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Partial Differential Equations — *Double Ball Property for non-divergence horizontally elliptic operators on step two Carnot groups*, by Giulio Tralli, communicated on 20 April 2012.

Abstract. — Let $\mathscr{L}$ be a linear second order horizontally elliptic operator on a Carnot group of step two. We assume $\mathscr{L}$ in non-divergence form and with measurable coefficients. Then, we prove the *Double Ball Property* for the nonnegative sub-solutions of $\mathscr{L}$. With our result, in order to solve the Harnack inequality problem for this kind of operators, it becomes sufficient to prove the so called *ε-Critical Density*.

Key words: Degenerate elliptic equation, invariant Harnack inequality, double ball property.

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1. Introduction

As it is well known, in the theory of fully nonlinear elliptic equations a crucial role is played by the Krilov-Safonov's Harnack inequality for nonnegative solutions to the linear equations in non-divergence form and measurable coefficients.

However, in several research areas, such as Complex or CR Geometry, there are fully nonlinear equations characterized by an underlying sub-Riemannian structure which are not elliptic at any point, see e.g. [8], [11], [9], [10], [2], [3], [7]. The existence theory for viscosity solutions to such equations is quite well settled, mainly thanks to the papers [11], [9], [3]. On the contrary, the problem of the solutions regularity is still widely open. This is mainly due to the lack of pointwise estimates for solutions to linear sub-elliptic equations with rough coefficients. In this context, a long standing open problem is the *invariant Harnack inequality* for positive solutions to horizontally elliptic equations on Lie groups, in non-divergence form and rough coefficients.

Di Fazio, Gutiérrez and Lanconelli in [4] found an axiomatic procedure to establish the scale invariant Harnack inequality in very general settings like doubling Hölder quasimetric spaces. Homogeneous Lie groups and, more generally, Carnot-Carathéodory spaces are remarkable examples of settings where their procedure applies. Di Fazio, Gutiérrez and Lanconelli proved that the *double-ball property* and the *ε-critical density* are sufficient conditions for the Harnack inequality to hold. Recently, this general approach has been used by Gutiérrez and Tournier to prove the Harnack inequality for a class of horizontally elliptic operators with measurable coefficients in the Heisenberg group.

In this paper, we establish the double ball property for non-divergence linear second order operators which are elliptic with respect to the generators of a step two Carnot group. To be precise, let us fix some definitions. Let $(\mathbb{R}^N, *, \delta_\lambda)$ be an homogeneous Carnot group of step two and $X_1, \dots, X_m$ be the vector fields generating the Lie algebra. The dilations $\{\delta_\lambda\}_{\lambda>0}$ are defined by $\delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$ for all $(x,t) \in \mathbb{R}^{m+n} = \mathbb{R}^N$. Consider the second order differential operator

$$\mathscr{L} = \sum_{i,j=1}^{m} a_{ij}(x,t) X_i X_j \tag{1}$$

where $A(x,t) = (a_{ij}(x,t))_{i,j \le m}$ is a $m \times m$ symmetric matrix with measurable entries. We say that $\mathscr{L}$ is horizontally elliptic on $\mathbb{R}^N$ if there exist $\Lambda > \lambda > 0$ such that

$$\lambda \|v\|^2 \le \langle A(x,t)v, v \rangle \le \Lambda \|v\|^2$$

for all $v \in \mathbb{R}^m$ and for all $(x,t) \in \mathbb{R}^N$.

Here $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{R}^m$, but we shall use the same notation for all the Euclidean norms. Moreover, we denote with $B_R(p_0)$ the homogeneous open ball of radius R centered at p_0, i.e.

$$B_R(p_0) = \{p_0 * (x,t) \in \mathbb{R}^N : \|x\|^4 + \|t\|^2 < R^4\}.$$

Following Di Fazio, Gutiérrez and Lanconelli, in the present context we can state the double ball property as follows.

Double Ball Property 1.1. *Let R be a positive constant and $p_0 \in \mathbb{R}^N$. We set*

$$K = \{u \in C^2(B_{3R}(p_0)) : u \ge 0 \text{ and } \mathscr{L}u \le 0 \text{ on } B_{3R}(p_0), u \ge 1 \text{ on } B_R(p_0)\}.$$

We say that $\mathscr{L}$ satisfies the Double Ball Property on $B_{3R}(p_0)$ if there exists a positive constant γ depending only on the ellipticity constants λ, Λ such that

$$u \ge \gamma \quad \text{on } B_{2R}(p_0)$$

for all $u \in K$.

