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**Probability Theory** — *Well posedness of Fokker–Planck equations for generators of time-inhomogeneous Markovian transition probabilities*, by GIUSEPPE DA PRATO and MICHAEL RÖCKNER, communicated on 20 April 2012.

ABSTRACT. — The aim of this paper is to study existence and uniqueness for Fokker–Planck equations for operators being generators of time-inhomogeneous Markovian transition probabilities.

KEY WORDS: Fokker–Planck equations, Markovian transition probabilities, differential stochastic equations.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 35Q84, 60J35, 47D07.

## 1. INTRODUCTION AND FRAMEWORK

In all this paper $E$ represents a Polish space with metric $d$. We denote by $\mathscr{B}(E)$ the $\sigma$-algebra of all Borel subsets of $E$ and by $\mathscr{P}(E)$ the set of all probability measures defined on $(E, \mathscr{B}(E))$. Moreover, $B_b(E)$, $C_b(E)$, $UC_b(E)$ is the Banach space of all real bounded functions, which are Borel measurable, continuous, uniformly continuous, respectively, endowed with the norm $\|\varphi\|_0 = \sup_{x\in E}|\varphi(x)|$.

Let $T > 0$ be fixed. While in Sections 1–3 of this paper $T$ is assumed to be finite and we consider the interval $[0, T]$, in Section 4 we work on all of $\mathbb{R}$. We say that $\pi = \pi_{s,t}(\cdot\,,\cdot)$, $0 \le s \le t \le T$, is a *Markovian transition probability* on $E$, if

(i) $\pi_{s,t}(x,\cdot)$ is a probability measure on $(E, \mathscr{B}(E))$ for each $0 \le s \le t \le T$, $x \in E$.
(ii) $\pi_{s,t}(\cdot, \Gamma)$ belongs to $B_b(E)$ for each $0 \le s \le t \le T$, $\Gamma \in \mathscr{B}(E)$.
(iii) $\pi_{s,t}(x,\Gamma) = \displaystyle\int_E \pi_{s,r}(x,dy)\pi_{r,t}(y,\Gamma)$, for each $0 \le s \le t \le T$, $\Gamma \in \mathscr{B}(E)$.
(iv) $\pi_{s,s}(x,\Gamma) = \mathbb{1}_\Gamma(x)$, for each $x \in E$, $\Gamma \in \mathscr{B}(E)$.
(v) $\pi$ is called *forward continuous*, if for all $u \in C_b(E)$, $x \in E$, $0 \le s \le t \le T$

$$\lim_{r\to t, r\in[s,T]} \int_E u(y)\pi_{s,r}(x,dy) = \int_E u(y)\pi_{s,t}(x,dy).$$

(vi) $\pi$ is called *parabolic Feller*, if for all $u \in C_b([0,T]\times E)$, $\tau > 0$

$$(s,x) \mapsto \int_E u(s,y)\pi_{s,s+\tau}(x,dy).$$

is uniformly continuous on $[0,T]\times E$.

Any Markovian transition probability $\pi$ on $E$ defines a family of linear operators $P_{s,t}$, $0 \le s \le t \le T$, on the space $B_b(E)$, by the formula

$$P_{s,t}\varphi(x) = \int_E \varphi(y)\pi_{s,t}(x,dy), \quad 0 \le s \le t \le T,\ x \in E,\ \varphi \in B_b(E). \tag{1.1}$$

$P_{s,t}$, $0 \le s \le t \le T$, is called the *Markovian transition evolution operator* associated to the transition function $\pi$. By (1.1) it follows that

$$P_{s,t} = P_{s,r}P_{r,t}, \quad \forall 0 \le s \le r \le t \le T, \quad P_{s,s} = I, \quad \forall s \in [0,T]. \tag{1.2}$$

REMARK 1.1. Clearly (vi) implies that $P_{s,t}(UC_b(E)) \subset UC_b(E)$ for all $0 \le s \le t \le T$. For the time homogeneous case this is even equivalent to (vi). Indeed, suppose $\pi$ is time homogeneous, i.e. $\pi_{s,s+\tau} = \pi_{0,\tau}$ for all $s \in [0,T]$, $\tau \in [0, T-s]$. Then $\pi$ is parabolic Feller (see property (vi) above) if and only if

$$P_{0,\tau}(UC_b(E)) \subset UC_b(E), \quad \forall \tau \ge 0.$$

That (vi) implies the latter is obvious. The converse follows by Lemma A.1 in the Appendix.

