinates, the cubic can be written as:

$$F = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + ax_1x_2x_5 = 0\}.$$

The fixed points of the three involutions of type I are 3 disjoint elliptic curves E, E', E'' and 81 isolated points, divided into 9 orbits of 9 elements, giving $9A_1$ singularities. The $9 \cdot 2 = 18$ isolated points of the 2 type III(2) automorphisms gives $3A_2$ singularities on S/G. We check easily that $q = 0$ and $p_g = 1$, moreover:

$$K_{S/G}^2 = \frac{1}{18}K'^2 = 1$$

for $K' = K_S - E - E' - E''$ and we deduce that $c_2 = 23$. As $K' = F + F' + F''$ (for F, F', F'' fibers of γ_E, $\gamma_{E'}$, $\gamma_{E''}$) is nef, Z is minimal. □

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