In [5] Gutiérrez and Tournier proved this property for the Heisenberg group. In this paper we prove that it holds for a general Carnot group of step two. We first recognize that, via weak Maximum Principle, the double ball property is a consequence of a kind of solvability of the Dirichlet problem for $\mathscr{L}$ in the exterior of any homogeneous ball $B_R(p_0)$. As a matter of fact, our main tool is the existence of suitable barrier functions in the interior of $B_R(p_0)$ at any point of the boundary. In Section 2 we show that, indeed, the existence of such kind of barriers implies the double-ball property. In Section 3 we find explicit barriers at every boundary point of the homogeneous balls. At the non-characteristic points (i.e. where the horizontal gradient does not vanish) we use some standard arguments, whereas at the characteristic points our construction requires the explicit knowledge of the vector fields $X_1, \dots, X_m$ and of the composition law for groups of step two.

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2. Interior barriers

We start by recalling the weak maximum principle for the operator $\mathscr{L}$ in (1).

Weak Maximum Principle 2.1. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $u \geq v$ on $\partial\Omega$ and $\mathscr{L}u \leq \mathscr{L}v$ in Ω. Then, $u \geq v$ in Ω.*

A proof of this principle for this kind of operators can be found in [6], Corollary 1.3.

We now give the definition of interior $\mathscr{L}$-barrier function.

Definition 2.2. *Let Ω be an open set of $\mathbb{R}^N$ with non-empty boundary. Fix $p \in \partial\Omega$. A function h is an interior $\mathscr{L}$-barrier function for Ω at p if*

- *h is a C^2 function defined on an open bounded neighborhood U of p,*
- *h and U depend only on Λ, λ and the vector fields X_j's,*
- *$\mathscr{L}h \leq 0$ on U,*
- *$h(p) = 0$,*
- *$\{(x,t) \in U : h \leq 0\} \backslash \{p\} \subseteq \Omega$.*

We are going to prove some lemmas.

Lemma 2.3. *Let T be a compact subset of an open set $O \subset \mathbb{R}^N$. There exists $v_0 > 1$ such that*

$$\delta_v T \subset O$$

for all $v \in [1, v_0]$.

Proof. The sets T and $\mathbb{R}^N \backslash O$ are close and disjoint. Therefore, their distance d is a positive number. If $(x,t) \in T$ and $\lambda > 0$, we have

$$\operatorname{dist}(\delta_\lambda(x,t), T) \leq \operatorname{dist}(\delta_\lambda(x,t), (x,t)) \leq |\lambda - 1| \sqrt{\|x\|^2 + (\lambda+1)^2 \|t\|^2}.$$

Since T is bounded, it is easy to choose $v_0 > 1$ such that

$$\sup_{(x,t) \in T} \operatorname{dist}(\delta_v(x,t), T) < d$$

for all $v \in [1, v_0]$. □

We set

$$K_0 = K_0(\mathscr{L}) = \{u \in C^2(B_{3/2}(0)) : u \geq 0 \text{ and } \mathscr{L}u \leq 0 \text{ on } B_{3/2}(0),\\ u \geq 1 \text{ on } B_1(0)\}.$$

The next lemma is an application of the weak maximum principle for the operator $\mathscr{L}$.

Lemma 2.4. *Suppose that, for every $p \in \partial B_1(0)$, there exists an interior $\mathscr{L}$-barrier function for $B_1(0)$ at p. Then, there exists $\nu \in \left(1, \frac{3}{2}\right)$ (not depending on the coefficients of the matrix A) such that*

$$u \geq \frac{1}{2} \quad \text{on } B_\nu(0)$$

for all $u \in K_0$.