EXAMPLE 1.2. Let us consider the stochastic differential equation

$$\begin{cases} dX(t) = (AX(t) + F(t, X(t))\,dt + B\,dW(t), & t \in [s,T] \\ X(s) = x \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; H), \end{cases} \tag{1.3}$$

on a separable real Hilbert space $H$, where $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $e^{tA}$ in $H$, $B \in L(H)$ and $F : D(F) \subset [0,T] \times H \to H$ is measurable possibly nonlinear. We assume that for any $t > 0$ we have $\operatorname{Tr} Q_t < +\infty$ where

$$Q_t x = \int_0^t e^{sA} BB^* e^{sA^*} x\,ds, \quad x \in H.$$

To express the dependence on $s$, $x$, we denote a solution of (1.3) by

$$X(t,s,x), \quad t \ge s.$$

Concerning $F$ we assume that problem (1.3) is well-posed. Define the corresponding transition evolution operator

$$P_{s,t}\varphi(x) := \mathbb{E}[\varphi(X(t,s,x))], \quad \varphi \in C_b(H). \tag{1.4}$$

Then denoting by $\pi_{s,t}(x,dy)$ the law of $X(t,s,x)$ it is clear that $\pi_{s,t}(\cdot,\cdot)$, $0 \le s \le t \le T$, is a Markovian transition probability, which is forward con-

tinuous, if $t \mapsto X(t,s,x)$ is $\mathbb{P}$-a.s. continuous. Note that for $u \in C_b([0,T] \times E)$, $(s,x) \in [0,T] \times E$ and $\tau > 0$

$$\mathbb{E}[u(s+\tau, X(s+\tau,s,x))] = \int_E u(s+\tau, y)\pi_{s,s+\tau}(x,dy).$$

Hence $\pi$ is parabolic Feller if the law of the space time process

$$Z(\tau,(s,x)) := (s+\tau, X(s+\tau,s,x)), \quad \tau \geq 0,$$

depends uniformly continuously on its initial condition $(s,x)$. To ensure this we need corresponding conditions on $F$ in assumptions (1.3), as e.g. that $D(F) = [0,T] \times H$, $F : [0,T] \times H \to H$ is uniformly continuous and there exists a constant $K > 0$ such that

$$|F(t,x) - F(t,y)| \leq K|x-y|$$

for a $x.y \in H$, $t \in [0,T]$.

## 2. The transition semigroup on $C_{b,T}([0,T] \times E)$

We fix $T > 0$ and a Markovian transition probability $\pi = \pi_{s,t}$, $0 \leq s \leq t \leq T$, satisfying (v) and (vi), with corresponding Markovian transition evolution operators $(P_{s,t})_{0 \leq s \leq t \leq T}$. Define a semigroup $S_\tau^{(T)}$, $\tau \geq 0$, of linear operators on

$$C_{b,T}([0,T] \times E) := \{u \in UC_b([0,T] \times E) : u(T,x) = 0, \forall x \in E\}$$

as follows

$$(S_\tau^{(T)}u)(t,x) = \begin{cases} (P_{t,t+\tau}(u(t+\tau,\cdot)))(x), & \text{if } 0 \leq t \leq T-\tau,\ x \in E, \\ 0, & \text{if } T-\tau \leq t \leq T,\ x \in E. \end{cases} \tag{2.1}$$

To show that $S_\tau^{(T)}u \in C_{b,T}([0,T] \times E)$, by (vi) and Lemma A.1 in the Appendix we have only to check that

$$[0,T] \ni t \mapsto P_{t,t+\tau}(u(t+\tau,\cdot)) \in UC_b(E),$$

is left continuous in $t := T-\tau$. So, let $t_n \in [0, T-t]$, $t_n \to T-t$, as $n \to \infty$. Then since $u$ is uniformly continuous on $[0,T] \times E$ and $u(t,x) = 0$ for $x \in E$ we have $\lim_{n\to\infty} \|u(t_n+\tau,\cdot)\|_0 = 0$, where $\|\cdot\|_0$ denotes the supremum norm. But

$$\|P_{t_n,t_n+\tau}u(t_n+\tau,\cdot))\|_0 \leq \|u(t_n+\tau,\cdot)\|_0,$$

so, in $UC_b(E)$

$$\lim_{n\to\infty} P_{t_n,t_n+\tau}u(t_n+\tau,\cdot)) = 0 = P_{T-\tau,T}(u(T,\cdot)).$$

Furthermore, we note that obviously for all $f \in C_{b,T}([0,T] \times E)$

$$S_0^{(T)} f = f$$

and

$$S_\tau^{(T)} f(t,x) = \mathbb{1}_{[0,T-\tau)}(t) S_\tau^{(T)} f(t,x), \quad \forall (t,x) \in [0,T] \times E.$$

If, in particular $u(t,x) = \alpha(t)\varphi(x)$, $t \in [0,T]$, $x \in E$, we have

$$(S_\tau^{(T)} u)(t,x) = \alpha(t+\tau)(P_{t,t+\tau}\varphi)(x), \quad t \in [0,T],\ x \in E. \tag{2.2}$$

By (1.2) it follows that $S_\tau^{(T)}$, $\tau \geq 0$, is a semigroup of linear operators on $C_{b,T}([0,T] \times E)$. However, it is not strongly continuous in general as is well known from the literature on Markov semigroups, see e.g. [11], [12], [5], [19], [10], [14], [15], [16] and [18].

Let us introduce the notion of $\pi$-convergence following [19]. Let $(u_n) \subset C_{b,T}([0,T] \times E)$ and $u \in C_{b,T}([0,T] \times E)$. We say that $(u_n)$ is $\pi$-convergent to $u$ and write $u_n \xrightarrow{\pi} u$ if

(i) $\lim_{n\to\infty} u_n(t,x) = u(t,x)$, $\forall (t,x) \in [0,T] \times E$.

(ii) $\sup_{n\in\mathbb{N}} \|u_n\|_0 < \infty$, (with $\|\cdot\|_0$ denoting the supremum norm).