Proof. Fix $p \in \partial B_1(0)$ and consider the barrier function $h = h_p$ defined on $U = U_p$. If we set $V = (U \cap B_{3/2}(0)) \backslash \overline{B_1(0)}$, we have that $h \geq 0$ and $\mathscr{L}h \leq 0$ on V. Let us now consider the boundary $\partial V = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \partial V \cap \partial B_1(0)$ and $\Gamma_2 = \partial V \backslash \Gamma_1$. The number $m = \inf_{\Gamma_2} h$ is strictly positive because $\{(x,t) \in \partial V : h(x,t) = 0\} = \{p\}$. So, the function $w = 1 - \frac{1}{m}h$ is well defined. We get

$$\mathscr{L}w = -\frac{1}{m}\mathscr{L}h \geq 0 \quad \text{on } V, \qquad w \leq 1 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad w \leq 0 \quad \text{on } \Gamma_2.$$

If $u \in K_0$, we deduce

$$\mathscr{L}u \leq \mathscr{L}w \quad \text{on } V, \qquad u \geq w \quad \text{on } \partial V.$$

By the Weak Maximum Principle for $\mathscr{L}$, $u \geq w$ on V. Since $w(p) = 1$, there exists an open neighborhood W_p of p contained in $U \cap B_{3/2}(0)$ where $w \geq \frac{1}{2}$. The sets W_p depend only on the barrier functions and so on the ellipticity constants.

The compact set $\partial B_1(0)$ is contained in the open set $O = \bigcup_{p \in \partial B_1(0)} W_p$. By the previous lemma, there exists $\nu > 1$ such that $(B_\nu(0) \backslash B_1(0)) \subset O$. Therefore, we deduce

$$u \geq \frac{1}{2} \quad \text{on } B_\nu(0)$$

for all $u \in K_0$. □

We are now ready to prove the double ball property under the assumptions of the previous lemma.

Proposition 2.5. *Suppose that there exists an interior $\mathscr{L}$-barrier function for $B_1(0)$ at every point of $\partial B_1(0)$. Then, the Double Ball Property* 1.1 *on $B_{3R}(p_0)$ is satisfied for all $R > 0$ and $p_0 \in \mathbb{R}^N$.*

PROOF. Fix $p_0 = 0$ and $R = 1$. If $u \in K$, in particular $u \in K_0$. By the previous lemma, $u \geq \frac{1}{2}$ on $B_\nu(0)$ for a fixed $1 < \nu < \frac{3}{2}$. Let us consider the function

$$v = 2u \circ \delta_\nu.$$

It is a non-negative function of class C^2 defined on $B_{3/\nu}(0) \supseteq B_{3/2}(0)$ (since $\nu < 2$). We have that $v \geq 1$ on $B_1(0)$. By setting $\tilde{\mathscr{L}} = \sum_{i,j} \tilde{A}_{i,j}(p) X_i X_j$ where $\tilde{A}(p) = A(\delta_\nu(p))$, we get

$$\tilde{\mathscr{L}} v(p) = 2\nu^2 (\mathscr{L} u)(\delta_\nu(p)) \leq 0$$

because of the homogeneity of the vector fields. This means that $v \in K_0(\tilde{\mathscr{L}})$, but $\tilde{A}(p)$ have the same ellipticity constants of $A(p)$ and ν depends only on these. So $v \geq \frac{1}{2}$ on $B_\nu(0)$, that implies $u \geq \frac{1}{4}$ on $B_{\nu^2}(0)$. If $\nu^2 \geq 2$, we have just proved the statement. If it is not, the argument can be reapplied. Since $\nu > 1$, there exists an integer n_0 such that $\nu^{n_0} \geq 2$. Therefore, we get

$$u \geq \frac{1}{2^{n_0}} =: \gamma \quad \text{on } B_2(0)$$

for all $u \in K$.

If p_0 and R are arbitrary, we can argue in the same way. As a matter of fact, we consider the function

$$\tilde{u}(p) = u(p_0 * \delta_R(p))$$

for $u \in C^2(B_{3R}(p_0))$. The homogeneity and the left-invariance of $\mathscr{L}$ imply that

$$\sum_{i,j} A_{i,j}(p_0 * \delta_R(p)) X_i X_j \tilde{u}(p) = R^2 (\mathscr{L} u)(p_0 * \delta_R(p)).$$

So, the argument above works with the same constant γ. □

3. Explicit barriers

It is known (see, e.g. [1]) that the m vector fields generating an N-dimensional Carnot group of step two are, up to isomorphism, of the form

$$X_i(x,t) = \partial_{x_i} + \frac{1}{2} \sum_{k=1}^{n} (B^k x)_i \partial_{t_k},$$

where $B^1, \ldots, B^n$ are suitable $m \times m$ linearly independent skew-symmetric matrices.