We call an operator $S$ on $C_{b,T}([0,T] \times E)$ $\pi$-continuous if $u_n \xrightarrow{\pi} u$ implies $Su_n \xrightarrow{\pi} Su$ for $u, u_n \in C_{b,T}([0,T] \times E)$.

PROPOSITION 2.1. *$S_\tau^{(T)}$ is $\pi$-continuous for all $\tau \geq 0$.*

PROOF. Obviously,

$$\sup_{n\in\mathbb{N}} \|S_\tau^{(T)} u_n\|_0 \leq \sup_{n\in\mathbb{N}} \|u_n\|_0 < \infty.$$

Moreover if $0 \leq t \leq T - \tau$, we have

$$|(S_\tau^{(T)} u)(t,x) - (S_\tau^{(T)} u_n)(t,x)| \leq (P_{t,t+\tau}|u(t+\tau,\cdot) - u_n(t+\tau,\cdot)|)(x).$$

Now the conclusion follows from the dominated convergence theorem. □

PROPOSITION 2.2. *Let $u \in C_{b,T}([0,T] \times E)$. Then whenever $\tau, \tau_n \in [0,\infty)$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \tau_n = \tau$, we have*

$$S_{\tau_n}^{(T)} u \xrightarrow{\pi} S_\tau^{(T)} u, \quad \text{as } n \to \infty.$$

PROOF. **Case 1**. Consider $(t,x) \in [0, T-\tau) \times E$. Then for large enough $n$, also $(t,x) \in [0, T-\tau_n) \times E$. Hence

$$|(S^{(T)}_{\tau_n}u)(t,x) - (S^{(T)}_{\tau}u)(t,x)|$$
$$= |(P_{t,t+\tau_n}(u(t+\tau_n,\cdot)))(x) - (P_{t,t+\tau}(u(t+\tau,\cdot)))(x)| \to 0 \quad \text{as } n \to \infty,$$

since

$$\lim_{n\to\infty} \|u(t+\tau_n,\cdot) - u(t+\tau,\cdot)\|_0 = 0.$$

**Case 2**. Consider $(t,x) \in (T-\tau, T] \times E$.
Since then for large $n$ also $(t,x) \in (T-\tau_n, T] \times E$ we have

$$(S^{(T)}_{\tau_n}u)(t,x) = 0 = (S^{(T)}_{\tau}u)(t,x).$$

**Case 3**. Consider $(T-\tau, x) \in [0,T] \times E$.
Then if $\tau_n \geq \tau$ we have

$$(S^{(T)}_{\tau_n}u)(T-\tau,x) = 0 = (S^{(T)}_{\tau}u)(T-\tau,x).$$

So, we may assume $\tau_n < \tau$ for all $n \in \mathbb{N}$.
Then

$$|(S^{(T)}_{\tau_n}u)(T-\tau,x)| = |(P_{T-\tau,T-\tau+\tau_n}(u(T-\tau+\tau_n,\cdot)))(x)|$$
$$\leq \|u(T-\tau+\tau_n,\cdot)\|_0 \to 0 \quad \text{as } n \to \infty,$$

hence

$$\lim_{n\to\infty} (S^{(T)}_{\tau_n}u)(T-\tau,x) = 0 = (S^{(T)}_{\tau}u)(T-\tau,x). \qquad \square$$

### *2.1. The infinitesimal generator of $S^{(T)}_{\tau}$, $\tau \geq 0$*

DEFINITION 2.3. *We say that $u \in C_{b,T}([0,T] \times E)$ belongs to the domain of $\mathscr{K}^{(T)}$ if*

(i) *For each $(t,x) \in [0,T] \times E$ there exists the limit*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}((S^{(T)}_{\varepsilon}u)(t,x) - u(t,x)) =: (\mathscr{K}^{(T)}u)(t,x)$$

*and $\mathscr{K}^{(T)}u \in C_{b,T}([0,T] \times E)$.*

(ii) $\sup_{\varepsilon\in(0,1]} \frac{1}{\varepsilon}\|S^{(T)}_{\varepsilon}u - u\|_0 < +\infty$.

*$\mathscr{K}^{(T)}$ is called the* infinitesimal generator *of $S^{(T)}_{\tau}$.*

In the following we set

$$\Delta_{\varepsilon} := \frac{1}{\varepsilon}(S^{(T)}_{\varepsilon} - 1).$$

PROPOSITION 2.4. *Let $u \in D(\mathscr{K}^{(T)})$ and let $\tau \geq 0$. Then $S_\tau^{(T)} u \in D(\mathscr{K}^{(T)})$ and we have*

$$\mathscr{K}^{(T)} S_\tau^{(T)} u = S_\tau^{(T)} \mathscr{K}^{(T)} u. \tag{2.3}$$

*Moreover, for each $(t,x) \in [0,T] \times E$, $(S_\tau^{(T)} u)(t,x)$ is differentiable at each $\tau \geq 0$ and*

$$\frac{d}{d\tau}(S_\tau^{(T)} u)(t,x) = (\mathscr{K}^{(T)} S_\tau^{(T)} u)(t,x) = (S_\tau^{(T)} \mathscr{K}^{(T)} u)(t,x). \tag{2.4}$$

PROOF. Let $u \in D(\mathscr{K}^{(T)})$ and $(t,x) \in [0,T] \times E$. Then we have

$$(\Delta_\varepsilon S_\tau^{(T)} u)(t,x) = (S_\tau^{(T)} \Delta_\varepsilon u)(t,x).$$

Since $\Delta_\varepsilon u \xrightarrow{\pi} \mathscr{K}^{(T)} u$, by Proposition 2.1 it follows that as $\varepsilon \to 0$

$$(\Delta_\varepsilon S_\tau^{(T)} u)(t,x) \xrightarrow{\pi} (S_\tau^{(T)} \mathscr{K}^{(T)} u)(t,x).$$

So, $S_\tau^{(T)} u \in D(\mathscr{K}^{(T)})$ and $\mathscr{K}^{(T)} S_\tau^{(T)} u = S_\tau^{(T)} \mathscr{K}^{(T)} u$. On the other hand,

$$(D_\tau^+ S_\tau^{(T)} u)(t,x) = \lim_{\varepsilon \to 0^+} (S_\tau^{(T)} \Delta_\varepsilon u)(t,x) = (S_\tau^{(T)} \mathscr{K}^{(T)} u)(t,x).$$

Since $(S_\tau^{(T)} \mathscr{K}^{(T)} u)(t,x)$ is continuous in $\tau$ by Proposition 2.2, we have, by an elementary result, that $(S_\tau^{(T)} u)(t,x)$ is continuously differentiable and $(D_\tau S_\tau^{(T)} u)(t,x) = S_\tau^{(T)} \mathscr{K} u(t,x)$. □

We shall denote by $\rho(\mathscr{K}^{(T)})$ the resolvent set of $\mathscr{K}^{(T)}$, i.e. the set of all $\lambda \in \mathbb{R}$ such that

$$\lambda - \mathscr{K}^{(T)} : D(\mathscr{K}^{(T)}) \to C_{b,T}([0,T] \times E)$$

is bijective and its resolvent $R(\lambda, \mathscr{K}^{(T)}) := (\lambda - \mathscr{K}^{(T)})^{-1}$ is $\pi$-continuous.

PROPOSITION 2.5. *$\rho(\mathscr{K}^{(T)}) = \mathbb{R}$. Moreover, for any $\lambda \in \mathbb{R}$ and any $f \in C_{b,T}([0,T] \times E)$ we have for $(t,x) \in [0,T] \times E$*

$$\begin{aligned}(R(\lambda, \mathscr{K}^{(T)}) f)(t,x) &= \int_0^\infty e^{-\lambda\tau} (S_\tau^{(T)} f)(t,x)\, d\tau \\ &= \int_0^{T-t} e^{-\lambda\tau} (S_\tau^{(T)} f)(t,x)\, d\tau.\end{aligned} \tag{2.5}$$

PROOF. Let $f \in C_{b,T}([0,T] \times E)$ and for any $\lambda \in \mathbb{R}$, $(t,x) \in [0,T] \times E$ set, following [5]

$$(F(\lambda) f)(t,x) = \int_0^\infty e^{-\lambda\tau} (S_\tau^{(T)} f)(t,x)\, d\tau.$$

It is easy to see that $F(\lambda)f \in C_{b,T}([0,T]\times E)$ and that $F(\lambda)$ is $\pi$-continuous. Now we show that $\lambda \in \rho(\mathscr{K}^{(T)})$. We write

$$\begin{aligned}(\Delta_\varepsilon F(\lambda)f)(t,x)&\\ &= \frac{1}{\varepsilon}\left[e^{\lambda\varepsilon}\int_\varepsilon^{T-t} e^{-\lambda\tau}(S_\tau^{(T)}f)(t,x)\,d\tau - \int_0^{T-t} e^{-\lambda\tau}(S_\tau^{(T)}f)(t,x)\,d\tau\right]\\ &= \frac{1}{\varepsilon}(e^{\lambda\varepsilon}-1)\int_\varepsilon^{T-t} e^{-\lambda\tau}(S_\tau^{(T)}f)(t,x)\,d\tau - \frac{1}{\varepsilon}\int_0^{\varepsilon} e^{-\lambda\tau}(S_\tau^{(T)}f)(t,x)\,d\tau.\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0}(\Delta_\varepsilon F(\lambda)f)(t,x) = \lambda(F(\lambda)f)(t,x) - f(t,x). \tag{2.6}$$

On the other hand,

$$\|\Delta_\varepsilon F(\lambda)f\|_0 \le \Big(\frac{e^{\lambda\varepsilon}-1}{\varepsilon\lambda}+1\Big)\|f\|_0 \max\{1, e^{-\lambda\tau}\}.$$

and therefore $F(\lambda)f \in D(\mathscr{K}^{(T)})$ and

$$\mathscr{K}^{(T)}F(\lambda)f = \lambda F(\lambda) - f. \tag{2.7}$$

It remains to show that

$$F(\lambda)(\lambda - \mathscr{K}^{(T)})\varphi = \varphi, \quad \forall \varphi \in D(\mathscr{K}^{(T)}), \tag{2.8}$$