In order to apply Proposition (2.5), we have to prove that there exists an interior $\mathscr{L}$-barrier function for $B_1(0)$ at every point of $\partial B_1(0)$. We are going to give a sufficient condition for the existence of these barriers.

LEMMA 3.1. *Let Ω be a bounded domain defined by*

$$\Omega = \{(x,t) \in \mathbb{R}^N : F(x,t) < 0\},$$

where F is a real-valued function. Fix $p = (x_0, t_0) \in \partial\Omega$. Suppose that F is smooth near p and

$$\nabla_X F := (X_1 F, \ldots, X_m F) \neq 0$$

at p. Then, there exists an interior $\mathscr{L}$-barrier function for Ω at p.

Proof. Let us denote by $B_e((\xi_0, \tau_0), \beta)$ the euclidean ball centered at (ξ_0, τ_0) with radius β. We choose

$$(\xi_0, \tau_0) = p - \beta \frac{\nabla F(p)}{\|\nabla F(p)\|}$$

and β small enough such that $B_e((\xi_0, \tau_0), \beta)$ is tangent to $\partial\Omega$ at p and contained in Ω. Let us now consider the function

$$h(x, t) = e^{-\alpha\beta^2} - e^{-\alpha(\|x-\xi_0\|^2 + \|t-\tau_0\|^2)}.$$

The positive constant α will be fixed later on. This function is strictly positive out of $B_e((\xi_0, \tau_0), \beta)$ and vanishing on the sphere. An easy computation shows that, for $j = 1, \ldots, m$,

$$\begin{aligned} X_j h(x, t) &= \alpha e^{-\alpha(\|x-\xi_0\|^2 + \|t-\tau_0\|^2)} \Big(2(x - \xi_0)_j + \sum_{k=1}^{n} (B^k x)_j (t - \tau_0)_k \Big) \\ &=: \alpha e^{-\alpha(\|x-\xi_0\|^2 + \|t-\tau_0\|^2)} v_j(x, t). \end{aligned}$$

We have

$$\begin{aligned} \mathscr{L} h(x, t) = \alpha e^{-\alpha(\|x-\xi_0\|^2 + \|t-\tau_0\|^2)} \sum_{i,j=1}^{m} a_{ij}(x, t) \\ \times \Big(2\delta_{ij} + \sum_{k=1}^{n} b_{ji}^k (t - \tau_0)_k + \frac{1}{2} \sum_{k=1}^{n} (B^k x)_j (B^k x)_i - \alpha v_i(x, t) v_j(x, t) \Big). \end{aligned}$$

The product of a symmetric matrix and a skew-symmetric matrix has zero trace, so $\mathrm{Tr}(A(x, t) B^k) = 0$ and we get

$$\begin{aligned} \mathscr{L} h(x, t) &= \alpha e^{-\alpha(\|x-\xi_0\|^2 + \|t-\tau_0\|^2)} \Big(2\,\mathrm{Tr}(A(x, t)) + \frac{1}{2} \sum_{k=1}^{n} \langle A(x, t) B^k x, B^k x \rangle \\ &\qquad - \alpha \langle A(x, t) v(x, t), v(x, t) \rangle \Big) \\ &\leq \alpha e^{-\alpha(\|x-\xi_0\|^2 + \|t-\tau_0\|^2)} \Big(2m\Lambda + \frac{\Lambda}{2} \sum_{k=1}^{n} \|B^k x\|^2 - \alpha\lambda \|v(x, t)\|^2 \Big) \\ &=: H(x, t). \end{aligned}$$

We stress that the function H depends on λ, Λ, but it does not depend on the coefficients of the matrix A. We also remark that

$$v(p) = \frac{2\beta}{\|\nabla F(p)\|}\nabla_X F(p) \neq 0.$$

Therefore, if we choose

$$\alpha > \frac{\Lambda}{\lambda}\left(2m + \frac{1}{2}\sum_{k=1}^{n}\|B^k x_0\|^2\right)\frac{\|\nabla F(p)\|^2}{4\beta^2\|\nabla_X F(p)\|^2},$$

we obtain $H(p) < 0$. Then, there exists an open bounded neighborhood U of p (depending only on the function H, namely on p, F, λ, Λ and on the matrices defining the vector fields) where $\mathscr{L}h \le H < 0$. The function h has all the properties required to be an interior $\mathscr{L}$-barrier function for Ω at p. □