which then implies that $\lambda \in \rho(\mathscr{K}^{(T)})$. Let us prove (2.8). If $\varphi \in D(\mathscr{K}^{(T)})$ taking into account Proposition 2.4, we have

$$\begin{aligned}(F(\lambda)\mathscr{K}^{(T)}\varphi)(t,x) &= \int_0^\infty e^{-\lambda\tau}(S_\tau^{(T)}\mathscr{K}^{(T)}\varphi)(t,x)\,d\tau\\ &= \int_0^{T-t} e^{-\lambda\tau}\frac{d}{d\tau}(S_\tau^{(T)}\varphi)(t,x)\,d\tau\\ &= -\varphi(t,x) + \lambda(F(\lambda)\varphi)(t,x), \quad \forall (t,x)\in[0,T]\times E.\end{aligned}$$

which implies (2.8). □

REMARK 2.6. The domain $D(\mathscr{K}^{(T)})$ of $\mathscr{K}^{(T)}$ is not dense in $C_{b,T}([0,T]\times E)$ in general; however, it is easy to check that it is $\pi$-dense in $C_{b,T}([0,T]\times E)$, that is for any $u \in C_{b,T}([0,T]\times E)$ there exists $(u_n) \subset D(\mathscr{K}^{(T)})$ such that $u_n \xrightarrow{\pi} u$. (See the proof of Proposition 2.8 below.)

REMARK 2.7. By (2.5) it follows that for $\lambda > 0$

$$\|R(\lambda, \mathscr{K}^{(T)})f\|_0 \leq \frac{1}{\lambda}\|f\|_0, \quad \forall f \in C_{b,T}([0,T] \times E).$$

Therefore $\mathscr{K}^{(T)}$ is $m$-dissipative in $C_{b,T}([0,T] \times E)$.

PROPOSITION 2.8. *$D(\mathscr{K}^{(T)})$ is $\pi$-dense in $C_{b,T}([0,T] \times E)$, i.e., for every $f \in C_{b,T}([0,T] \times E)$ there exist $u_n \in D(\mathscr{K}^{(T)})$, $n \in \mathbb{N}$, such that $u_n \overset{\pi}{\to} f$.*

PROOF. Let $f \in C_{b,T}([0,T] \times E)$ and define $u_n := nR(n, \mathscr{K}^{(T)})f$, i.e. for $(t,x) \in [0,T] \times E$

$$\begin{aligned} u_n(t,x) &:= n\int_0^\infty e^{-n\tau}(S_\tau^{(T)}f)(t,x)\,d\tau \\ &= \int_0^\infty e^{-\tau}(S_{\tau/n}^{(T)}f)(t,x)\,d\tau \\ &\to (S_0^{(T)}f)(t,x) = f(t,x), \end{aligned}$$

by Proposition 2.2. Now the assertion follows by Remark 2.7. □

## 3. FOKKER–PLANCK EQUATIONS

A *probability kernel* on $[0,T] \times E$ is a mapping

$$[0,T] \to \mathscr{P}(E), \quad t \mapsto \mu_t$$

such that for any $I \in \mathscr{B}(E)$ the mapping

$$[0,T] \to \mathbb{R}, \quad t \mapsto \mu_t(I)$$

is measurable.

We shall identify a probability kernel $(\mu_t)_{t\in[0,T]}$ with the Borel measure on $[0,T] \times E$ defined by

$$\boldsymbol{\mu}(A) = \int_{[0,T]\times E} \mathbb{1}_A(t,x)\mu_t(dx)\,dt, \quad A \in \mathscr{B}([0,T] \times E).$$

DEFINITION 3.1. *We say that a probability kernel $\boldsymbol{\mu}$ is a solution of the Fokker–Planck equation if*

$$\int_{[0,T]\times E} (\mathscr{K}^{(T)}u)(t,x)\mu_t(dx)\,dt = -\int_E u(0,x)\mu_0(dx), \quad \forall u \in D(\mathscr{K}^{(T)}). \tag{3.1}$$

REMARK 3.2. Let $u \in D(\mathscr{K}^{(T)})$ and $\tau \geq 0$. Then by (3.1), replacing $u$ by $S_\tau^{(T)}u$ we deduce

$$\int_{[0,T]\times E} (\mathscr{K}^{(T)} S_\tau^{(T)} u)(t,x)\mu_t(dx)\,dt = -\int_E (S_\tau^{(T)}u)(0,x)\mu_0(dx).$$

By Proposition 2.4 integrating with respect to $\tau$, yields

$$\int_{[0,T]\times E} (S_\tau^{(T)}u)(t,x)\mu_t(dx)\,dt - \int_{[0,T]\times E} u(t,x)\mu_t(dx)\,dt = -\int_0^\tau dt \int_E (P_{0,t}u(t,\cdot))(x)\mu_0(dx). \tag{3.2}$$

Therefore, if $\boldsymbol{\mu} \in \mathscr{P}([0,T]\times E)$ is a solution to the Fokker–Planck equation (3.1), then (3.2) holds for any $u \in C_{b,T}([0,T]\times E)$, because $D(\mathscr{K}^{(T)})$ is $\pi$-dense in $C_{b,T}([0,T]\times E)$. Conversely, it is easy to see that if (3.2) holds for any $u \in C_{b,T}([0,T]\times E)$, then (3.1) is fulfilled. So, (3.1) and (3.2) are equivalent.

There are in general several solutions to (3.1); to select one of them one has to specify the initial value $\mu_0$.

THEOREM 3.3. *For any $\zeta \in \mathscr{P}(E)$ there exists a unique solution $\boldsymbol{\mu}$ to* (3.1) *such that $\mu_0 = \zeta$.*

PROOF. *Existence*. Let $\zeta \in \mathscr{P}(E)$ and for any $t \in (0,T]$ set $\mu_t =: P_{0,t}^*\zeta$ (where $P_{0,t}^*$ is the dual operator of $P_{0,t}$ in the dual space $C_b^*(E)$ of $C_b(E)$), that is

$$\int_H \varphi(x)\mu_t(dx) = \int_E (P_{0,t}\varphi)(x)\zeta(dx), \quad \forall \varphi \in C_b(E).$$

We claim that $\boldsymbol{\mu}(dt,dx) := \mu_t(dx)\,dt$ is a solution to (3.1). For this it is enough to check (3.2). We have in fact

$$\begin{aligned}
&\int_0^{T-\tau} dt \int_E (P_{t,t+\tau}u(t+\tau,\cdot))(x)\mu_t(dx) \\
&\quad = \int_0^{T-\tau} dt \int_E (P_{0,t}P_{t,t+\tau}u(t+\tau,\cdot))(x)\zeta(dx) \\
&\quad = \int_0^{T-\tau} dt \int_E (P_{0,t+\tau}u(t+\tau,\cdot))(x)\zeta(dx) \\
&\quad = \int_\tau^T dt \int_E (P_{0,t}u(t,\cdot))(x)\zeta(dx)
\end{aligned}$$

and (3.2) follows.

*Uniqueness.* Assume that $\boldsymbol{\mu^1}$ and $\boldsymbol{\mu^2}$ are two solutions of (3.1) such that $\mu_0^1 = \mu_0^2 = \zeta$. We claim that $\boldsymbol{\mu^1} = \boldsymbol{\mu^2}$. In fact from (3.1) it follows that

$$\int_{[0,T]\times E} (\mathscr{K}^{(T)}u)(t,x)(\mu_t^1(dx) - \mu_t^2(dx))\,dt = 0,$$

for all $u \in D(\mathscr{K}^{(T)})$. On the other hand, the range of $\mathscr{K}^{(T)}$ is $C_{b,T}([0,T]\times E)$ because 0 is in the resolvent set of $\mathscr{K}^{(T)}$. So, $\boldsymbol{\mu^1} = \boldsymbol{\mu^2}$. □

REMARK 3.4. For time homogeneous Markov semigroups Theorem 3.3 has been proved in [18].

PROPOSITION 3.5. *Let $\zeta \in \mathscr{P}(E)$, $\mu_t =: P_{0,t}^*\zeta$ for any $t \in (0,T]$ and $\boldsymbol{\mu}(dt,dx) = \mu_t(dx)\,dt$. Then $S_\tau^{(T)}$, $\tau > 0$, is uniquely extendable to a strongly continuous semigroup of contractions $S_\tau^{(T,\boldsymbol{\mu})}$, $\tau > 0$, in $L^1(H\times[0,T],\boldsymbol{\mu})$. By $\mathscr{K}^{(T,\boldsymbol{\mu})}$ we shall denote its infinitesimal generator.*

PROOF. We claim that

$$\int_{[0,T]\times E} |S_\tau^{(T,\boldsymbol{\mu})}u|\,d\boldsymbol{\mu} \le \int_{[0,T]\times E} |u|\,d\boldsymbol{\mu}, \quad \forall u \in C_{b,T}([0,T]\times E). \tag{3.3}$$

It is enough to show (3.3) for any $u \ge 0$. In this case (3.3) follows from (3.2). □

REMARK 3.6. It is clear that $D(\mathscr{K}^{(T)})$ is a core for $\mathscr{K}^{(T,\boldsymbol{\mu})}$.

### *3.1. Comparison with other notions of Fokker–Planck equations*

Let $E$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$. In the literature one is generally concerned with a different concept of Fokker–Planck equations. Namely, one considers a *concrete* differential operator

$$\mathscr{N}u(t,x) = D_t u(t,x) + \frac{1}{2}[BB^* D_x^2 u(t,x)] + \langle Ax + F(t,x), D_x u(t,x)\rangle,$$

defined in some space $D(\mathscr{N})$ of smooth functions. Then, given a suitable $\zeta_0 \in \mathscr{P}(E)$, one looks for a probability kernel $\mu_t(dx)\,dt$ such that

$$\int_{[0,T]\times E} \mathscr{N}u(t,x)\mu_t(dx)\,dt = -\int_E u(0,x)\mu(0)(dx), \quad u \in D(\mathscr{N}). \tag{3.4}$$

To explain the difference, let us go back to Example 1.2 assuming again that problem (1.3) is well posed. In this case the well posedness of the problem is equivalent to saying that $D(\mathscr{N})$ is a core for $\mathscr{K}^{(T,\boldsymbol{\mu})}$ for every solution of (3.4).

It is important to consider the case when it is not known that the SDE corresponding to $\mathscr{N}$ is well posed. In this case solving the Fokker–Planck equation will produce a kind of weak solution. See [7], [1], [2], [3], [4].