Remark 3.2. If we denote with N the defining function of $B_1(0)$, i.e. $N(x,t) = \|x\|^4 + \|t\|^2 - 1$, we have

$$\nabla_X N(x,t) = 4\|x\|^2 x + \sum_{k=1}^{n} t_k B^k x.$$

Since the matrices B^k's are skewsymmetric, the vectors x and $B^k x$ are orthogonal for every $k = 1,\dots,n$. So, we can state that

$$\nabla_X N(x,t) = 0 \quad \Leftrightarrow \quad x = 0.$$

Proposition 3.3. *For every $p \in \partial B_1(0)$, there exists an interior $\mathscr{L}$-barrier function for $B_1(0)$ at p.*

Proof. By Lemma 3.1 and the last remark, it remains only to prove the existence of a barrier at the points $(0,t_0) \in \partial B_1(0)$. So, let us fix $t_0 = (t_1^0,\dots,t_n^0)$ with $\|t_0\| = 1$. Denote with P the orthogonal projector on $\mathrm{Range}(\sum_{k=1}^n t_k^0 B^k) = \mathrm{Ker}(\sum_{k=1}^n t_k^0 B^k)^\perp$ and with Q the orthogonal projector on $\mathrm{Ker}(\sum_{k=1}^n t_k^0 B^k)$. We remark that $x = Px + Qx$ and

$$\left\|\sum_{k=1}^{n} t_k^0 B^k x\right\| \ge \sigma\|Px\|, \quad \sigma > 0,$$

for all $x \in \mathbb{R}^m$. Since the matrices B^k's are linearly independent, the matrix P has got a positive rank N_1, $0 < N_1 \le m$. Moreover, we put $M = \max_k\|B^k\|$. For a fixed

$$\gamma > \frac{\Lambda}{\lambda}\left(\frac{5m}{2N_1} + \frac{15 + m - N_1}{N_1} + \frac{5nM^2}{16N_1}\right)$$

(in particular we note that $\gamma > 2$ and $\gamma > \frac{\Lambda}{\lambda}\frac{m-N_1}{N_1}$), we set

$$f(x,t) = \|x\|^4 + (\|Qx\|^2 - \gamma\|Px\|^2)^2 + \|t'\|^2 + \langle t, t_0\rangle,$$

where $t' = t - \langle t, t_0\rangle t_0$. Finally, for a positive constant β to be fixed later on, we put

$$h(x,t) = e^{-\beta} - e^{-\beta f(x,t)}.$$

The function h vanishes at $(0, t_0)$ and it is negative if and only if $f < 1$. So, we have

$$\{(x,t) \in \mathbb{R}^N : h(x,t) \le 0, \langle t, t_0\rangle > 0\}\backslash\{(0,t_0)\} \subset B_1(0).$$

A straightforward calculation shows that

$$\begin{aligned} X_j h(x,t) = \beta e^{-\beta f(x,t)}\Big(&4\|x\|^2 x_j + 4(\|Qx\|^2 - \gamma\|Px\|^2)(Qx - \gamma Px)_j \\ &+ \sum_{k=1}^n t'_k(B^k x)_j + \frac{1}{2}\sum_{k=1}^n t_k^0 (B^k x)_j\Big) = \beta e^{-\beta f(x,t)} X_j f(x,t). \end{aligned}$$

Then we get

$$\begin{aligned} \mathscr{L}h(x,t) = \beta e^{-\beta f(x,t)}\Bigg(&4\|x\|^2\,\mathrm{Tr}(A(x,t)) + 8\langle A(x,t)x, x\rangle \\ &+ 4(\|Qx\|^2 - \gamma\|Px\|^2)(\mathrm{Tr}(A(x,t)Q) - \gamma\,\mathrm{Tr}(A(x,t)P)) \\ &+ 8\langle A(x,t)(Qx - \gamma Px), Qx - \gamma Px\rangle \\ &+ \frac{1}{2}\sum_{k=1}^n \langle A(x,t)B^k x, B^k x\rangle - \left\langle A(x,t)\sum_{k=1}^n t_k^0 B^k x, \sum_{k=1}^n t_k^0 B^k x\right\rangle \\ &- \beta\langle A(x,t)\nabla_X f(x,t), \nabla_X f(x,t)\rangle\Bigg) \\ \le \beta e^{-\beta f(x,t)}\Bigg(&4\Lambda\|x\|^2(m+2) + 8\Lambda(\|Qx\|^2 + \gamma^2\|Px\|^2) \\ &+ 4(\|Qx\|^2 - \gamma\|Px\|^2)(\mathrm{Tr}(A(x,t)Q) - \gamma\,\mathrm{Tr}(A(x,t)P)) \\ &+ \frac{\Lambda}{2}\sum_{k=1}^n \|B^k x\|^2 - \lambda\left\|\sum_{k=1}^n t_k^0 B^k x\right\|^2 - \beta\lambda\|\nabla_X f(x,t)\|^2\Bigg). \end{aligned}$$