## 4. Asymptotic behavior

We are here concerned with a Markovian transition probability $\pi = \pi_{s,t}(x, \cdot)$ on $E$, with $-\infty < s \le t < +\infty$ satisfying (i)–(vi) from Section 1 with $\mathbb{R}$ replacing the interval $[0, T]$. In this case we can define a semigroup in $C_b(\mathbb{R} \times E)$ setting

$$(S_\tau u)(t, x) = (P_{t,t+\tau} u)(t + \tau, \cdot))(x), \quad u \in C_b(\mathbb{R} \times E). \tag{4.1}$$

It is easy to prove several properties for $S_\tau$ similar to those seen for $S_\tau^{(T)}$. In particular, we can define the infinitesimal generator $\mathscr{K}$ of $S_\tau$ through its resolvent. We prove again that $\mathscr{K}$ is $m$-dissipative, however, we can only say that its resolvent set contains $[0, +\infty)$.

Following [8], [9], we say that a family $\nu_t$, $t \in \mathbb{R}$, is an evolution system of measures if

$$\int_E P_{s,t} \varphi \, d\nu_s = \int_E \varphi \, d\nu_t, \quad \forall \varphi \in C_b(E), \ -\infty < s \le t < +\infty. \tag{4.2}$$

(4.2) is equivalent to

$$P^*_{s,t} \nu_s = \nu_t, \quad -\infty < s \le t < +\infty. \tag{4.3}$$

Note that from property (v) it follows from (4.3) that $\nu_t$, $t \in \mathbb{R}$, is *continuous* in the sense that

$$t \mapsto \int_E \varphi \, d\nu_t$$

is continuous on $\mathbb{R}$ for all $\varphi \in C_b(E)$ (or, equivalently, for all $\varphi \in UC_b(E)$).

Evolution systems of measures are naturally connected to invariant measures of $S_\tau$, $\tau \ge 0$, as the following proposition shows.

Proposition 4.1. *Let $\nu_t$, $t \in \mathbb{R}$, be continuous. Then $\nu_t$, $t \in \mathbb{R}$, is an evolution system of measures if and only if we have*

$$\int_{\mathbb{R} \times E} S_\tau u \, d\boldsymbol{\nu} = \int_{\mathbb{R} \times E} u \, d\boldsymbol{\nu}, \quad \forall u \in C_b(\mathbb{R} \times E) \cap L^1(\mathbb{R} \times E; \boldsymbol{\nu}), \ \tau > 0, \tag{4.4}$$

*where $\boldsymbol{\nu}(dt, dx) = \nu_t(dx) \, dt$.*[1]

Proof. Assume that $\nu_t$, $t \in \mathbb{R}$, is an evolution system of measures. It is enough to show (4.4) when

$$u(t, x) = \alpha(t) \varphi(x), \quad t \in \mathbb{R}, \ x \in E,$$

[1] $\boldsymbol{\nu}$ is not a probability measure on $\mathbb{R} \times E$.

where $\alpha \in C_b(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\varphi \in C_b(E)$. In this case we have, taking into account that $v_t = P^*_{s,t} v_s$

$$\begin{aligned}\int_{\mathbb{R}\times E} S_\tau u \, d\boldsymbol{\nu} &= \int_{\mathbb{R}} \alpha(t+\tau) \int_E P_{t,t+\tau}\varphi(x) v_t(dx)\, dt \\ &= \int_{\mathbb{R}} \alpha(t+\tau) \int_E P_{s,t} P_{t,t+\tau}\varphi(x) v_s(dx)\, dt \\ &= \int_{\mathbb{R}} \alpha(t+\tau) \int_E P_{s,t+\tau}\varphi(x) v_s(dx)\, dt \\ &= \int_{\mathbb{R}} \alpha(t) \int_E P_{s,t}\varphi(x) v_s(dx)\, dt \\ &= \int_{\mathbb{R}} \int_E \alpha(t)\varphi(x) v_t(dx)\, dt,\end{aligned}$$

and (4.4) follows.

Conversely, assume that

$$\int_{\mathbb{R}\times E} S_\tau u \, d\boldsymbol{\nu} = \int_{\mathbb{R}} \alpha(t+\tau) \int_E P_{t,t+\tau}\varphi(x) v_t(dx)\, dt = \int_{\mathbb{R}\times E} u \, d\boldsymbol{\nu},$$

for all $\tau > 0$, where $u(t,x) = \alpha(t)\varphi(x)$ and $\alpha$ and $\varphi$ are as before. Then we have

$$\begin{aligned}\int_{\mathbb{R}} \alpha(t) \int_E \varphi(x) v_t(dx)\, dt &= \int_{\mathbb{R}} \alpha(t+\tau) \int_E P_{t,t+\tau}\varphi(x) v_t(dx)\, dt \\ &= \int_{\mathbb{R}} \alpha(t+\tau) \int_E \varphi(x) P^*_{t,t+\tau} v_t(dx)\, dt \\ &= \int_{\mathbb{R}} \alpha(s) \int_E \varphi(x) P^*_{s-\tau,s} v_{s-\tau}(dx)\, ds.\end{aligned}$$