Since $\gamma > \frac{\Lambda}{\lambda}\frac{m-N_1}{N_1}$, we have

$$\mathrm{Tr}(A(x,t)Q) - \gamma\,\mathrm{Tr}(A(x,t)P) \le (m - N_1)\Lambda - \gamma N_1\lambda < 0.$$

If $\|Px\|^2 \le \frac{1}{\gamma^2}\|Qx\|^2$, then in particular $\|Qx\|^2 - \gamma\|Px\|^2 \ge \frac{2}{5}\|x\|^2$ (since $\gamma > 2$) and so we deduce

$$\mathscr{L}h(x,t) \le \beta e^{-\beta f(x,t)}\|x\|^2\Big(4m\Lambda + 24\Lambda + \frac{8}{5}((m-N_1)\Lambda - \gamma N_1\lambda) + \Lambda n\frac{M^2}{2}\Big) < 0$$

because of our choice of γ. Otherwise, if $\|Px\|^2 > \frac{1}{\gamma^2}\|Qx\|^2$, then $\|Px\|^2 \ge \frac{1}{1+\gamma^2}\|x\|^2$ and we have

$$\begin{aligned}\|\nabla_X f(x,t)\| &\ge \left\|4\|x\|^2 x + 4(\|Qx\|^2 - \gamma\|Px\|^2)(Qx - \gamma Px) + \frac{1}{2}\sum_{k=1}^n t_k^0 B^k x\right\| \\ &\quad - \left\|\sum_{k=1}^n t_k' B^k x\right\| \ge \frac{1}{2}\left\|\sum_{k=1}^n t_k^0 B^k x\right\| - \|t'\|\sum_{k=1}^n \|B^k x\| \\ &\ge \frac{\sigma}{2}\|Px\| - \|t'\| nM\|x\| \ge \Big(\frac{\sigma}{2\sqrt{1+\gamma^2}} - \|t'\|nM\Big)\|x\|.\end{aligned}$$

Here we used the fact that the vector $\sum_{k=1}^n t_k^0 B^k x$ is orthogonal to Px and Qx. Hence, if in addition $\|t'\| < \frac{\sigma}{4nM\sqrt{1+\gamma^2}}$, then

$$\|\nabla_X f(x,t)\| \ge \frac{\sigma}{4\sqrt{1+\gamma^2}}\|x\|$$

and so we deduce

$$\begin{aligned}\mathscr{L}h(x,t) \le \beta e^{-\beta f(x,t)}\|x\|^2\Big(&4\Lambda(m+2) + 4\gamma(\gamma N_1\Lambda - (m-N_1)\lambda) \\ &+ 16\Lambda\gamma^2 + \Lambda n\frac{M^2}{2} - \lambda\frac{\sigma^2}{1+\gamma^2} - \beta\lambda\frac{\sigma^2}{16(1+\gamma^2)}\Big).\end{aligned}$$

By choosing β big enough, we obtain $\mathscr{L}h < 0$. Summing up, the function h is an interior $\mathscr{L}$-barrier function for $B_1(0)$ at $(0, t_0)$ if we consider it on the domain $\big\{(x,t) : \langle t, t_0\rangle > 0, \|t'\| < \frac{\sigma}{4nM\sqrt{1+\gamma^2}}\big\}$.

We stress that, if $m = N_1$ (that is $Q = 0$), we can choose a simpler barrier like

$$e^{-\beta} - e^{-\beta(\|x\|^4 + \|t'\|^2 + \langle t, t_0\rangle)}.$$

The condition $m = N_1$ for all $(0, t_0) \in \partial B_1(0)$ means exactly that the group is an H-group in the sense of Metivier (in particular the groups of Heisenberg type satisfy this condition). □

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