By the arbitrariness of $\alpha$ it follows that

$$\int_E \varphi(x) v_t(dx) = \int_E P_{t-\tau,t}\varphi(x) v_{t-\tau}(dx), \quad \text{for } dt\text{-a.s. } t \in \mathbb{R}.$$

To complete the proof we have to show that this holds for every $t \in \mathbb{R}$. Since the left hand side is continuous in $t$ by assumption, it remains to show that for all $\tau \geq 0$

$$\mathbb{R} \ni s \mapsto \int_E P_{s,s+\tau}\varphi \, dv_s$$

is continuous. But this is an immediate consequence of the continuity of $v_s$, $s \in \mathbb{R}$, and property (vi) of $\pi$. □

### 4.1. *Asymptotic behavior*

Concerning the asymptotic behavior of $P_{s,t}$ both for $s \to -\infty$ and $t \to +\infty$ there are interesting situations where there is a unique evolution system of measures $\nu_t$, $t \in \mathbb{R}$, which, in addition, enjoys the following properties

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_E \varphi \, d\nu_t, \quad \forall \varphi \in C_b(E),\ t \in \mathbb{R}, \tag{4.5}$$

and

$$\lim_{t\to+\infty} \left[ P_{s,t}\varphi(x) - \int_E \varphi \, d\nu_t \right] = 0, \quad \forall \varphi \in C_b(E),\ s \in \mathbb{R}. \tag{4.6}$$

Equations (4.5) and (4.6) were first proved in [8] for reaction-diffusion equations, then in [7] and [13] for Ornstein–Uhlenbeck semigroups with time dependent coefficients and in [6] for the 2D Navier–Stokes equation, see also [17].

(4.6) gives also some information about the Fokker–Planck equation (3.1). Namely if $\mu(t, dx) = \mu_t(dx)\, dt$ is the solution of (3.1) with $\mu_0 = \zeta$, we have

$$\lim_{t\to+\infty} \left[ \int_E \varphi \, d\mu_t - \int_E \varphi \, d\nu_t \right] = 0, \quad \forall \varphi \in C_b(E). \tag{4.7}$$

This shows that asymptotically for $t \to \infty$, $\mu_t$ is close to $\nu_t$ independently of $\mu_0$.

## A. Appendix

Lemma A.1. *We have*

$$C([0,T]; UC_b(E)) = UC_b([0,T] \times E),$$

*with the same norms.*

Proof. "$\supset$" Let $u \in UC_b([0,T] \times E)$ and let $d$ denote the metric in $E$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} |t-s| + d(x,y) < \delta &\Rightarrow |u(t,x) - u(s,y)| < \varepsilon \\ &\Rightarrow |u(t,x) - u(s,x)| < \varepsilon \\ &\Rightarrow \|u(t,\cdot) - u(s,\cdot)\|_0 < \varepsilon \\ &\Rightarrow u \in C([0,T]; UC_b(E)). \end{aligned}$$

"$\subset$" Let $u \in C([0,T]; UC_b(E))$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|t-s| < \delta \Rightarrow \|u(t,\cdot) - u(s,\cdot)\|_0 < \varepsilon.$$

Furthermore, $\{u(s,\cdot) : s \in [0,T]\}$ as a continuous image of the compact set $[0,T]$ is compact in $UC_b(E)$. Hence for any $\varepsilon > 0$ there exists $s_1, \dots, s_n \in [0,T]$ such that

$$\{u(s,\cdot) : s \in [0,T]\} \subset \bigcup_{i=1}^{n} \{v \in UC_b(E) : \|v - u(s_i,\cdot)\|_0 < \varepsilon\}.$$

CLAIM. $\{u(s,\cdot) : s \in [0,T]\}$ is uniformly equicontinuous.

PROOF OF THE CLAIM (see proof of Ascoli–Arzelà).

Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$d(x,y) < \delta \Rightarrow |u(s_i,x) - u(s_i,y)| < \frac{\varepsilon}{3}, \quad i = 1, \dots, n.$$

Then for any $s \in [0,T]$ there exists $i \in \{1, \dots, n\}$ such that

$$\|u(s,\cdot) - u(s_i,\cdot)\|_0 < \frac{\varepsilon}{3}.$$

Hence we have

$$\begin{aligned} |u(s,x) - u(s,y)| \le\ & |u(s,x) - u(s_i,x)| \\ & + |u(s_i,x) - u(s_i,y)| \\ & + |u(s_i,y) - u(s,y)| < \varepsilon, \end{aligned}$$

provided $d(x,y) < \delta$, which proves the claim.

Now let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|t - s| < \delta \Rightarrow \|u(t,\cdot) - u(s,\cdot)\|_0 < \frac{\varepsilon}{2}, \quad i = 1, \dots, n$$

and (by Claim)

$$d(x,y) < \delta \Rightarrow |u(s,x) - u(s,y)| < \frac{\varepsilon}{2} \quad \forall s \in [0,T].$$

Therefore

$$\begin{aligned} |t - s| + d(x,y) < \delta \Rightarrow\ & |u(t,x) - u(s,y)| \\ & \le |u(t,x) - u(s,x)| \\ & \quad + |u(s,x) - u(s,y)| \le \varepsilon. \end{aligned}$$

□

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